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ESTIMATES OF THE ACTION ORDER OF SOME JET GROUPS

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Abstract. As we know, if the general k-jet group $G(n,k)$ in $\mathbf{R}^n$ acts continuously on a manifold of finite dimension $m > 0$ and the highest k-th order derivatives act nontrivially, then k is bounded by a known function of m and n [4]. In geometric interpretation the estimate function translates into the order of the natural bundle in which the group $G(n,k)$ acts on the standard fibre (meant as the space of geometric objects). On manifolds with extra structure the same question arises for corresponding subgroups of $G(n,k)$, preserving the structure. In this note the sharp upper estimates are found for actions of unimodular, homothetic and symplectic subgroups.

1. Introduction

Given a volume form ω on a smooth n-dimensional manifold, the k-jet group of local diffeomorphisms of $\mathbf{R}^n$ preserving the form $\omega_o = dx^1 \wedge \cdots \wedge dx^n$ is the *unimodular* differential group $Sl(n,k)$. It may be seen as group of polynomial diffeomorphisms of $\mathbf{R}^n$ of degree k, without constant term, with determinant equal to 1. Its Lie algebra $sl(n,k)$ consists of polynomial, of degree $\leq k$, vector fields of divergence zero.

A complementary subgroup to $Sl(n,k)$ in $G(n,k)$ is the group of *nonlinear homotheties* of variable scale $\rho(x) \neq 0$. Denote it by $H(n,k)$. Its elements are polynomial diffeomorphisms of the form $f(x) = \rho(x)x$.

The Lie algebra $h(n,k)$ is made up of all vector fields $X = aX_o$ with $X_o = \sum_i x_i \frac{\partial}{\partial x_i}$ and $a(x) = \sum_{1\leq|\alpha|\leq k} a_\alpha x^\alpha$.

The Lie algebras of $Sl(n,k)$ and $H(n,k)$ are direct summands of the Lie algebra of $G(n,k)$,

$$sl(n,k) \oplus h(n,k) = g(n,k)\,.$$

The *symplectic* k-jet group $Sp(n,k)$ is formed by all local polynomial (of degree $\leq k$) diffeomorphisms in $\mathbf{R}^{2n}$ preserving the origin and the symplectic form $\omega =$

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$\sum_{1\le i\le n} dx_i \wedge dx_{n+i}$. Its Lie algebra is formed by Hamiltonian vector fields

$$X_f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_{n+i}} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_{n+i}}\,, \tag{1.1}$$

where f is an arbitrary polynomial of the form $\sum_{2\le|\alpha|\le k+1} a_\alpha x^\alpha$.

A k-jet group acts *k-essentially* means that the action does not factorize to an action of its truncation to order $k-1$.

Theorem 1 (Main Result)**.** *Let $\mathbf{G} \subset G(n,k)$ be one of the following k-jet groups*

1. *$Sl(n,k)$ of unimodular diffeomorphisms of $\mathbf{R}^n$, $(n \ge 2)$,*
2. *$H(n,k)$ of nonlinear homotheties $(n \ge 1)$,*
3. *$Sp(2n,k)$ of symplectic diffeomorphisms.*

Suppose $\mathbf{G}$ acts continuously and k-essentially on a manifold of dimension $m \ge 1$. Then, respectively,

(1) $k \le \frac{m}{n-1}$, (2) $k \le 2m+1$, (3) $k \le \frac{m}{n}$,

and the estimates are sharp.

The main idea of the procedure to get the estimates is as for the action of the general group $G(n,k)$ (then $k \le \max\{\frac{m}{n-1}, \frac{m}{n}+1\}$). For details see the monograph [2] or the article [4]. It consists in the estimation of the lowest possible codimension d of "admissible" Lie subalgebras g of the Lie algebra $\mathbf{g}$ of $\mathbf{G}$.

Roughly we justify it as follows. Let m be the dimension of the manifold M on which the group $\mathbf{G}$ acts continuously. If $y \in M$ and G_y is the isotropy subgroup at the point y then

$$m \ge \dim(\mathbf{G}/G_y) \ge d\,. \tag{1.2}$$

This is so because the homogeneous space $\mathbf{G}/G_y$ is mapped injectively and continuously into M. Solving the extreme inequality we get an estimation of k.

For technical reason, we determine the lower bound d by passing to the graded subalgebra

$$Gr(g) = g_o + g_1 + \cdots + g_{k-1}, \quad \dim Gr(g) = \dim g$$

induced by the natural filtration in jet groups. The elements of g_p are represented by homogeneous polynomial vector fields of degree $p+1$, which belong to g modulo polynomials of higher degree (if any). The admissibility of g means that $g_{k-1} \ne \mathbf{g}_{k-1}$, i.e. it is not maximal. By the above equality of dimensions, it suffices to consider only graded subalgebras of $Gr(\mathbf{g})$. The results will be true also for the original Lie algebras. Another essential simplification is the use of the following algebraic lemma ([2], [4]).

Lemma 1. *Let E, F be finite dimensional real vector spaces and let $f : E\times F \to R$ be a bilinear form . Denote by E^o or F^o the annihilators of E or F related to f, respectively. Let $M \subset E$ and $N \subset F$ be subspaces satisfying $f|(M\times N) = 0$. Then*

$$\operatorname{codim} M + \operatorname{codim} N \ge \operatorname{codim} E^o\,.$$

If g_{k-1} is not maximal, there is a non-zero linear form C on $\mathbf{g}_{k-1}$ with $g_{k-1} \subset \ker C$. Given p, q, $p+q=k-1$, we define a bilinear form $f : \mathbf{g}_p \times \mathbf{g}_q \to R$ by $f(X,Y) = C([X,Y])$ and we study the dimensions of the annihilators. By Lemma 1 we get the estimate

$$\operatorname{codim} g_p + \operatorname{codim} g_q \geq \operatorname{codim} h$$

where $h \subset \mathbf{g}_p$ is the annihilator of $\mathbf{g}_q$ related to f.

For the coordinate expression, let

$$X = \sum_{i,\lambda} x^i_\lambda \frac{\partial}{\partial x^i} \in g_p, \qquad Y = \sum_{i,\mu} y^i_\mu \frac{\partial}{\partial x^i} \in \mathbf{g}_q\,.$$

Then

$$[X,Y]^i_\gamma = \sum_{\substack{\gamma=\lambda+\mu-1_j\\|\lambda|=p+1\\|\mu|=q+1}} (\lambda_j x^i_\lambda y^j_\mu - \mu_j y^i_\mu x^j_\lambda),$$

$$C([X,Y]) = \sum_{i,\gamma} C^\gamma_i [X,Y]^i_\gamma, \quad |\gamma| = k\,.$$

Thus we shall study the equation

$$\sum_{\substack{i,j\\|\lambda|=p+1\\|\mu|=q+1}} \left(C^{\mu+\lambda-1_j}\lambda_j - C^{\mu+\lambda-1_i}\mu_i\right) x^i_\lambda y^j_\mu = 0\,. \tag{1.3}$$

The required codimension of h is bounded below by the number of linearly independent coefficients at independent y^j_μ, i.e. by the rank of the linear system

$$\sum_{\substack{1\leq i\leq n\\|\lambda|=p+1}} \left(C^{\mu+\lambda-1_j}\lambda_j - C^{\mu+\lambda-1_i}\mu_i\right) x^i_\lambda = 0. \tag{1.4}$$

where the indices j,μ are fixed and stay at independent y's.

2. The case g=$sl(n,k)$

In this case the polynomial vector fields X, Y have divergence zero. In coordinates it writes

$$\sum_i (\mu_i + 1) y^i_{\mu+1_i} = 0 \qquad \text{for each } \mu. \tag{2.1}$$

Therefore the variables y^j_μ with $j \notin \mu$ are independent. Hence the equations (1.4) hold for such couples (j,μ). We shall use the symbol (1) for a multiindex μ with $\mu_1 = |\mu|$.

Denote by E_s the (LHS of) $(s,(1))$-equation for $2 \leq s \leq n$ and by F_k the $(n,(1)+1_k)$-equation for $1 \leq k < n$. They are

$$E_s = \sum_{i,\lambda} \left(C^{(1)+\lambda-1_s}_i \lambda_s - C^{(1)+\lambda-1_i}_s \delta^1_i (q+1)\right) x^i_\lambda\,.$$

$$F_k = \sum_{i,\lambda} \big(C_i^{(1)+1_k+\lambda-1_n}\lambda_n - C_n^{(1)+1_k+\lambda-1_i}(\delta_i^1 q + \delta_i^k)\big) x_\lambda^i \,.$$

Now, let E_1 be the joint coefficient at $y^1_{(1)}$ after elimination

$$y^n_{(1)+1_n} = -(q+1)y^1_{(1)} - y^2_{(1)+1_2} - \cdots - y^{n-1}_{(1)+1_{n-1}}$$

(see (2.1)). We have

$$\begin{aligned} E_1 = \sum_{i,\lambda} \Big[& \big(C_i^{(1)+\lambda-1_1}\lambda_1 - C_1^{(1)+\lambda-1_i}(\delta_i^1(q+1)\big) \\ & - (q+1)\big(C_i^{(1)+1_n+\lambda-1_n}\lambda_n - C_n^{(1)+1_n+\lambda-1_i}(\delta_i^1 q + \delta_i^k)\big)\Big] x_\lambda^i \,. \end{aligned}$$

We claim that the subsystem $E_1 = 0,\ E_s = 0,\ F_k = 0$ of $2n-1$ equations is of rank $\geq 2n-2$. For this consider a linear combination

$$a^1 E_1 + \sum_{2\leq s\leq n} a^s E_s + \sum_{1\leq k\leq n-1} b^k F_k = 0 \qquad a^i, b^k \in \mathbf{R}\,. \tag{2.2}$$

We may assume that $C_n^{(1)} \neq 0$ (any linear form C which does not annihilates the whole subalgebra $sl(n,k)$ can satisfy this condition in appropriate coordinate system; we deduce it from properties established in [2] or [4]).

The couple $(i,\lambda) = (n,(1))$ stands at independent variable $x^n_{(1)}$; on substituting it in (2.2) we get

$$C_n^{(1)}(p+q+1)a^1 = 0\,,$$

which gives $a^1 = 0$. Then, if we substitute $\lambda = (1) + 1_v$ and $i = n$ in (2.2) we obtain

$$\sum_{2\leq s\leq n} C_n^{(1)+1_v-1_s}\lambda_s a^s = 0\,.$$

For $s \neq v$ $\lambda_s = 0$, and for $s = v$ $\lambda_v = 1$, so it gives $a^v = 0$ for $2 \leq v \leq n-1$.

Assume now $\lambda = (1)$ and $2 \leq i \leq n-1$. From (2.2) it remains

$$\sum_{2\leq k\leq n-1} C_n^{(1)+1_k-1_i}\delta_i^k b^k = 0\,.$$

Hence $b^i = 0$ for $2 \leq i \leq n-1$. Consequently, (2.2) reduces to

$$a^n E_n + b^1 F_1 = 0$$

or explicitely

$$\sum_{i,\lambda} \big(C_i^{(1)+\lambda-1_n}\lambda_n - C_n^{(1)+\lambda-1_i}\delta_i^1(q+1)\big)(a^n + b^1)x_\lambda^i = 0\,.$$

Taking account of condition (2.1) we find the joint coefficient at $x^1_{(1)+1_n}$ to be $-(p+q+2)C_n^{(1)}(a^n + b^1)$. Hence $a^n + b^1 = 0$.

All together we proved that

$$a^1 = \ldots = a^{n-1} = 0,\ b^2 = \ldots = b^{n-1} = 0,\ a^n + b^1 = 0\,,$$

so that the rank of the system (2.2) is $\geq 2n-2$.

In view of Lemma 1 we get the estimation

$$\operatorname{codim} g_p + \operatorname{codim} g_{k-p-1} \geq 2n-2, \qquad 0 \leq p \leq k-1\,,$$

and consequently we have proved

Proposition 1. *Let g be a subalgebra of $\mathbf{g} = sl(n,k)$ which does not contain $\mathbf{g}_{k-1}$. Then*

$$codim\, g \geq k(n-1)\,.$$

The following example shows that the estimate is the best possible.

Example I

Let $g \subset sl(n,k)$ be defined by

$$g = \{\sum_{i,\nu} a^i_\nu x^\nu \frac{\partial}{\partial x_i} \in sl(n,k);\; a^j_{(1)} = 0, \quad \text{for } \; j = 2,...,n,\; 1 \leq |(1)| \leq k\}\,.$$

The codimension of this subalgebra is $k(n-1)$.

3. The case $\mathbf{g} = h(n,k)$

We apply the same procedure as in the previous section. In coordinates the homogeneous elements $X \in \mathbf{g}_p$, $Y \in \mathbf{g}_q$, $p+q = k-1$, write now

$$X = \sum_{i,\lambda} a_\lambda x^{\lambda+1_i} \frac{\partial}{\partial x_i}, \qquad Y = \sum_{i,\mu} b_\mu y^{\mu+1_i} \frac{\partial}{\partial x_i}$$

and

$$[X,Y] = \sum_{\substack{1\leq i\leq n\\ |\lambda|=p,\,|\mu|=q}} a_\lambda b_\mu \big(|\lambda| - |\mu|\big) x^{\lambda+\mu+1_i} \frac{\partial}{\partial x_i} \in \mathbf{g}_{k-1}\,. \tag{3.1}$$

Assume, correspondingly,

$$C([X,Y]) = \sum_{\lambda+\mu} C^{\lambda+\mu} a_\lambda b_\mu \big(|\lambda| - |\mu|\big) = 0,$$

for $X \in g_p$ and $Y \in \mathbf{g}_q$ with $p \neq q$. Since b_μ are arbitrary, we get the system of linear equations in x_λ:

$$\sum_{|\lambda|=p} C^{\lambda+\mu} a_\lambda = 0, \quad \text{for all } \; |\mu| = q\,.$$

For any non-zero form $C \in S^k(R^{n^*})$, this system has rank at least 1. Hence we deduce

$$\operatorname{codim} g_p + \operatorname{codim} g_{k-p-1} \geq 1, \quad p \neq \frac{k-1}{2}\,.$$

Summing up over p yields

$$2\operatorname{codim} g \geq \begin{cases} k & \text{if } k \text{ is even,}\\ k-1 & \text{if } k \text{ is odd.}\end{cases}$$

It gives $\operatorname{codim} g \geq \left[\frac{k}{2}\right]$.

Proposition 2. *Any admissible Lie subalgebra of $h(n,k)$ has codimension at least* $\left[\frac{k}{2}\right]$.

The following example shows that the estimate is sharp.

Example II

$$g = \{\sum_{i,\nu} a_\nu x^{\nu+1_i} \frac{\partial}{\partial x_i} \in h(n,k); a_{(1)} = 0, \text{ for } |(1)| = 1, ..., [\frac{k}{2}]\}.$$

Its codimension is $\left[\frac{k}{2}\right]$.

4. The case $\mathbf{g} = sp(2n,k)$

The map $f \to X_f$ defined in (1.1) is an isomorphism of $sp(2n,k)$ and the Lie algebra $\mathcal{F}(2n,k+1)$ of all polynomials of degree $k+1$ without terms of degree ≤ 1, with the Poisson bracket

$$\{f,g\} = \sum_{1\leq i\leq n} \frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_{n+i}} - \frac{\partial f}{\partial x_{n+i}}\frac{\partial g}{\partial x_i}. \tag{4.1}$$

Then $X_{\{f,g\}} = [X_f, X_g]$.

To introduce filtration and the grading

$$\mathcal{F}(n,k+1) = h_o + h_1 + ... + h_{k-1},$$

corresponding to that of $sp(n,k)$, we define $ord(f)$ by that for monomials $ord(x^\alpha) := |\alpha| - 2$. For any polynomial function $f \in \mathcal{F}$, $ord(f)$ is defined to be the minimum of orders of non-zero monomials.

Equivalently, it suffices to find the lowest codimension of admissible Lie subalgebras h in $\mathcal{F}(n,k+1)$. The algorithm is analogous as before.

Take homogeneous polynomials

$$f = \sum_{|\mu|=p+2} a_\mu x^\mu \in h_p, \quad g = \sum_{|\nu|=q+2} b_\nu x^\nu \in h_q, \quad p+q = k-1$$

and compute the Poisson bracket

$$\{f,g\} = \sum_{\substack{1\leq i\leq n\\ |\mu|=p+2\\ |\nu|=q+2}} (\mu_i\nu_{n+i} - \mu_{n+i}\nu_i) a_\mu b_\nu x^{\mu+\nu-1_i-1_{n+i}} \in h_{k-1}.$$

Take a non-zero form $C \in S^{k+1}(\mathbf{R}^{n^*})$ and suppose that $C^\gamma\{f,g\}_\gamma = 0$. Explicitely, it follows

$$\sum_{i,\mu,\nu} C^{\mu+\nu-1i-1n+i}(\mu_i\nu_{n+i} - \mu_{n+i}\nu_i) a_\mu b_\nu = 0.$$

Hence, as b_ν are arbitrary, we get the linear system

$$\sum_{i,\mu} C^{\mu+\nu-1_i-1_{n+i}}(\mu_i\nu_{n+i} - \mu_{n+i}\nu_i) a_\mu = 0, \quad |\nu| = q+2. \tag{4.2}$$

Notice that, up to a linear change of coordinates in $\mathbf{R}^n$, we may assume $C^{(1)} \neq 0$.

Denote by E_1, F_i, G_i, H_i the ν-equations of (4.1) for ν equal respectively to

$$(1),\ \ (1)+1_i+1_{n+i},\ \ (1)+1_i,\ \ (1)+1_{n+i}\,.$$

We claim that the subsystem is of rank $\geq 2n$.

For this consider linear combination

$$e_1E_1+f_iF_i+g_iG_i+h_iH_i=0 \tag{4.3}$$

with real coefficients. Set $\mu=(1)+1_i+1_{n+i}$, then the combination (4.3) reduces to

$$C^{(1)}(-q-2)e_1a_{(1)+1_{n+1}}=0\,.$$

As a_μ are arbitrary and $C^{(1)}\neq 0$, it follows $e_1=0$. Set $\mu=(1)$. From (4.2) we get $C^{(1)}(p+2)h_1a_{(1)}=0$, whence $h_1=0$. For $\mu=(1)+1_{n+i}$, using (4.2) and the so far results, we have

$$\sum_{2\leq i\leq n} C^{(1)}g_ia_{(1)+1_i}=0\,,$$

hence $g_i=0$ for $2\leq i\leq n$.

For $\mu=(1)+1_i$ we obtain

$$\sum_{2\leq i\leq n} C^{(1)}f_ia_{(1)+1_i}=0\,,$$

hence $f_i=0$ for $2\leq i\leq n$.

Thus, jointly, $2n$ coefficients in the linear combination (4.3) vanish, so that the rank of the considered subsystem is at least $2n$. It gives $\operatorname{codim} h_p+\operatorname{codim} h_{k-p-1}\geq 2n$ for $0\leq p\leq k-1$. Hence $\operatorname{codim} h\geq kn$. It translates into

Proposition 3. *Any admissible Lie subalgebra of $sp(2n,k)$ has codimension at least kn*

The following example makes the estimate the best possible.

Example III

Let $g\subset sp(2n,k)$ be defined by

$$g=\{X_f;\ \ f=\sum_\mu a_\mu x^\mu;\ \ a_{(1)+1_i}=0\ \text{ for } 1\leq i\leq n,\ \ 1\leq|(1)|\leq k\}\,.$$

Its codimension is kn.

In view of (1.2), the Propositions 1,2 and 3 followed by the examples imply immediately the assertions of the Theorem 1.

Some consequences

(i) If $m<n-1$, the unimodular k-jet group $SL(n,k)$ acts trivially on m-dimensional manifolds. The first positive case $m=n-1$ is represented by the natural action of $SL(n,k)$ on the projective space $P^{n-1}(\mathbf{R}^n)$.

In unimodular geometry on n-manifolds every field of geometric objects valued in $\mathbf{R}^m$, $m<n-1$, is a collection of functions.

If $m < n-1$, the linear group $GL(n)$ can act nontrivially on m-manifolds only via the determinant.

It has been known, that for linear representations of $Gl(n)$, this assertion is true already for $m < n$.

(ii) Similar conclusions can be formulated for the action of the symplectic k-jet group $Sp(2n, k)$ in the case $m < n$.

(iii) Since for $n = 1$ the homothetic jet group $H(1, k)$ coincides with the general differential group $G(1, k)$, the Theorem 1 translates into the following known theorem, proved first by J. E. Penzov (1950) [3] and independently by D. B. A. Epstein and W. P. Thurston (1979) [1].

If a differential group G(1,k) *acts continuously on a manifold of dimension m, then $k \leq 2m+1$ and the estimate is sharp.*

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