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THE CONTACT SYSTEM ON THE SPACES OF (m, ℓ)-VELOCITIES

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ABSTRACT. In this paper we define the contact system on the space M_m^ℓ of the (m, ℓ)-velocities of a smooth manifold M. For each velocity $p_m^\ell \in M_m^\ell$, the tangent space $T_{p_m^\ell} M_n^\ell$ and the $\mathbf{R}_m^\ell$-module $\mathrm{Der}_{\mathbf{R}}(C^\infty(M), \mathbf{R}_m^\ell)$ are canonically isomorphic; as a consequence, p_m^ℓ gives rise to a morphism p_{m*}^ℓ between the $\mathbf{R}_m^{\ell-1}$-modules $\mathrm{Der}_{\mathbf{R}}(\mathbf{R}_m^\ell, \mathbf{R}_m^{\ell-1})$ and $T_{p_m^{\ell-1}} M_m^{\ell-1}$ which is injective if and only if p_m^ℓ is regular. If X is an m-dimensional submanifold of M and p_m^ℓ is a regular point of X_m^ℓ, then the image of the above morphism is the tangent space to $X_m^{\ell-1}$ at $p_m^{\ell-1}$; in this sense, p_m^ℓ is a frame for $X_m^{\ell-1}$ at $p_m^{\ell-1}$.

Each smooth differential form on M can be prolonged to a form on M_m^ℓ with values in $\mathbf{R}_m^\ell$; the inner product of the lift of each $(m+1)$-form ω on M to $M_m^{\ell-1}$ with the image by each p_{m*}^ℓ of a basis of $\mathrm{Der}_{\mathbf{R}}(\mathbf{R}_m^\ell, \mathbf{R}_m^{\ell-1})$ gives rise to an $\mathbf{R}_m^{\ell-1}$-valued 1-form defined on M_m^ℓ. The Pfaff system generated by the real components of those 1-forms, when ω runs through the set of $(m+1)$-forms on M, is the contact system on M_m^ℓ.

1. The spaces of (m, ℓ)-velocities

In this section we fix the notations used along the paper and recall the basic definitions and properties of the spaces of A-points and (m, ℓ)-velocities of a smooth manifold. A more detailed exposition may be found in [5] (see also [6, 4, 3, 2]).

By a *local algebra* (also called *Weil algebra* in [3]) we shall mean a finite dimensional local commutative $\mathbb{R}$-algebra A with a unit.

If A is a local algebra and $\mathfrak{m}$ its maximal ideal, then there is a nonnegative integer ℓ such that $\mathfrak{m}^\ell \neq 0$ and $\mathfrak{m}^{\ell+1} = 0$; this integer is called the *height* of A, according to Weil [6]. The *width* of A is the dimension of the vector space $\mathfrak{m}/\mathfrak{m}^2$.

Let us denote $\mathbb{R}_m^\infty = \mathbb{R}[[X_1, \dots, X_m]]$ and let $\mathfrak{m}(\mathbb{R}_m^\infty)$ be its maximal ideal; the quotient ring $\mathbb{R}_m^\ell = \mathbb{R}_m^\infty \big/ \mathfrak{m}(\mathbb{R}_m^\infty)^{\ell+1}$ is a local algebra of height ℓ. In general, if $m_1, \dots, m_k, \ell_1, \dots, \ell_k$ are positive integers, then the tensor product

$$\mathbb{R}_{m_1,\dots,m_k}^{\ell_1,\dots,\ell_k} = \mathbb{R}_{m_1}^{\ell_1} \otimes \cdots \otimes \mathbb{R}_{m_k}^{\ell_k}$$

is a local algebra of height $\ell_1 + \cdots + \ell_k$. Each local algebra A is a quotient of $\mathbb{R}_m^\infty$ by an ideal of finite codimension (for a proof see [3]).

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Definition 1.1. Let M be a smooth manifold and A a local algebra. An *A-point* or *near point of type* A of M is an algebra homomorphism $p^A : C^\infty(M) \longrightarrow A$. A near point $p^A \in M^A$ is said to be *regular* if the algebra homomorphism $p^A : C^\infty(M) \longrightarrow A$ is onto. We will denote by M^A the set of A-points of M; the set of regular A-points of M will be denoted by $\check{M}^A$.

Examples. (1) The space of algebra homomorphisms $\mathrm{Hom}_{\mathbb{R}}(C^\infty(M), \mathbb{R})$ is well known to be M, hence the $\mathbb{R}$-points of M are the usual points of M. Thus, if A is a local algebra, the composition of each A-point $p^A \in M^A$ with the homomorphism $A \longrightarrow A/\mathfrak{m} \approx \mathbb{R}$ is a point $p \in M$. We say that p^A is an A-point *near* p and that p is the *projection of* p^A *into* M.

(2) If $\mathbb{D} = \mathbb{R}_1^1$, the algebra of dual numbers, then $M^{\mathbb{D}} = TM$, the tangent bundle to M.

(3) When $A = \mathbb{R}_m^\ell$, the space of $\mathbb{R}_m^\ell$-points of M will be denoted by M_m^ℓ; it agrees with the space $J_0^\ell(\mathbb{R}^m, M)$ of (m, ℓ)-velocities on M defined by Ehresmann [1]. Moreover, the regular (m, ℓ)-points of M are the regular (m, ℓ)-velocities; more concretely: Let $p_m^\ell \in M_m^\ell$, where $\ell \geq 1$, and $\varphi : \mathbb{R}^m \longrightarrow M$ a mapping such that $j_0^\ell \varphi = p_m^\ell$. Then p_m^ℓ is regular if and only if φ defines a local diffeomorphism between a neighbourhood of the origin of $\mathbb{R}^m$ and a locally closed submanifold of M. $\check{M}_m^\ell$ is an open subset of M_m^ℓ; for $\ell > 0$ and $m > n = \dim M$, $\check{M}_m^\ell = \emptyset$. If $m \leq n$, then $\check{M}_m^\ell$ is a dense subset of M_m^ℓ (see [5]).

Let M and A be as above; each function $f \in C^\infty(M)$ can be prolonged to a mapping $f^A : M^A \longrightarrow A$ defined by $f^A(p^A) = p^A(f)$. We will simply write f instead of f^A when no confusion can arise.

Let $\{a_1, \ldots, a_d\}$ be a basis of A as a vector space; $f(p^A)$ can be written in the form

$$f(p^A) = \sum_{k=1}^{d} f_k(p^A) a_k,$$

$f_1, \ldots, f_d$ being real-valued functions defined on M^A, called the *real components* of f in M^A with respect to the basis $\{a_1, \ldots, a_N\}$. The set M^A can be given a smooth structure canonically determined by the condition that each $f \in C^\infty(M)$ be smooth when considered as a mapping from M^A to A.

Let $y_1, \ldots, y_n \in C^\infty(M)$ be a coordinate system on an open subset U of M; set $A = \mathbb{R}_m^\ell$ and take the basis $\{\frac{1}{\alpha!} x^\alpha : |\alpha| \leq \ell\}$ of A. If for each $p_m^\ell \in U_m^\ell$ we write

$$y_i(p_m^\ell) = \sum_{|\alpha| \leq \ell} \frac{1}{\alpha!} y_{i\alpha}(p_m^\ell) x^\alpha \qquad i = 1, \ldots, n,$$

the functions $y_{i\alpha}$ $(1 \leq i \leq n;\ |\alpha| \leq \ell)$ form a coordinate system in U_m^ℓ.

If A is a local algebra, the mapping which assigns to M the manifold M^A is a covariant functor from the category of finite dimensional smooth manifolds into itself; in fact, each smooth mapping $\varphi : M \longrightarrow N$ gives a mapping $\varphi^A : M^A \longrightarrow N^A$

which associates with each $p^A \in M^A$ the algebra homomorphism

$$\varphi^A(p^A)\colon C^\infty(N) \longrightarrow A$$
$$f \longmapsto (\varphi^*(f))(p^A) = (p^A \circ \varphi^*)(f)$$

It follows easily that if $\varphi\colon M \longrightarrow N$, $\psi\colon N \longrightarrow N_1$ are smooth maps, then $(\psi\circ\varphi)^A = \psi^A \circ \varphi^A$. We will simply write φ instead of φ^A when no confusion can arise. As for each $f \in C^\infty(N)$ and $p^A \in M^A$ we have

$$(\varphi^*(f))(p^A) = f(\varphi^A(p^A)),$$

if we fix a basis $a_1, \ldots, a_d$ of A it follows that

$$(\varphi^*(f))_k = f_k \circ \varphi^A = (\varphi^A)^*(f_k)$$

for $k = 1, \ldots, d$, hence $(\varphi^A)^*(f_k) \in C^\infty\left(M^A\right)$ for each $f \in C^\infty(N)$ and $1 \le k \le d$. Since the f_k determine the smooth structure in N^A, the mapping $\varphi^A : M^A \longrightarrow N^A$ is smooth.

As a special case, each smooth automorphism of M gives a smooth automorphism of M^A, and the same is true for each one-parameter group of automorphisms of M.

The following theorem, due to Weil [6], is fundamental in the theory of Weil bundles:

Theorem 1.2 (Weil). *Let M be a smooth manifold, and A, B local algebras. The manifolds $(M^B)^A$ and $M^{A\otimes B}$ are canonically diffeomorphic.*

2. Lift of tangent vector fields and differential forms

As a direct consequence of Weil's theorem and the fact that $M^{\mathbb{D}} = TM$ we have another important result:

Theorem 2.1. *For each point $p^A \in M^A$ there exists a canonical linear isomorphism between $T_{p^A}M^A$, the tangent space to M^A at p^A, and $\mathrm{Der}_{\mathbb{R}}(C^\infty(M), A)$, where A is considered as a $C^\infty(M)$-module through the homomorphism p^A.*

Because of this theorem the tangent space $T_{p^A}M^A$ can be understood as a space of derivations from $C^\infty(M)$ into A; in this case it will be denoted by $\mathcal{T}_{p^A}M^A$ and called *tangent module to M^A at p^A*. It is a free A-module of rank $n = \dim M$.

Fix a basis $a_1, \ldots, a_d$ in A; if we think a function $f \in C^\infty(M)$ as a mapping from M^A into A, we write it as $f = \sum_{k=1}^d f_k a_k$, where $f_k \in C^\infty(M^A)$. For each $\bar{D}_{p^A} \in T_{p^A}M^A$, the derivation $D_{p^A} \in \mathcal{T}_{p^A}M^A$ attached to it according to the above theorem maps each $f \in C^\infty(M)$ into the element

$$D_{p^A} f = \sum_{k=1}^{d} \left(\bar{D}_{p^A} f_k\right) a_k.$$

From now on we will consider only the spaces M_m^ℓ, although some results remain valid in the general case (see [5] for details). Let D be a tangent vector field on M; for each $p_m^\ell \in M_m^\ell$ the mapping $D_{p_m^\ell} : C^\infty(M) \longrightarrow \mathbb{R}_m^\ell$ defined as $D_{p_m^\ell}(f) = (Df)(p_m^\ell)$ is an element of $\mathcal{T}_{p_m^\ell}M_m^\ell$ which we call *value of D at p_m^ℓ*. Thus we obtain a vector field $\overline{D}$ on M_m^ℓ which will be called the *prolongation of D to M_m^ℓ*. If $f \in C^\infty(M)$ and $\{f_\alpha, |\alpha| \leq \ell\}$ are its real components on M_m^ℓ, then $\overline{D}(f_\alpha) = (Df)_\alpha$.

Proposition 2.2. *A point $p_m^\ell \in M_m^\ell$ is regular if and only if each tangent vector to M_m^ℓ at p_m^ℓ is the value at p_m^ℓ of a vector field on M.*

For a proof see [5].

For each point $p_m^\ell \in M_m^\ell$, $\mathcal{T}_{p_m^\ell} M_m^\ell$ is a free $\mathbb{R}_m^\ell$-module with rank $n = \dim M$; let us denote by $\mathcal{T}^*_{p_m^\ell} M_m^\ell$ its dual $\mathbb{R}_m^\ell$-module. Given a function $f \in C^\infty(M)$ we can define a map

$$d_{p_m^\ell} f : \mathcal{T}_{p_m^\ell} M_m^\ell \longrightarrow \mathbb{R}_m^\ell$$

by associating to each derivation $D_{p_m^\ell} \in \mathcal{T}_{p_m^\ell} M_m^\ell$ the element

$$(d_{p_m^\ell} f)(D_{p_m^\ell}) = D_{p_m^\ell} f.$$

Then $d_{p_m^\ell} f$ is $\mathbb{R}_m^\ell$-linear and the map which assigns to each $f \in C^\infty(M)$ the form $d_{p_m^\ell} f$ is a derivation from $C^\infty(M)$ into the $C^\infty(M)$-module $\mathcal{T}^*_{p_m^\ell} M_m^\ell$. We will call $d_{p_m^\ell} f$ the *differential of f at p_m^ℓ*.

If $y_1, \ldots, y_n \in C^\infty(M)$ is a coordinate system around $p = p_m^0$, for each $f \in C^\infty(M)$ we have:

$$d_{p_m^\ell} f = \sum_{i=1}^n \left(\frac{\partial f}{\partial y_i}\right)(p_m^\ell) d_{p_m^\ell} y_i,$$

as we can see by applying both sides of this equality to

$$\left(\frac{\partial}{\partial y_i}\right)_{p_m^\ell} \qquad i = 1, \ldots, n,$$

and having in mind that these derivations are a basis of $\mathcal{T}_{p_m^\ell} M_m^\ell$.

Let $\mathcal{E}^1(M)$ be the $C^\infty(M)$-module of the 1-forms in M. If $\omega \in \mathcal{E}^1(M)$ and $p \in M$, around p we can write $\omega = \sum_{i=1}^n g_i dy_i$. The germs at p of the g_i are completely determined by ω, so the same is true for the $g_i(p_m^\ell)$; then we can give the following

Definition 2.3. The *value of ω at p_m^ℓ* is

$$\omega_{p_m^\ell} = \sum_{i=1}^n g_i(p_m^\ell) d_{p_m^\ell} y_i.$$

The expression $\omega_{p_m^\ell}$ belongs to $\mathcal{T}^*_{p_m^\ell} M_m^\ell$ and the map which assigns to each $\omega \in \mathcal{E}^1(M)$ its value at p_m^ℓ is a morphism of $C^\infty(M)$- modules from $\mathcal{E}^1(V)$ into $\mathcal{T}^*_{p_m^\ell} M_m^\ell$ which agrees with the map $df \longrightarrow d_{p_m^\ell} f$ on the exact 1-forms and it is completely determined by this condition.

Proposition 2.2 asserts that, if p_m^ℓ is regular, then each element of $\mathcal{T}_{p_m^\ell} M_m^\ell$ is the value at p_m^ℓ of some vector field D tangent to M; this allows us to give in this case an alternative definition of $\omega_{p_m^\ell}$: If $D_{p_m^\ell} \in \mathcal{T}_{p_m^\ell} M_m^\ell$, then

$$\omega_{p_m^\ell}(D_{p_m^\ell}) = [\omega(D)](p_m^\ell),$$

where D is any vector field on M whose value at p_m^ℓ is $D_{p_m^\ell}$. This definition agrees with the previous one, because both of them are the same for exact 1-forms.

Our next proposition follows in a straightforward way:

Proposition 2.4. *The morphism $\mathcal{E}^1(M) \longrightarrow \mathcal{T}^*_{p_m^\ell} M_m^\ell$ can be prolonged in a natural way to a $C^\infty(M)$-algebra morphism from the covariant tensor algebra on M into the tensor algebra of the free $\mathbb{R}_m^\ell$-module $\mathcal{T}^*_{p_m^\ell} M_m^\ell$.*

Definition 2.5. If T is a covariant tensor field over M, we will call *value of T at* p_m^ℓ the image $T_{p_m^\ell}$ of T by the morphism of the previons proposition.

It is clear that if T has an homogeneous degree then $T_{p_m^\ell}$ has the same degree as T and that, if T es symmetric or skew-symmetric, the same is true for $T_{p_m^\ell}$.

3. The regular (m, ℓ)-velocities as frames

Let $p_m^\ell \in M_m^\ell$; for each derivation $\xi : \mathbb{R}_m^\ell \longrightarrow \mathbb{R}_m^{\ell-1}$ we will write $\xi_{(p_m^\ell)} = \xi \circ p_m^\ell$. It is clear that $\xi_{(p_m^\ell)} \in \mathcal{T}_{p_m^{\ell-1}} M_m^{\ell-1}$ and that the map

$$\begin{aligned} p_{m*}^\ell \colon \mathrm{Der}_{\mathbb{R}}(\mathbb{R}_m^\ell, \mathbb{R}_m^{\ell-1}) &\longrightarrow \mathcal{T}_{p_m^{\ell-1}} M_m^{\ell-1} \\ \xi &\longmapsto p_{m*}^\ell(\xi) = \xi_{(p_m^\ell)} \end{aligned}$$

is a morphism of $\mathbb{R}_m^{\ell-1}$-modules.

Proposition 3.1. *Each point $p_m^\ell \in M_m^\ell$ is completely determined by the couple (p_m^0, p_{m*}^ℓ).*

Proof. It follows from the fact that each $P(x) \in \mathbb{R}_m^\ell$ is completely determined by its projection on $\mathbb{R}$ and by the polynomials $\frac{\partial P(x)}{\partial x_i} \in \mathbb{R}_m^{\ell-1}$ $(1 \leq i \leq m)$. □

Proposition 3.2. *The point p_m^ℓ is regular if and only if p_{m*}^ℓ is injective.*

Proof. The necessity of the condition is inmediate. On the other hand, if $p_m^\ell \in M_m^\ell$ is not regular, its image is a proper subalgebra of $\mathbb{R}_m^\ell$, hence the proposition is a consecuence of the following

Lemma 3.3. *If B is a proper subalgebra of $\mathbb{R}_m^\ell$, then there is a nonzero derivation from $\mathbb{R}_m^\ell$ into $\mathbb{R}_m^{\ell-1}$ whose restriction to B vanishes.*

Proof. The m derivations $(\frac{\partial}{\partial x_i})_0 \colon \mathbb{R}_m^\ell \longrightarrow \mathbb{R}$ $(1 \leq i \leq m)$ cannot have linearly independent restrictions to B: on the contrary the "inverse function theorem module $O^{\ell+1}$" would imply that $B = \mathbb{R}_m^\ell$; hence there exist constants $\lambda_1, \dots, \lambda_m$, not all equal to zero, such that the derivation $\overline{\xi} = \lambda_1 \frac{\partial}{\partial x_1} + \cdots + \lambda_m \frac{\partial}{\partial x_m}$ from $\mathbb{R}_m^\ell$ into $\mathbb{R}_m^{\ell-1}$ applies B into $\mathfrak{m}(\mathbb{R}_m^{\ell-1})$. Therefore, if $P(x) \in \mathfrak{m}(\mathbb{R}_m^\ell)^{\ell-1}$, $P(x) \notin \mathfrak{m}(\mathbb{R}_m^\ell)^\ell$, the derivation $\xi = P(x)\overline{\xi}$ from $\mathbb{R}_m^\ell$ into $\mathbb{R}_m^{\ell-1}$ vanishes on B. □

Proposition 3.4. *Let W be a closed submanifold of M, I its ideal in $C^\infty(M)$, $p_m^\ell \in W_m^\ell$, $\overline{D}_{p_m^\ell}$ a tangent vector to M_m^ℓ at p_m^ℓ and $D_{p_m^\ell} \in \mathcal{T}_{p_m^\ell} M_m^\ell$ the derivation attached to it according to theorem 2.1. A necessary and sufficent condition for $\overline{D}_{p_m^\ell}$ to be tangent to W_m^ℓ is that the derivation $D_{p_m^\ell}$ annihilates I.*

Proof. It is straightforward. ☐

Proposition 3.5. *Let W be an m-dimensional submanifold of M, p^ℓ_m a regular point of W^ℓ_m and $p^{\ell-1}_m$ its projection into $M^{\ell-1}_m$. The tangent $\mathbb{R}^{\ell-1}_m$-module $\mathcal{T}_{p^{\ell-1}_m} W^{\ell-1}_m$ agrees with the image of* $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^{\ell-1}_m)$ *by the map p^ℓ_{m*}.*

Proof. We can suppose W closed in M; if I is its ideal in $C^\infty(M)$, from its definition and proposition 3.4 follows that the image of p^ℓ_{m*} is a subspace of $\mathcal{T}_{p^{\ell-1}_m} W^{\ell-1}_m$. As p^ℓ_m is regular in W^ℓ_m, from proposition 3.2 we conclude, being both of them free $\mathbb{R}^{\ell-1}_m$-modules of rank m. ☐

According to proposition 3.5 we can say that p^ℓ_m is a *frame for* $W^{\ell-1}_m$ at $p^{\ell-1}_m$. As a particular case, when $W = M$ and $\ell = 1$, each point $p^1_n \in \check{M}^1_n$ gives an isomorphism $p^1_{n*}\colon \mathrm{Der}_{\mathbb{R}}(\mathbb{R}^1_n, \mathbb{R}) \longrightarrow T_pM$ and by proposition 3.1 it is completely determined by the couple (p, p^1_{n*}). Thus, the projection $\check{M}^1_n \longrightarrow M \approx \mathcal{J}^1_n(M)$ is the usual frame bundle on M (note that $\mathrm{Aut}(\mathbb{R}^1_n) \approx \mathrm{GL}(n, \mathbb{R})$).

The following proposition will be useful to deal with the contact system on the higher order Grassmann bundles $\mathcal{J}^\ell_m(M)$, because the mapping $\check{M}^\ell_m \longrightarrow \mathcal{J}^\ell_m(M)$ is a fibre bundle with $\mathrm{Aut}(\mathbb{R}^\ell_m)$ as structural group (see [5, 2]).

Proposition 3.6. *Let $p^\ell_m, q^\ell_m \in \check{M}^\ell_m$; if $p^{\ell-1}_m = q^{\ell-1}_m$ and the mappings p^ℓ_{m*} and q^ℓ_{m*} have the same image in $\mathcal{T}_{p^{\ell-1}_m} W^{\ell-1}_m$, then p^ℓ_m and q^ℓ_m belong to the same orbit of the group* $\mathrm{Aut}(\mathbb{R}^\ell_m)$.

Proof. By the "inverse function theorem module $O^{\ell+1}$" it suffices to show that p^ℓ_m and q^ℓ_m have the same jet, so we will show that $\ker p^\ell_m \subseteq \ker q^\ell_m$.

Let $f \in \ker p^\ell_m$; as p^ℓ_{m*} and q^ℓ_{m*} have the same range, for each $\xi \in \mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^{\ell-1}_m)$ we have $\xi(f(q^\ell_m)) = \xi_{(q^\ell_m)} f = 0$, hence $f(q^\ell_m)$ is constant, but then

$$f(q^\ell_m) = f(q^0_m) = f(p^0_m) = 0,$$

that is to say, $f \in \ker q^\ell_m$. ☐

Example. Let $p^1_m \in \check{M}^1_m$; the map p^1_{m*} is one to one, therefore its range is an m-dimensional vector subspace of T_pM; furthermore, in this case the converse of the former proposition holds, that is to say, if two points p^1_m and q^1_m lay in the same orbit of $\mathrm{Aut}\left(\mathbb{R}^1_m\right) = Gl(m, 1)$, then $p^0_m = q^0_m$ and p^1_{m*} and q^1_{m*} have the same range in T_pM. Thus, the orbits of $\mathrm{Aut}\left(R^1_m\right)$ in $\check{M}^1_m$ can be identified with the ones of $Gl(m, 1)$ in the set of m-dimensional subspaces of T_pM, where p runs through M; hence $\check{M}^1_m$ is the m-Stiefel manifold of M and $\mathcal{J}^1_m(M)$ its m-Grassmannian.

4. The contact system on M^ℓ_m

Let $\{\xi_1, \ldots, \xi_m\}$ be a basis of the free $\mathbb{R}^{\ell-1}_m$-module $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}^\ell_m, \mathbb{R}^{\ell-1}_m)$. For each exterior differential form ω of degree $m+1$ on M and each $p^\ell_m \in M^\ell_m$ we can define a map $\hat{\omega}_{p^\ell_m}$ from $\mathcal{T}_{p^\ell_m} M^\ell_m$ into $\mathbb{R}^{\ell-1}_m$ by

$$\hat{\omega}_{p^\ell_m}\left(D_{p^\ell_m}\right) = \omega_{p^{\ell-1}_m}\left(\xi_{1(p^\ell_m)}, \ldots, \xi_{m(p^\ell_m)}, D_{p^{\ell-1}_m}\right), \tag{4.1}$$

where $p_m^{\ell-1} \in M_m^{\ell-1}$ is the projection of p_m^ℓ and $D_{p_m^{\ell-1}}$ is the one of $D_{p_m^\ell}$. It is obvious that $\hat{\omega}_{p_m^\ell}$ is $\mathbb{R}_m^\ell$-linear.

For each tangent vector field D on M we define a mapping

$$\hat{\omega}(D)\colon M_m^\ell \longrightarrow \mathbb{R}_m^{\ell-1}$$

as follows:

$$[\hat{\omega}(D)](p_m^\ell) = \hat{\omega}_{p_m^\ell}\left(D_{p_m^\ell}\right),$$

where for each $p_m^\ell \in M_m^\ell$ the value $D_{p_m^\ell}$ of D at p_m^ℓ is considered as an element of $\mathcal{T}_{p_m^\ell} M_m^\ell$. Thus, $\hat{\omega}$ is a smooth vector field of 1-forms on M_m^ℓ with values in $\mathbb{R}_m^{\ell-1}$. Its real components are a collection of $\binom{m+\ell-1}{m}$ smooth 1-forms on M_m^ℓ.

When the basis $\{\xi_1,\ldots,\xi_m\}$ in (4.1) is changed, the new $\hat{\omega}$ differs from the previous one in a constant factor belonging to $\mathbb{R}_m^{\ell-1}$. Accordingly, the Pfaff system on M_m^ℓ spanned by the real components of $\hat{\omega}$ does not depend on the basis $\{\xi_1,\ldots,\xi_m\}$.

Definition 4.1. The Pfaff system $\Omega(M_m^\ell)$ spanned in M_m^ℓ by the real components of the 1-forms $\hat{\omega}$, when ω runs through the set of $(m+1)$-forms on M, is called the *contact system* on M_m^ℓ.

Proposition 4.2. *$\Omega(M_m^\ell)$ is regular with rank $r = (n-m)\binom{m+\ell-1}{m}$ on the open subset $\check{M}_m^\ell$ and lower than or equal to r at the other points of M_m^ℓ.*

Proof. Let $p_m^\ell \in \check{M}_m^\ell$ and fix local coordinates $y_1,\ldots,y_n$ around $p = p_m^0$ in M such that

$$y_i(p_m^\ell) = x_i \qquad (i = 1,\ldots,m)$$
$$y_{m+j}(p_m^\ell) = 0 \qquad (j = 1,\ldots,n-m)$$

Then the range of p_{m*}^ℓ is the $\mathbb{R}_m^{\ell-1}$-submodule of $\mathcal{T}_{p_m^{\ell-1}} M_m^{\ell-1}$ spanned by the derivations $\left(\frac{\partial}{\partial y_i}\right)_{p_m^{\ell-1}}$, $(1 \le i \le m)$, and from formula (4.1) follows that for each $(m+1)$-form ω on M the corresponding $\hat{\omega}_{p_m^\ell}$ is a linear span, with coefficients in $\mathbb{R}_m^{\ell-1}$, of the $n-m$ 1-forms (valued in $\mathbb{R}_m^{\ell-1}$)

$$\left(d_{p_m^{\ell-1}} y_{m+j}\right) \circ {\pi_\ell^{\ell-1}}_* \qquad (j = 1,\ldots,n-m).$$

From theorem 2.1 follows that the real components of the forms $d_{p_m^{\ell-1}} y_{m+j}$ are $d_{p_m^{\ell-1}} y_{m+j,\alpha}$ $(|\alpha| \le \ell-1)$; as, on the other hand, the tangent linear map ${\pi_\ell^{\ell-1}}_*\colon \mathcal{T}_{p_m^\ell} M_m^\ell \longrightarrow \mathcal{T}_{p_m^{\ell-1}} M_m^{\ell-1}$ is onto, when ω varies the real components of $\hat{\omega}_{p_m^\ell}$ run through a vector subspace of $T_{p_m^\ell}^* M_m^\ell$ of dimension $r = (n-m)\binom{m+\ell-1}{m}$.

As $\check{M}_m^\ell$ is dense in M_m^ℓ and the rank of a Pfaff system is a lower semicontinuous function, the rank of $\Omega(M_m^\ell)$ is lower than or equal to r at every point of M_m^ℓ. □

Corollary 4.3. *For each $p_m^\ell \in \check{M}_m^\ell$ the value at p_m^ℓ of the contact system $\Omega(M_m^\ell)$ is the $\mathbb{R}_m^{\ell-1}$-submodule of $\mathcal{T}^*_{p_m^{\ell-1}} M_m^{\ell-1}$ orthogonal to the image of p_{m*}^ℓ.*

Proof. If ω is a $(m+1)$-form on M, the 1-form over $\mathcal{T}_{p_m^{\ell-1}} M_m^{\ell-1}$ (with values in $\mathbb{R}_m^{\ell-1}$) which assigns to each $D_{p_m^{\ell-1}}$ the right side of (4.1) belongs to the submodule of $\mathcal{T}^*_{p_m^{\ell-1}} M_m^{\ell-1}$ orthogonal to the image of p^ℓ_{m*}; as by proposition 3.2 the dimension of this submodule over $\mathbb{R}_m^{\ell-1}$is $n-m$, and hence $(n-m)\binom{m+\ell-1}{m}$ over $\mathbb{R}$, the former proposition allows to conclude. □

Proposition 4.4. *If* $\pi_\ell^{\ell-1} : M_m^\ell \longrightarrow M_m^{\ell-1}$ *is the canonical projection, then* $(\pi_\ell^{\ell-1})^*\Omega(M_m^{\ell-1}) \subseteq \Omega(M_m^\ell)$.

Proof. Each derivation $\xi : \mathbb{R}_m^\ell \longrightarrow \mathbb{R}_m^{\ell-1}$ applies $\mathfrak{m}(\mathbb{R}_m^\ell)^\ell$ in $\mathfrak{m}(\mathbb{R}_m^{\ell-1})^{\ell-1}$, hence it gives a derivation $\overline{\xi} : \mathbb{R}_m^{\ell-1} \longrightarrow \mathbb{R}_m^{\ell-2}$; if $\{\xi_1, \ldots, \xi_m\}$ is a basis if the $\mathbb{R}_m^{\ell-1}$-module $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}_m^\ell, \mathbb{R}_m^{\ell-1})$, then $\{\overline{\xi}_1, \ldots, \overline{\xi}_m\}$ is a basis of the $\mathbb{R}_m^{\ell-2}$-module $\mathrm{Der}_{\mathbb{R}}(\mathbb{R}_m^{\ell-1}, \mathbb{R}_m^{\ell-2})$. Then, for each $D_{p_m^\ell} \in \mathcal{T}_{p_m^\ell} M_m^\ell$ and each $(m+1)$-form ω on M we have:

$$\begin{aligned}\langle (\pi_\ell^{\ell-1})^* \left(\widehat{\omega}_{p_m^{\ell-1}}\right), D_{p_m^\ell} \rangle &= \langle \widehat{\omega}_{p_m^{\ell-1}}, D_{p_m^{\ell-1}} \rangle = \\ \omega_{p_m^{\ell-2}} &\left(\overline{\xi}_{1 p_m^{\ell-1}}, \ldots, \overline{\xi}_{m p_m^{\ell-1}}, D_{p_m^{\ell-2}} \right) = \\ &= \text{projection of } \langle \widehat{\omega}_{p_m^\ell}, D_{p_m^\ell} \rangle \text{ in } \mathbb{R}_m^{\ell-2},\end{aligned}$$

hence the set of real components of $(\pi_\ell^{\ell-1})^* \left(\hat{\omega}_{p_m^{\ell-1}}\right)$ is contained in the set of real components of $\hat{\omega}_{p_m^\ell}$, which finishes the proof. □

Theorem 4.5. *If* W *is an* m*-dimensional submanifold of* M, *then* W_m^ℓ *is a solution of the contact system* $\Omega(M_m^\ell)$. *Furthermore, it is a locally maximal solution, in the following sense: if* U^ℓ *is a submanifold of* M_m^ℓ *solution of* $\Omega(M_m^\ell)$ *and it contains an open subset of* W_m^ℓ, *then* $\dim U^\ell = \dim W_m^\ell$.

Proof. From proposition 3.5 and corollary 4.3 it follows that $\check{W}_m^\ell$ is a solution of $\Omega(M_m^\ell)$, and hence that W_m^ℓ is a solution of $\Omega(M_m^\ell)$, because $\check{W}_m^\ell$ is a dense open subset of W_m^ℓ. Thus, it remains to show the local maximality of W_m^ℓ.

Let U^ℓ be the submanifold of W_m^ℓ cited in the statement, and let us denote by $\overline{\pi}_\ell^j$ the restriction to U^ℓ of the projection $\pi_\ell^j : M_m^\ell \longrightarrow M_m^j$. Let $p_m^\ell \in U^\ell \cap \check{W}_m^\ell$ (this set is not empty, because U^ℓ contains an open subset of W_m^ℓ and $\check{W}_m^\ell$ is dense in W_m^ℓ); the rank of the linear map $\overline{\pi}^0_{\ell*} : T_{p_m^\ell} U^\ell \longrightarrow T_p M$ is m. In fact, from proposition 4.4 follows that U^ℓ is a solution of $\Omega(M_m^1)$, hence by corollary 4.3 the projection $\overline{\pi}^0_{\ell*}\left(D_{p_m^\ell}\right)$ of each vector $D_{p_m^\ell} \in T_{p_m^\ell} U^\ell$ belongs to the image of p^1_{m*}, which, according to proposition 3.5, is isomorphic to $T_p W$ and consequently has dimension m; as on the other hand $T_{p_m^\ell} U^\ell \supseteq T_{p_m^\ell} W_m^\ell$, each vector $D_p \in T_p W$ is the image under $\overline{\pi}^0_{\ell*}$ of some $D_{p_m^\ell} \in T_{p_m^\ell} U^\ell$, and we conclude.

The former discussion shows also that the rank of $\overline{\pi}^0_{\ell*}$ is lower than or equal to m at every point of U^ℓ, hence it is equal the greatest possible at p_m^ℓ and from its semicontinuity follows that it is equal to m on a neighborhood of p_m^ℓ in U^ℓ. Then, from the rank theorem follows that the image under π_ℓ^0 of a suitable neighborhood $U^\ell_{(0)}$ of p_m^ℓ in U^ℓ is an m-dimensional locally closed submanifold of M. The image under $\overline{\pi}_\ell^0$ of $U^\ell_{(0)}$ contains the one of $U^\ell_{(0)} \cap W_m^\ell$ and, as both of them have the same dimension, we can suppose that $\overline{\pi}_\ell^0(U^\ell_{(0)}) \subseteq W$, taking $U^\ell_{(0)}$ smaller if necessary. Thus we have proved the case $j = 0$ of the following assertion:

(P_j) U^ℓ contains an open subset $U^\ell_{(j)}$ whose projection by π_ℓ^j is contained in W_m^j and which contains a nonempty open subset of W_m^ℓ.

The case $j = \ell$ is precisely our theorem, which we prove by induction on j.

Let us assume that $j \geq 1$ and that (P_{j-1}) holds. Then, as $\overline{\pi}_\ell^{j-1}$ applies $U^\ell_{(j-1)}$ in W_m^{j-1}, its rank at each point of $U^\ell_{(j-1)}$ is lower than or equal to $\dim W_m^{j-1}$; $U^\ell_{(j-1)}$ contains a nonempty open subset of $\check{W}^\ell_m$, at whose points the rank of $\overline{\pi}_\ell^{j-1}$ is greater than or equal to $\dim W_m^{j-1}$, hence the set $U^\ell_{(j)}$ of those points of $U^\ell_{(j-1)}$ belonging to $\check{W}^\ell_m$ for which the rank of $\overline{\pi}_\ell^{j-1}$ is the greatest one $=\dim W_m^{j-1}$ is a nonempty open subset of $U^\ell_{(j-1)}$, hence of U^ℓ, and it contains a nonempty open subset of W_m^ℓ. If we show that $\pi_\ell^j\left(U^\ell_{(j)}\right) \subseteq W_m^j$, we will finish the proof.

Let $p_m^\ell \in U^\ell_{(j)}$, $\overline{D}_{p_m^\ell} \in T_{p_m^\ell} U^\ell$ and $D_{p_m^\ell} \in \mathcal{T}_{p_m^\ell} M_m^\ell$ the derivation attached to $\overline{D}_{p_m^\ell}$ by theorem 2.1. As, by proposition 4.4, U^ℓ is a solution of $\Omega(M_m^j)$, from corollary 4.3 follows that $D_{p_m^{j-1}}$ belongs to the image of p^j_{m*}. The condition on the rank of $\overline{\pi}_\ell^{j-1}$ at p_m^ℓ implies that, when $\overline{D}_{p_m^\ell}$ runs through $T_{p_m^\ell} U^\ell$, the corresponding $D_{p_m^{j-1}}$ runs through $\mathcal{T}_{p_m^{j-1}} W_m^{j-1}$, which is a free $\mathbb{R}_m^{j-1}$-module with rank m. As p_m^j is regular, the image of p^j_{m*} is also a free $\mathbb{R}_m^{j-1}$-module with rank m; but then, from our remark about the vectors $D_{p_m^{j-1}}$ follows that $\mathcal{T}_{p_m^{j-1}} W_m^{j-1} \subseteq \operatorname{Im} p^j_{m*}$, hence they must agree, because both of them are vector $\mathbb{R}$-spaces with the same dimension.

On the other hand, as $p_m^{j-1} \in \check{W}_m^{j-1}$, there exists a point $q_m^j \in \check{W}_m^j$ such that $q_m^{j-1} = p_m^{j-1}$; then $\operatorname{Im} q^j_{m*} = \mathcal{T}_{p_m^{j-1}} W_m^{j-1} = \operatorname{Im} p^j_{m*}$, by proposition 3.5, hence from proposition 3.6 follows that p_m^j and q_m^j lay in the same orbit of $\operatorname{Aut}(\mathbb{R}_m^j)$ in W_m^j, hence $p_m^j \in \check{W}_m^j$. Thus we have shown that $\pi_\ell^j\left(U^\ell_{(j)}\right) \subseteq \check{W}_m^j$, and (P_j).

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