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ON A DIAGRAM IN THE THEORY OF CONNECTIONS

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ABSTRACT. We study the correspondence between Christoffel and Cartan symbols and construct a commutative diagram which relates the nonlinear Janet sequences for the k^{th} order Christoffel and Cartan symbols and the nonlinear gauge sequence for the k^{th} order coframe bundle.

1. Introduction

This note consists of three parts. In the first part, we define Christoffel symbols and Cartan symbols as coordinates on some coset spaces and study the relations between them. We give some local formulas which are used in the third part. In the second part, we indicate that the above constructions can be carried out in a more general setting. This part is motivated by our attempt to understand the underlying reasons why the theory of higher order connections is intractible as indicated by some specialists (see [4]). We suggest that things may simplify if we slightly change our view of a connection on a natural bundle.

We pursue our point of view in the second part also in the last part which contains the main core of this note. In [8], we constructed a three term exact differential sequence with symmetric Christoffel symbols. The meaning of the second operator in this sequence was clarified in [9] in a framework of covariant differentiarion of geometric objects. We indicate that an analogous sequence can be defined for Cartan symbols. We supply some local formulas and complete proofs will appear in [2]. These sequences are indeed nonlinear Janet sequences for Christoffel and Cartan symbols and we relate these sequences by a commutative diagram where the vertical maps are bijections. The nonlinear Janet sequence is introduced in [11] and studied in further detail in [12], [13], [14]. We refer to these works also for a detailed history of this subject.

We next relate the above Janet sequences to the nonlinear gauge sequence for the k^{th} order coframe bundle, another three term exact sequence, by a commutative diagram. The latter sequence is also studied in [13], [14] for arbitrary principal

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bundles and is shown to be isomorphic to the tail of the nonlinear Spencer sequence, first introduced in [7]. It follows from this diagram that nonlinear Janet sequence, as it stands in [11], [12], [13], [14], requires further investigation as it does not incorporate the integrability conditions of the first and second kinds $I_{*}=0, I_{**}=0$ into a diagram but interprets them seperately. Consequently, it seems to us that some assertions in the extraordinary work [13], elaborated further in [14], are still in need of further clarification.

2. Christoffel and Cartan symbols

This note is based on the explicit local formulas which describe the group operations of $G^{k}(n)$ and $J^{1}(G^{k}(n))$, where $G^{k}(n)$ is the jet group of order k. We will shortly write G^{k} for $G^{k}(n)$. For the invariant definitions of the concepts used here, we refer to [15]. Some of our local formulas here are contained also in [3], but we will include them here for completeness.

Choosing standard coordinates in $\Re^{n}$, an element of G^{k} can be written in the form $(f^{i}_{j_1}, f^{i}_{j_2 j_1}, \ldots, f^{i}_{j_k \cdots j_1})$ which we will shortly denote by $\left(f^{i}_{\nu}\right), 1 \leq |\nu| \leq k$. To express the group operation $\left(h^{i}_{\nu}\right) = \left(f^{i}_{\nu}\right)\left(g^{i}_{\nu}\right)$, we simply differentiate $\frac{\partial z^{i}}{\partial x^{j}} = \frac{\partial z^{i}}{\partial y^{a}}\frac{\partial y^{a}}{\partial x^{j}}$ $k-1$ times and substitute $h^{i}_{j_s \cdots j_1} = \frac{\partial^{s} z^{i}}{\partial x^{j_s} \ldots \partial x^{j_1}}$, $f^{i}_{j_s \cdots j_1} = \frac{\partial^{s} z^{i}}{\partial y^{j_s} \ldots \partial y^{j_1}}$, $g^{i}_{j_s \cdots j_1} = \frac{\partial^{s} y^{i}}{\partial x^{j_s} \ldots \partial x^{j_1}}$, $1 \leq s \leq k$. For later use, we write these formulas for $k=3$:

$$
\begin{aligned}
E_1: \quad h^{i}_{j} &= f^{i}_{a} g^{a}_{j} \\
E_2: \quad h^{i}_{j_2 j_1} &= f^{i}_{a_2 a_1} g^{a_2}_{j_2} g^{a_1}_{j_1} + f^{i}_{a} g^{a}_{j_2 j_1} \\
E_3: \quad h^{i}_{j_3 j_2 j_1} &= f^{i}_{a_3 a_2 a_1} g^{a_3}_{j_3} g^{a_2}_{j_2} g^{a_1}_{j_1} + f^{i}_{a_2 a_1} g^{a_2}_{j_3 j_2} g^{a_1}_{j_1} + f^{i}_{a_2 a_1} g^{a_2}_{j_2} g^{a_1}_{j_3 j_1} \\
&\quad + f^{i}_{a_2 a_1} g^{a_2}_{j_3} g^{a_1}_{j_2 j_1} + f^{i}_{a} g^{a}_{j_3 j_2 j_1}
\end{aligned}
$$

It is therefore straightforward to write E_k. The order of indices in the above formulas is written according to the order of differentiation. This point will be important in the next section.

We will now consider a group, which will be denoted by $J^{1}G^{k}$. An element of this group is of the form $(f^{i}_{j_1}, f^{i}_{j_2 j_1}, \ldots, f^{i}_{j_k \cdots j_1} \mid f^{i}_{r,j_1}, f^{i}_{r,j_2 j_1}, \ldots, f^{i}_{r,j_k \cdots j_1})$ which we will denote by $\left(f^{i}_{\nu} \mid f^{i}_{r,\nu}\right)$. To write the group operation, we note that some terms in E_s arise by differentiating certain terms in E_{s-1} for $s \geq 2$. Indicating differentiation by a comma in those terms, we obtain

$$
\begin{aligned}
E'_2: \quad h^{i}_{j_2, j_1} &= f^{i}_{a_2, a_1} g^{a_2}_{j_2} g^{a_1}_{j_1} + f^{i}_{a} g^{a}_{j_2, j_1} \\
E'_3: \quad h^{i}_{j_3, j_2 j_1} &= f^{i}_{a_3, a_2 a_1} g^{a_3}_{j_3} g^{a_2}_{j_2} g^{a_1}_{j_1} + f^{i}_{a_2 a_1} g^{a_2}_{j_3, j_2} g^{a_1}_{j_1} + f^{i}_{a_2 a_1} g^{a_2}_{j_2} g^{a_1}_{j_3, j_1} \\
&\quad + f^{i}_{a_2, a_1} g^{a_2}_{j_3} g^{a_1}_{j_2 j_1} + f^{i}_{a} g^{a}_{j_3, j_2 j_1}
\end{aligned}
$$

Now the equations $E_1, \ldots, E_k, E'_2, \ldots, E'_{k+1}$ describe the group operation of $J^{1}G^{k}$.

The above elementary descriptions have three easy but important consequences:

1. There exists an injective homomorphism $i_1 : G^1 \to G^k$ given by $(f^i_j) \to (f^i_j, 0, \ldots, 0)$. This is shown by an induction using chain rule.

2. There exists an injective homomorphism $i_2 : G^k \to J^1G^k$ given by $(f^i_{j_1}, f^i_{j_2j_1}, \ldots, f^i_{j_k\cdots j_1}) \to (f^i_{j_1}, f^i_{j_2j_1}, \ldots, f^i_{j_k\cdots j_1} \mid 0, \ldots, 0)$. This statement is also shown by an easy induction using chain rule.

3. There exists an injective homomorphism $i_3 : G^{k+1} \to J^1G^k$ given by $(f^i_{j_1}, f^i_{j_2j_1}, \ldots, f^i_{j_{k+1}\cdots j_1}) \to (f^i_{j_1}, f^i_{j_2j_1}, \ldots, f^i_{j_k\cdots j_1} \mid f^i_{j_2j_1}, \ldots, f^i_{j_{k+1}\cdots j_1})$. The reason is that if we omit the commas in the expressions in $E'_2, \ldots, E'_{k+1}$, then the resulting equations become identical with $E_2, \ldots, E_{k+1}$ by construction.

Let us now consider the right coset spaces G^{k+1}/i_1G^1 and J^1G^k/i_2G^k. These spaces admit global coordinates $\Gamma^i_{j_2j_1}, \ldots, \Gamma^i_{j_{k+1}\cdots j_1}$ and $C^i_{j_2,j_1}, \ldots, C^i_{j_{k+1},j_k\cdots j_1}$ as follows. Let $[(f^i_\nu)], [(\,g^i_\mu \mid g^i_{r,\mu})]$ denote the equivalence classes of $\left(f^i_\nu\right) \in G^{k+1}$ and $(g^i_\mu \mid g^i_{r,\mu}) \in J^1G^k$. We define Γ^i_ν by $i_1(f^i_\nu)^{-1}\left(f^i_\nu\right) = \left(id, \Gamma^i_\nu\right)$ and $C^i_{r,\mu}$ by $i_2(g^i_\mu)^{-1}(g^i_\mu \mid g^i_{r,\mu}) = (id \mid C^i_{r,\mu})$. It is easily checked that these maps are well defined bijections. Note that $id =$ identity has different meanings in the last two formulas. We will call these coordinates Christoffel and Cartan symbols respectively.

The groups G^{k+1} and i_3G^{k+1} act on G^{k+1}/i_1G^1 and J^1G^k/i_2G^k on the right respectively. These actions are given in terms of Christoffel and Cartan symbols respectively by

$$i_1(f^i_\nu)^{-1}\left(id, \Gamma^i_\nu\left(f^i_\nu\right)\right), \quad i_2(f^i_\mu)^{-1}(id \mid C^i_{r,\mu})(f^i_\mu \mid f^i_{r,\mu}) \tag{1}$$

Also, $i_3 : G^{k+1} \to J^1G^k$ induces a map

$$G^{k+1}/i_1G^1 \to J^1G^k/i_2G^k \tag{2}$$

in the obvious way. We have

Proposition 1. *The map given by* (2) *is an injection.*

Let $\pi : G^{k+1}/i_1G^1 \to G^k/i_1G^1$ be the projection. It is easily checked using definitions that the map in (2) is given in terms of Christoffel and Cartan symbols by

$$i_2\left(\pi\left(\Gamma^i_\nu\right)\right)^{-1} i_3\left(\Gamma^i_\nu\right) = (id \mid C^i_\nu) \tag{3}$$

In (3) we assume $\Gamma^i_j = \delta^i_j$ for simplicity of notation. In view of E_2, E'_2 and E'_3, (3) gives

$$\begin{aligned} C^i_{j_2,j_1} &= \Gamma^i_{j_2j_1} \\ C^i_{j_3,j_2j_1} &= \Gamma^i_{j_3j_2j_1} - \Gamma^i_{aj_1}\Gamma^a_{j_3j_2} - \Gamma^i_{j_2a}\Gamma^a_{j_3j_1} \end{aligned} \tag{4}$$

and more generally, $C^i_{r,\mu} = \Gamma^i_{r\mu}+$ terms which contain Γ^i_ν with $|\nu| \leq |\mu|$. Therefore we can solve Christoffel symbols inductively in terms of Cartan symbols. Since i_3 is not onto, it is clear that (2) is not onto.

Now let $F^k \to M$ be the k^{th} order holonomic coframe bundle of some differentiable manifold M and let $J^1F^k \to M$ be its first jet bundle. We prefer here coframes as some local formulas below are somewhat easier in this case, but clearly everything can be done also with frames. Let (x^i, U) and (y^i, V) be two coordinate systems on M with $U \cap V \neq \emptyset$ and let $\partial^{(k)}_{x^v}(y)$ denote the k-jet $(\frac{\partial y^i}{\partial x^j}, \ldots, \frac{\partial^k y^i}{\partial x^{j_k} \ldots \partial x^{j_1}})$ of the coordinate change $(x) \to (y)$. The definitions of these bundles show that their sections over (x^i) are respectively of the form $(f^i_\nu(x)), (f^i_\nu(x) \mid f^i_{r,\nu}(x))$ and are subject to transformation rules $f^i_\nu(x) = f^i_\nu(y) \cdot \partial^{(k)}_{x^v}(y)$ and $(f^i_\nu(x) \mid f^i_{r,\nu}(x)) = (f^i_\nu(y) \mid f^i_{r,\nu}(y)) \cdot i_3(\partial^{(k+1)}_{x^v}(y))$. We act on these transformation rules on the left by the elements of G^k and J^1G^k which makes $F^k \to M$ and $J^1F^k \to M$ principal G^k and J^1G^k bundles respectively. We will not be interested in J^1F^k as a principle bundle in this note. We now form the associated bundles of F^{k+1} with fibers G^{k+1}/i_1G^1 and J^1G^k/i_2G^k with respect to the actions given by (1). We will denote these bundles by $(h)\, CH^{k+1} \to M$, where (h) stands for holonomic, and $CA^{k+1} \to M$. Christoffel and Cartan symbols become local coordinates on these bundles. Their transformation rules are determined by (1). Choosing coordinates, (3) gives an injection $\lambda : (h)\, CH^{k+1} \to CA^{k+1}$ which is independent of coordinates as the map (2) commutes with the actions of G^{k+1} and i_3G^{k+1}. We will denote the image of λ by $(t)\, CA^{k+1}$. The correspondence given in the following result will occupy a central position in this note.

Proposition 2 ([4]). *Let Γ be a section of $(h)\, CH^{k+1} \to M$. Then $\lambda(\Gamma)$ defines a torsion-free connection on $F^k \to M$. All torsion-free connections on $F^k \to M$ arise uniquely in this way.*

Proposition 2 justifies our notation $(t)\, CA^{k+1}$. Note that $C^i_{j_3,j_2j_1}$ is not symmetric even though it belongs to a torsion-free connection. A careful inspection of E'_3 shows that this is due to the 0's in $i_2(G^3) \subseteq J^1(G^2)$. It is easy to show that $C^i_{j_k,j_{k-1}\cdots j_1}$ is symmetric in $j_{k-1}, \ldots, j_1$.

3. Connections on (co)frame bundles

The constructions above can be done in a more general setting by replacing G^{k+1} (which we will now denote by $(h)\, G^{k+1}$) by the semi-holonomic jet group $(s)\, G^{k+1}$. The formulas $E_2, \ldots, E_k, E'_2, \ldots, E'_{k+1}$ continue to hold but we must now be careful with the order of indices. In this way, we arrive at $(s)\, G^{k+1}/i_1G^1$ and $J^1((s)G^k)/i_2((s)G^k)$. The proof of Proposition 1 gives this time a *bijection* between these spaces which in turn induces a bijection $(s)\, CH^{k+1} \to (s)\, CA^{k+1}$ given locally by (3) (compare to [15]). We also have the obvious injections

$$(5) \qquad (h)\, G^{k+1}/i_1G^1 \xrightarrow{\lambda} J^1((h)G^k)/i_2((h)G^k) \longrightarrow J^1((s)G^k)/i_2((s)G^k)$$

In (5), the image of the second space in the third defines quasi-holonomic connections on $(s)\,F^k \to M$ and the image of the first in the third defines quasi-holonomic connections which are torsion-free ([15]). We may now fix an intermediate group $(h)\,G^{k+1} \subseteq I^{k+1} \subseteq (s)\,G^{k+1}$. There exists an abundance of such groups which define principal bundles $(I)\,F^{k+1} \to M$ with $(h)\,F^{k+1} \subseteq (I)\,F^{k+1} \subseteq (s)\,F^{k+1}$. As in (5), the spaces I^{k+1}/i_1G^1 and J^1I^k/i_2I^k inject in $J^1((s)G^k)/i_2((s)G^k)$, where I^k is the projection of I^{k+1} in G^k. We may now have $I^{k+1}/i_1G^1 \subseteq J^1I^k/i_2I^k$ or equivalently $I^{k+1} \subset J^1I^k$. In this case, we obtain connections on $(I)\,F^k$ with some particular property. If I^{k+1} is not contained in J^1I^k, we nevertheless obtain some particular connections on $(s)\,F^k$ which however do not reduce to $(I)\,F^k$. It is easy to give examples of groups satisfying either condition.

We obtain in this way an abundance of bundles and connections and we hope to have made clear that it is not possible to seperate the bundle and the connection as they are two sides of the same coin. In addition, we will see below that none of these connections except $(t)\,CA^k$, which is identified with $(h)\,CH^k$, will make the nonlinear Janet sequence (7) exact. Complications arise also in the nonlinear gauge sequence (21) (see below) and in the diagram (22). Also, it is clear that this abundance of "particular connections" is unavoidable as long as the primary concept for us is the abstract connection rather than covariant differentiation itself.

In order to further clarify our point of view, we will consider the following simple example: Consider

$$\begin{array}{lccc} (6) & J^1F^1 = (s)\,F^2 & \longrightarrow & (s)\,CH^2 \\ & (f^i_j, f^i_{jk}) & \longrightarrow & (f^{-1})^i_a f^a_{jk} = \Gamma^i_{jk} \end{array}$$

If we view a connection as an object by itself through its abstract definition, it is natural to identify Γ^i_{jk} with the connection rather than studying Γ^i_{jk} for its own sake. As such, the possible lack of symmetry of Γ^i_{jk} is called the torsion of the connection. On the other hand, this property is attributed to the bundle on the LHS of (6) through (f^i_{jk}) with the name semi-holonomic. However, it is clear that this situation is caused by J^1 and has the same origin on both sides of (6). We thus end up with two totally different interpretations of a single phenomenon. We believe that this diversity, which becomes indeed too rich in higher orders, unifies if we shift our attention from abstract connections to jet sections which in particular may be used to define this abstract concept in the category of geometric objects. This point of view gives an interesting interpretation for G-connections [10]. In this respect, our point of view paralells the one in [13], [14].

4. A diagram

In [8], we constructed the differential sequence

$$(7) \qquad Diff(M, \Re^n) \longrightarrow (h)CH^{k+1} \xrightarrow{\nabla} T^* \otimes V((h)CH^k)$$

which is exact for $k \geq 2$. The meaning of the second operator is clarified in [9] in a framework of covariant differentiation of geometric objects. For the technical details and some interpretations, we refer to [8], [9].

The idea of (7) is simple: Sections of $(h)CH^{k+1} \to M$ which come from $Diff(M, \Re^n)$ satisfy some easy differential identities. Replacing derivatives with jet sections in these identities, the resulting expressions define the second operator in (7). Composition of the maps in (7) vanishes due to the identities and exactnes is established easily.

The question is now whether one can construct an analogue of (7) for $(t)CA^{k+1}$. Recall the bijection $\lambda : (h)CH^k \to (t)CA^k$. Now λ induces the bijection $id \otimes T(\lambda)$: $T^* \otimes V((h)CH^k) \to T^* \otimes V((t)CA^k)$ and we obtain the diagram

$$
\begin{array}{ccccc}
Diff(M, \Re^n) & \longrightarrow & (h)CH^{k+1} & \xrightarrow{\nabla} & T^* \otimes V((h)CH^k) \\
\downarrow & & \downarrow & & \downarrow \\
Diff(M, \Re^n) & \xrightarrow{\alpha} & (t)CA^{k+1} & \xrightarrow{\beta} & T^* \otimes V((t)CA^k)
\end{array} \tag{8}
$$

where α and β are defined such that (8) commutes. However, we will compute α, β explicitly as these formulas will be essential for our main purpose here.

(3) shows that α is given by

$$
i_2(\pi(\partial_{x^\nu}(z^i)))^{-1} i_3(\partial_{x^\nu}(z^i)) = (id \mid C^i_\nu) \tag{9}
$$

where $\left(z^i\right)$ are the standart coordinates in $\Re^n$ and $\left(x^i\right)$ are local coordinates on M. It follows from (9) that

$$
\begin{aligned}
C^i_{j_2,j_1}(x) &= \frac{\partial x^i}{\partial z^a} \frac{\partial^2 z^a}{\partial x^{j_2} \partial x^{j_1}} \\
C^i_{j_3,j_2 j_1}(x) &= \frac{\partial^2 x^i}{\partial z^{a_2} \partial z^{a_1}} \frac{\partial^2 z^{a_2}}{\partial x^{j_3} \partial x^{j_2}} \frac{\partial z^{a_1}}{\partial x^{j_1}} + \frac{\partial^2 x^i}{\partial z^{a_2} \partial z^{a_1}} \frac{\partial z^{a_2}}{\partial x^{j_2}} \frac{\partial^2 z^{a_1}}{\partial x^{j_3} \partial x^{j_1}} \\
&\quad + \frac{\partial x^i}{\partial z^a} \frac{\partial^3 z^a}{\partial x^{j_3} \partial x^{j_2} \partial x^{j_1}}
\end{aligned} \tag{10}
$$

Clearly, (10) reduces to (4) if we let $\frac{\partial x^i}{\partial z^a} \frac{\partial^s z^a}{\partial x^{j_s} \ldots \partial x^{j_1}} = \Gamma^i_{j_s \cdots j_1}(x)$ in view of (3) and (9). We will write C^i_{jk} for $C^i_{j,k}$.

A straightforward computation using (10) shows that we have the identity

$$
\partial_{j_3} C^i_{j_2 j_1} + C^i_{j_3 a} C^a_{j_2 j_1} - \Gamma^i_{j_3 j_2 j_1} = 0 \tag{11}
$$

where $\Gamma^i_{j_3 j_2 j_1} = C^i_{a j_2} C^a_{j_3 j_1} - C^i_{j_1 a} C^a_{j_3 j_2} - C^i_{j_3, j_2 j_1}$ as can be seen from (4).

Some more patience reveals the identity

$$
\begin{aligned}
&\partial_{j_4} C^i_{j_3,j_2j_1} + C^i_{j_4a} C^a_{j_3,j_2j_1} + C^a_{j_3j_2} C^i_{j_4,aj_1} + C^i_{aj_1} C^a_{j_4,j_3j_2} \\
&+C^a_{j_3j_1} C^i_{j_4,j_2a} + C^i_{ab} C^b_{j_4j_1} C^a_{j_3j_2} + C^i_{aj_1} C^a_{bj_2} C^b_{j_4j_3} \\
&+C^i_{aj_1} C^a_{j_3b} C^b_{j_4,j_2} + C^i_{j_2a} C^a_{j_4,j_3j_1} + C^a_{j_3j_1} C^i_{ba} C^b_{j_4,j_2} \\
&+C^i_{j_2a} C^a_{bj_1} C^b_{j_4,j_3} + C^i_{j_2a} C^a_{j_3b} C^b_{j_4,j_1} + \Gamma^i_{j_4j_3j_2j_1} = 0
\end{aligned}
\tag{12}
$$

These identities, which we will denote by $\Phi(j_k j_{k-1} \ldots j_1) = 0$, can be written in a compact form ([2]). As in (7), the expressions Φ now define β in local coordinates.

There is a fundamental difference between the two sequences in (8) even though they are isomorphic: The upper sequence uses only G^1-invariance whereas the lower sequence uses G^k-invariance, that is, the extra twist given by (3) turns out to be redundant. Consequently, Cartan symbols are more intricate objects than Christoffel symbols. For instance, we do not know a direct and simple method of covariant differentiation of geometric objects with respect to (torsion-free) Cartan symbols without transforming them first to (symmetric) Christoffel symbols (see [9]).

There are two problems with the sequences in (8) :

a) The order of jets is not stable.

b) ∇ and β vanish for $k = 1$, and the sequences are not exact for $k = 1$.

These difficulties are easily remedied as follows: We lift a section of, say, $(t)CA^k \to M$ to $(t)CA^{k+1} \to M$ and then map it by β. The difference between the images of two lifts by β belongs to $T^* \otimes V((t)CA^k)$. This difference is actually a $(1,k)$-tensor as all the lower order terms cancel. In fact, it is *symmetric* because the skewsymmetrization of the top order Cartan symbol can be expressed in terms of lower order Cartan symbols due to torsion-freeness (see (16) below). Taking the quotient of $T^* \otimes V((t)CA^k)$ by this tensor bundle and denoting it by $(t)CA_1(k)$, we obtain a well defined map $(t)CA^k \to (t)CA_1(k)$ for $k \geq 2$ which makes the lower sequence in (8) exact. Similar construction applies to the upper sequence. It is not difficult to show that the right vertical map in (8) induces a bijection $(h)CH_1(k) \to (t)CA_1(k)$ and we obtain the commutative diagram

$$
\begin{array}{ccccc}
\mathit{Diff}(M,\Re^n) & \longrightarrow & (h)CH^k & \stackrel{\nabla}{\longrightarrow} & (h)CH_1(k) \\
\downarrow & & \downarrow & & \downarrow \\
\mathit{Diff}(M,\Re^n) & \stackrel{\alpha}{\longrightarrow} & (t)CA^k & \stackrel{\beta}{\longrightarrow} & (t)CA_1(k)
\end{array}
\tag{13}
$$

Proposition 3 ([1]). *The upper and lower sequence in* (13) *are nonlinear Janet sequences for the spaces* $(h)CH^k$ *and* $(t)CA^k$ *respectively.*

Note that with patience (better, with the help of a computer) we are in position to compute anything in sight locally in view of $E_1, \ldots, E_k, E'_2, \ldots, E'_{k+1}$.

However, we warn the reader who is willing to do some of these elementary computations: Some expressions which look totally different may in fact be the same. For instance, a careful inspection of (10) shows that we also have the identity

$$\partial_{j_3} C^i_{j_2 j_1} - C^i_{a j_1} C^a_{j_3 j_2} - C^i_{j_2, j_3 j_1} = 0 \tag{14}$$

In fact the reader may check the identity

$$\partial_{j_4} C^i_{j_3, j_2 j_1} - C^i_{a, j_2 j_1} C^a_{j_4 j_3} - C^i_{j_3, j_2 j_1 j_4} = 0 \tag{15}$$

However, (14) is equivalent to (11) : the second formula in (4) gives

$$C^i_{j_3, j_2 j_1} - C^i_{j_2, j_3 j_1} = C^i_{a j_3} C^a_{j_2 j_1} - C^i_{j_2 a} C^a_{j_3 j_1} \tag{16}$$

Substituting (16) into (14) gives (11). Similarly (15) is equivalent to (12) as the reader can check. The reason for such peculiarity is the following: Recall that to construct the second operator in nonlinear Janet sequence one chooses a minimal set of functionally independent differential invariants ([6]) to eliminate the jet coordinates of top order ([11], [12], [13]). However, there is no unique way of choosing this minimal set. Different choices give rise to expressions which may look different but are of course equivalent as they all define the same invariant operator (see pg.231, 234 in [13]).

We now come to a crucial point: Replacing the derivatives in (9) by jet sections we obtain a first order differential operator

$$\begin{array}{rcl} F^k & \longrightarrow & CA^{k+1} \\ (f^i_\nu) & \longrightarrow & i_2 \left(f^i_\nu\right)^{-1} \left(f^i_\nu \mid \partial_r f^i_v\right) = \left(id \mid C^i_{r,\nu}\right) \end{array} \tag{17}$$

It follows from (17), E_2' and E_3' that

$$\begin{array}{ll} C^i_{j_2, j_1} & = g^i_a \partial_{j_2} f^a_{j_1} \\ C^i_{j_3, j_2 j_1} & = g^i_{a_2 a_1} \partial_{j_3} f^{a_2}_{j_2} f^{a_1}_{j_1} + g^i_{a_2 a_1} f^{a_2}_{j_2} \partial_{j_3} f^{a_1}_{j_1} + g^i_a \partial_{j_3} f^a_{j_2 j_1} \end{array} \tag{18}$$

where $(f^i_\nu)^{-1} = (g^i_\nu)$. Note the relation between (18) and (10) in view of (9) and (17). Also, note the comma in $C^i_{j_2, j_1}$ in (18) which is now essential.

We can not search now for differential identites because there are no differental relations between the components of $\left(f^i_\nu\right)$. This time, however, we can search for compatibility conditions that must be satisfied by $C^i_{s,\nu}$. Differentiating the equations in (18), substituting back from (18), making use of E_2, E_3 and E_3' along the way and skewsymmetrizing, we arrive at

$$\begin{array}{c} \left(\partial_{j_3} C^i_{j_2, j_1} + C^i_{j_3, a} C^a_{j_2, j_1}\right)_{[j_3, j_2]} = 0 \\ \left(\partial_{j_4} C^i_{j_3, j_2 j_1} + C^i_{j_4, a} C^a_{j_3, j_2 j_1} - C^i_{j_3, j_2 a} C^a_{j_4, j_1} - C^i_{j_3, a j_1} C^a_{j_4, j_2}\right)_{[j_4, j_3]} = 0 \end{array} \tag{19}$$

This computation is analogous to those in [3]. The meaning of (19) will be more apparent below. For the moment, let us denote the compatibility conditions obtained in this way by

$$\Xi(r,s,\mu) - \Xi(s,r,\mu) = 0, \qquad 1 \leq |\mu| \leq k \tag{20}$$

Let g^k (also) denote associated bundle of F^k by the adjoint action of G^k on its Lie algebra g^k and let $\xi^i_{r,s,\mu}$ be the local coordinates on $\Lambda^2(T^*) \otimes g^k$. Not surprisingly, we now have

Proposition 4. *There exists an exact sequence*

$$F^k \longrightarrow CA^{k+1} \xrightarrow{\kappa} \Lambda^2(T^*) \otimes g^k \tag{21}$$

where the first operator is given by (17) and the second operator is given by $\xi^i_{r,s,\mu} = \Xi(r,s,\mu) - \Xi(s,r,\mu)$, $1 \leq |\mu| \leq k$.

The sequence in (21) is contained in [14] (pg. 189) for an arbitrary principal bundle $P \to M$ and is called the (global) gauge sequence (compare, however, to [13], pg.324). It is shown in [13], [14] that this sequence is isomorphic to the tail of the nonlinear Spencer sequence.

As in the second section, we may want to replace now F^k in (21) by $(I)\,F^k$. The first operator, which is given by (17) where (f^i_ν) is now a section of $(I)\,F^k \to M$, produces arbitrary connections on $(I)\,F^k$. Now as a very crucial fact, a particular connection on $(I)\,F^k$ and its curvature do not fit into (21) if we insist in the first operator as we do here. For instance, it is not possible to replace CA^{k+1} in (21) by $(t)\,CA^{k+1}$.

We now come to the crux of the whole matter: We want to relate (21) to (8) and (13). First, note that the skewsymmetrization of (11) (or (14)) gives the first formula in (19). Similarly, the reader can directly verify that the skewsymmetrization of (12) (or (15)) gives the second formula in (19). In fact, we have

Proposition 5. *There exists a skewsymmetrization map $\delta : T^* \otimes V((t)CA^{k+1}) \to \Lambda^2(T^*) \otimes g^k$ defined by $\eta^i_{r,s,\mu} \to \eta^i_{r,s,\mu} - \eta^i_{s,r,\mu} = \xi^i_{r,s,\mu}$. With this map the following diagram*

$$\begin{array}{ccccc} Diff(M,\Re^n) & \longrightarrow & (t)CA^{k+2} & \xrightarrow{\beta} & T^* \otimes V((t)CA^{k+1}) \\ \downarrow & & \downarrow & & \downarrow^{\delta} \\ F^k & \longrightarrow & CA^{k+1} & \xrightarrow{\kappa} & \Lambda^2(T^*) \otimes g^k \end{array} \tag{22}$$

commutes for $k \geq 1$, where the first vertical map is obvious injection and the middle vertical map is the composition of the projection $(t)CA^{k+2} \to (t)CA^{k+1}$ and the injection $(t)CA^{k+1} \to CA^{k+1}$.

It is of utmost importance that the composition $\delta \circ \beta$ in (22) does not depend on the top order Cartan symbols, as it should be, *due to torsion-freeness* as indicated above.

Since the symmetric tensor disappears through skewsymmetrization, δ induces a map $\delta : (t)CA_1\,(k+1) \to \Lambda^2(T^*) \otimes g^k$ and we also have the commutative diagram

$$\begin{array}{ccccc} \mathit{Diff}(M, \Re^n) & \longrightarrow & (t)CA^{k+1} & \xrightarrow{\beta} & (t)CA_1\,(k+1) \\ \downarrow & & \downarrow & & \downarrow^{\delta} \\ F^k & \longrightarrow & CA^{k+1} & \xrightarrow{\kappa} & \Lambda^2(T^*) \otimes g^k \end{array} \tag{23}$$

where the middle column is injection.

It is shown in [1] that with the addition of a torsion space, the middle column in (23) extends one step to below to an exact sequence, indicating interesting relations to the study of torsions in [5].

We will now turn to some assertions in the extraordinary work [13]. First, it has been asserted in various places in [13], [14] that a connection is almost never a geometric object (except for $\Gamma^i_{jk} = C^i_{jk}$) as it belongs to Spencer sequence and not to Janet sequence. However, (13) shows that a *torsion-free* connection on the (co)frame bundle, viewed in terms of its components, is in fact also a geometric object and as such, it always belongs to Janet sequence. Second, the operator κ in (22), called gauge curvature in [13], [14], arises from the composition $\delta \circ \beta$ and thus from the Janet sequence. Therefore, there is no sharp distinction between the so called gauge curvature and classical curvature , in contrast to the claims in [13], [14]. Third, the integrability conditions of the second kind $I_{**} = 0$ are again incorporated by $\delta \circ \beta$, whereas they do not belong to a diagram in [11], [12], [13], [14] and are interpreted seperately. The second square in (22) shows that the Maurer-Cartan condition, which is shown to fit perfectly well with the Janet sequence (Vessiot framework) in these works, fits perfectly well also with the gauge sequence (Cartan framework), again in contrast to the claims in these works.

Leaving further elaborations on this matter to the reader, we will pass over to another problem (the author is indebted E. Abadoğlu for communicating this idea): Suppose that we replace the upper sequence in (23) by the Janet sequence of an arbitrary object bundle H. Now the question is whether we still have an analogue of (23) , that is, whether we have two spaces $C_1\,(H)$, $C_2\,(H)$, an injection $H \to C_1\,(H)$ and a diagram like

$$\begin{array}{ccccc} \mathit{Diff}\,(M, \Re^n) & \longrightarrow & H & \xrightarrow{\nabla} & H_1 \\ \downarrow & & \downarrow^{i} & & \downarrow^{\delta} \\ F & \longrightarrow & C_1\,(H) & \xrightarrow{\kappa} & C_2\,(H) \end{array} \tag{24}$$

We believe that the answer is affirmative. In fact, we believe that the dichotomy between Spencer and Janet sequences in the above works arises because the particular Janet sequences in (13) and the lower sequence in (24) , except for the particular case in (23) , are absent in these works.

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References

[1] Abadoğlu, E., preprints.
[2] Abadoğlu, E., Ph. D Theses.
[3] Abadoğlu, E., *An elementary approach to nonlinear Spencer sequence*, preprint.
[4] Kolář, I., *Torsion-free connections on higher order frame bundles*, New Developments In Differential Geometry, Proceedings, Kluwer (1996), 233–241.
[5] Kolář, I., Modugno, M., *Torsions of connections on some natural bundles*, Differential Geometry and its Applications 2, (1992), 1–16.
[6] Kumpera, A., *Invariants differentiels d'un pseudogroupe de Lie I, II*, J. Differential Geometry, 10, 1975, 347–416.
[7] Kumpera, A., Spencer, D.C., *Lie Equations, I, General Theory*, Ann. Math. Studies 73, Princeton University Press, 1972.
[8] Ortaçgil, E., *On a differential sequence in geometry*, Tr. J. Math., 20, 1996, 473–478.
[9] Ortaçgil, E., *Classical differential geometry with Christoffel symbols of Ehresmann ε-connections*, Arch. Math.(Brno), vol. 34, No. 2(1998), 229–237.
[10] Ortaçgil, E., *G-connections as twisted formal solutions of systems of PDE's related to geometric structures*, preprint.
[11] Pommaret, J.F., *Systems of Partial Differential Equations and Lie Pseudogroups*, Gordon&Breach, London, New York, 1978.
[12] Pommaret, J.F., *Differential Galois Theory*, Gordon&Breach, London, New York, 1978.
[13] Pommaret, J.F., *Lie Pseudogroups and Mechanics*, Gordon&Breach, London, New York, 1988.
[14] Pommaret, J.F., *Partial Differential Equations and Group Theory: New perspectives and Applications*, Kluwer Academic Publishers, 1994.
[15] Yuen, P.C., *Higher order frames and linear connections*, Cahiers Topol. Geom. Diff. 12, 1971, 333–371.

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