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THE CANONICAL ISOMORPHISM BETWEEN THE PROLONGATION OF THE SYMBOLS OF A NONLINEAR LIE EQUATION AND ITS ATTACHED LINEAR LIE EQUATION

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ABSTRACT. Let M be a smooth manifold and let us denote by $I^\ell(M)$ the fibre bundle of invertible ℓ-jets from M into M. For each point $\mathfrak{p}^\ell$ the vertical tangent space, for the source projection α, to $I^\ell(M)$ at $\mathfrak{p}^\ell$ is isomorphic to the module of $\alpha^*C^\infty(M)$-derivations from $C^\infty(M\times M)$ into $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$.

Let $\mathcal{R}^\ell$ be a nonlinear Lie equation and H^ℓ its attached linear Lie equation; the composition of each vertical tangent vector to $\mathcal{R}^\ell$ at $\mathfrak{p}^\ell$ with $(\mathfrak{p}^\ell)^{-1}$, when it is meant as an $\mathbb{R}$-algebra isomorphism from $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$ onto $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\beta(\mathfrak{p}^\ell)}$, gives a natural isomorphism between the vertical fibre bundle $V\mathcal{R}^\ell$ and the pullback $\mathcal{R}^\ell\times_M H^\ell$ of H^ℓ by the projection β.

From the interpretation of invertible jets and vertical vectors as homomorphisms and derivations, respectively, from $C^\infty(M\times M)$ into some Weil algebras follows that the r-prolongation preserves the translation which identifies the spaces $V_{\mathfrak{p}^\ell}\mathcal{R}^\ell$ and $H^\ell_{\beta(\mathfrak{p}^\ell)}$, therefore $V_{\mathfrak{p}^{\ell+r}}\mathcal{R}^{\ell+r}\approx H^{\ell+r}_{\beta(\mathfrak{p}^{\ell+r})}$, without any assumptions about regularity of $\mathcal{R}^{\ell+r}$, and the symbol of $\mathcal{R}^{\ell+r}$ at $\mathfrak{p}^{\ell+r}$ is isomorphic to the symbol of $H^{\ell+r}$ at $\beta(\mathfrak{p}^{\ell+r})$.

1. Invertible jets

The prolongation of a nonlinear Lie equation is not, in general, a nonlinear Lie equation. Certain conditions, such as manifold condition, of its definition can fall down. However, the prolongation preserves many relationships between a nonlinear equation and its attached linear equation. If we consider Weil's theory of jets no assumptions about regularity are necessary to prove these results. Firstly, we summarize our definitions.

As a result from Weil's theory of near points [5, 2], in [4] we give a characterization of the ℓ-jets of local cross-sections of a fibre bundle $\pi: M\longrightarrow X$ as $\mathbb{R}$-algebra homorphisms from $C^\infty(M)$ onto the algebras $C^\infty(M)/\mathfrak{m}_x^{\ell+1}$ $(x\in X)$ whose restriction to $C^\infty(X)$ is the natural morphism. In particular, each invertible ℓ-jet $\mathfrak{p}^\ell\in I^\ell(M)$ with source $\alpha(\mathfrak{p}^\ell)$ and target $\beta(\mathfrak{p}^\ell)$ is an $\mathbb{R}$-algebra homomorphism $\mathfrak{p}^\ell : C^\infty(M\times M)\longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$, where $\mathfrak{m}_{\alpha(\mathfrak{p}^\ell)}$ is the maximal ideal of functions vanishing at $\alpha(\mathfrak{p}^\ell)$; the homomorphism $\mathfrak{p}^\ell$ is regular over the two copies of

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the algebra of differentiable functions on M, given by source and target projection. The restriction of the homomorphism $\mathfrak{p}^\ell$ to the first copy of $C^\infty(M)$ is the natural homomorphism onto the quotient of $C^\infty(M)$ module $\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$, and its restriction to the second copy of $C^\infty(M)$ is a homomorphism

$$\mathfrak{p}^\ell|_{\beta^*(C^\infty(M))} : C^\infty(M) \longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}\,,$$

which splits up as the composition of the natural homomorphism onto $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\beta(\mathfrak{p}^\ell)}$ and an isomorphism, which we also denote

$$\mathfrak{p}^\ell : C^\infty(M)/\mathfrak{m}^{\ell+1}_{\beta(\mathfrak{p}^\ell)} \longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}\,.$$

This result characterizes invertible jets as isomorphisms of Weil algebras.

Our global definition coincides with the local definition of invertible jets, because a local diffeomorphism ϕ from M into M gives by the transpose homomorpism ϕ^* an isomorphism $\phi^*_p : C^\infty(M)/\mathfrak{m}^{\ell+1}_{\phi(p)} \longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_p$ for each $p \in \operatorname{Dom}\phi$. If ψ is another local diffeomorphism from M into M then, ϕ and ψ have a ℓ-th order contact at p if and only if $\phi^*_p = \psi^*_p$.

We write $(\mathfrak{p}^\ell)^{-1}$ for the inverse algebra isomorphism of $\mathfrak{p}^\ell \in I^\ell(M)$. We denote by $i^\ell : M \longrightarrow I^\ell(M)$ the section which applies each $p \in M$ to $i^\ell(p)$, the identity automorphism of $C^\infty(M)/m^{\ell+1}_p$, the Weil algebra at p of height ℓ. If $\alpha(\mathfrak{p}^\ell) = p$ and $\beta(\mathfrak{p}^\ell) = q$, $\mathfrak{q}^\ell \in I^\ell(M)$ with $\beta(\mathfrak{q}^\ell) = p$ and $\alpha(\mathfrak{q}^\ell) = x$, then the composition of jets $\mathfrak{q}^\ell \circ \mathfrak{p}^\ell : C^\infty(M)/\mathfrak{m}^{\ell+1}_q \longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_x$ is the *composition of algebra isomorphisms* of $\mathfrak{p}^\ell$ and $\mathfrak{q}^\ell$.

For each $\mathfrak{p}^\ell \in I^\ell(M)$ we denote by $R_{\mathfrak{p}^\ell}$ the diffeomorphism between the fibres at p and q for the source projection, $I^\ell_p(M)$ and $I^\ell_q(M)$ respectively, defined by $R_{\mathfrak{p}^\ell}(\bar{\mathfrak{p}}^\ell) = (\mathfrak{p}^\ell)^{-1} \circ \bar{\mathfrak{p}}^\ell$; it is *the translation attached to* $\mathfrak{p}^\ell$.

We denote by $\mathcal{J}^\ell TM \longrightarrow M$ the *vector bundle of all ℓ–jets of local cross-sections of $TM \longrightarrow M$*. For each $p \in M$ the points of the fibre $\mathcal{J}^\ell_p T\,M$ are algebra homomorphisms from $C^\infty(TM)$ onto $C^\infty(M)/\mathfrak{m}^{\ell+1}_p$ whose restrictions to $C^\infty(M)$ coincide with the natural homomorphism $C^\infty(M) \longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_p$. By composing each algebra homomorphism $j^\ell_p D : C^\infty(TM) \longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_p$ belonging to $\mathcal{J}^\ell_p TM$ with the differential

$$\begin{aligned} d : C^\infty(M) &\longrightarrow C^\infty(T\,M) \\ f &\longrightarrow df \qquad \text{where} \quad df(p, D_p) = D_p f \end{aligned}$$

the homomorphism $j^\ell_p D$ is identified with a derivation from $C^\infty(M)$ into $C^\infty(M)/\mathfrak{m}^{\ell+1}_p$, where $C^\infty(M)/\mathfrak{m}^{\ell+1}_p$ is a $C^\infty(M)$-module by the natural homomorphism, hence we have the isomorphism $\mathcal{J}^\ell_p TM \approx \operatorname{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M)/\mathfrak{m}^{\ell+1}_p)$.

Using this characterization of the ℓ–jets of vector fields as derivations we describe the vertical tangent space to $\alpha : I^\ell(M) \longrightarrow M$ at the point $\mathfrak{p}^\ell$. The vertical vectors $D_{\mathfrak{p}^\ell} \in V_{\mathfrak{p}^\ell} I^\ell(M)$ are $\alpha^* C^\infty(M)$-derivations $C^\infty(M \times M) \longrightarrow C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$ (see [4]); their specializations to $\beta^* C^\infty(M)$ are $\mathbb{R}$-derivations from $C^\infty(M)$ into $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$, where $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$ is endowed with a structure of $C^\infty(M)$-module by restricting to $\beta^* C^\infty(M)$ the homomorphism $\mathfrak{p}^\ell : C^\infty(M \times M) \longrightarrow$

$C^\infty(M)/\mathfrak{m}^{\ell+1}_{\alpha(\mathfrak{p}^\ell)}$. In particular $V_{i^\ell(\beta(\mathfrak{p}^\ell))}I^\ell(M) \approx \mathrm{Der}_{\mathbb{R}}(C^\infty(M), C^\infty(M)/\mathfrak{m}^{\ell+1}_{\beta(\mathfrak{p}^\ell)})$, where $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\beta(\mathfrak{p}^\ell)}$ is a $C^\infty(M)$-module as a quotient algebra with respect to $\mathfrak{m}^{\ell+1}_{\beta(\mathfrak{p}^\ell)}$, and as we have just seen this space of derivations is isomorphic to the fibre at $\beta(\mathfrak{p}^\ell)$ of the ℓ-th prolongation of the tangent bundle, $\mathcal{J}^\ell_{\beta(\mathfrak{p}^\ell)}TM$.

For each vector $D_{\mathfrak{p}^\ell} \in V_{\mathfrak{p}^\ell}I^\ell(M)$, its composition with the isomorphism $(\mathfrak{p}^\ell)^{-1}$ is a derivation from $C^\infty(M)$ into $C^\infty(M)/\mathfrak{m}^{\ell+1}_{\beta(\mathfrak{p}^\ell)}$, where the module structure is given by the natural homomorphism; this composition is the image of $D_{\mathfrak{p}^\ell}$ under the tangent mapping at $\mathfrak{p}^\ell$ of $R_{\mathfrak{p}^\ell}$, and we can write

$$D_{\mathfrak{p}^\ell} = \mathfrak{p}^\ell \circ D_{i^\ell \circ \beta(\mathfrak{p}^\ell)}$$

where $D_{i^\ell\circ\beta(\mathfrak{p}^\ell)} = (\mathfrak{p}^\ell)^{-1} \circ D_{\mathfrak{p}^\ell} = (R_{\mathfrak{p}^\ell})_*(D_{\mathfrak{p}^\ell}) \in V_{i^\ell\circ\beta(\mathfrak{p}^\ell)}I^\ell(M)$.

By these tangent mappings we can identify the vertical tangent bundle of the fibre bundle of invertible ℓ-jets with the pullback of the ℓ-th prolongation of the tangent bundle by the target projection.

Proposition 1.1. $VI^\ell(M) \approx I^\ell(M) \times_M \mathcal{J}^\ell TM$

If we think each global section of the ℓ-th prolongation of the tangent bundle as a vertical vector field on the manifold of invertible ℓ-jets, invariant by the translations $R_{\mathfrak{p}^\ell}$, the space of all global sections is endowed with a Lie algebra structure, with the natural Lie bracket (see [3]).

2. The vertical tangent space to $\mathcal{J}^r(I^\ell(M))$

Our next goal is to show that, in the same way as the vectors of $\mathcal{J}^\ell T\,M$ give, by means of the translations $R_{\mathfrak{p}^\ell}$, the vertical tangent space to $I^\ell(M)$, the vectors of $\mathcal{J}^r(\mathcal{J}^\ell T\,M)$ determine the vertical tangent space to $\mathcal{J}^r(I^\ell(M))$. We will need a well known result of Goldschmidt and Sternberg [1] which we prove in an intrinsical way to illustrate the general situation.

Lemma 2.1. *Let $\pi : Y \longrightarrow M$ be a surjective submersion. Then $V_{\mathfrak{p}^\ell}\mathcal{J}^\ell(Y) \approx \mathcal{J}^\ell_{\mathfrak{p}^\ell}(VY)$ for each $\mathfrak{p}^\ell \in \mathcal{J}^\ell(Y)$, where $\mathcal{J}^\ell_{\mathfrak{p}^\ell}(VY)$ is the fibre at $\mathfrak{p}^\ell$ of the natural projection $\mathcal{J}^\ell(VY) \longrightarrow \mathcal{J}^\ell(Y)$ induced by $VY \longrightarrow Y$.*

Proof. Let $\mathfrak{p}^\ell \in \mathcal{J}^\ell(Y)$ with $\pi^\ell(\mathfrak{p}^\ell) = y$ and $\pi(y) = p$; $\mathfrak{p}^\ell$ is an algebra homomorphism from $C^\infty(Y)$ onto $C^\infty(M)/\mathfrak{m}_p^{\ell+1}$ whose restriction to $C^\infty(M)$ is the natural homomorphism.

Each algebra homomorphism $C^\infty(VY) \longrightarrow C^\infty(M)/\mathfrak{m}_p^{\ell+1}$ whose restriction to $C^\infty(Y)$ is the jet $\mathfrak{p}^\ell$ determines, by composing it with the differential

$$\begin{aligned} d : C^\infty(Y) &\longrightarrow C^\infty(V\,Y) \\ f &\longrightarrow df \qquad \text{where} \quad df(p, D_p) = D_p f \end{aligned}$$

a $C^\infty(M)$-derivation at $\mathfrak{p}^\ell$ from $C^\infty(Y)$ to $C^\infty(M)/\mathfrak{m}_p^{\ell+1}$. This derivation and the jet $\mathfrak{p}^\ell$ characterize the considered homomorphism. □

The surjective submersion

$$\begin{aligned} VI^\ell(M) &\longrightarrow \mathcal{J}^\ell T\,M \\ (\mathfrak{p}^\ell, D_{\mathfrak{p}^\ell}) &\longrightarrow (\mathfrak{p}^\ell)^{-1}\circ D_{\mathfrak{p}^\ell}|_{\beta^* C^\infty(M)} \end{aligned}$$

is a morphism of fibre bundles over M when the target projection $\beta\colon I^\ell(M)\longrightarrow M$ is considered, and it identifies the algebra $C^\infty(\mathcal{J}^\ell T\,M)$ with a subalgebra of $C^\infty(VI^\ell(M))$.

If we keep in mind Goldshmidt and Sternberg's result, each vertical vector at $\mathfrak{p}^{\ell,r}$ on $\mathcal{J}^r(I^\ell(M))$, $D_{\mathfrak{p}^{\ell,r}}$, can be thought of as an element of $\mathcal{J}^r(VI^\ell(M))$ over $\mathfrak{p}^{\ell,r}\in\mathcal{J}^r(I^\ell(M))$, that is, as an algebra homomorphism from $C^\infty(VI^\ell(M))$ onto $C^\infty(M)/\mathfrak{m}^{r+1}_{\alpha(\mathfrak{p}^{\ell,0})}$, where $\mathfrak{p}^{\ell,0}=\mathfrak{p}^\ell$. Then the restriction of $D_{\mathfrak{p}^{\ell,r}}$ to $\beta^*C^\infty(M)$ is the composition of the natural homomorphism onto $C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})}$ and the projection $\mathfrak{p}^{0,r}: C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})}\longrightarrow C^\infty(M)/\mathfrak{m}^{r+1}_{\alpha(\mathfrak{p}^{\ell,0})}$,

$$\begin{array}{ccc} \beta^*C^\infty(M) & \longrightarrow & C^\infty(VI^\ell(M)) \\ {\scriptstyle\text{natural homomorphism}}\big\downarrow & & \big\downarrow{\scriptstyle D_{\mathfrak{p}^{\ell,r}}} \\ C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})} & \xrightarrow{\mathfrak{p}^{0,r}} & C^\infty(M)/\mathfrak{m}^{r+1}_{\alpha(\mathfrak{p}^{\ell,0})} \end{array}$$

As we can see in the following conmutative diagram

$$\begin{array}{ccccc} \beta^*C^\infty(M) & \longrightarrow & C^\infty(\mathcal{J}^\ell TM) & \longrightarrow & C^\infty(VI^\ell(M)) \\ {\scriptstyle\text{natural hom.}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle D_{\mathfrak{p}^{\ell,r}}} \\ C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})} & = & C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})} & \xrightarrow{\mathfrak{p}^{0,r}} & C^\infty(M)/\mathfrak{m}^{r+1}_{\alpha(\mathfrak{p}^{\ell,0})} \end{array}$$

the restriction of $D_{\mathfrak{p}^{\ell,r}}$ to $C^\infty(\mathcal{J}^\ell T\,M)$, as a subalgebra of $C^\infty(VI^\ell(M))$, is the composition of a homomorphism from $C^\infty(\mathcal{J}^\ell T\,M)$ onto $C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})}$ with $\mathfrak{p}^{0,r}: C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})}\longrightarrow C^\infty(M)/\mathfrak{m}^{r+1}_{\alpha(\mathfrak{p}^{\ell,0})}$; as on the oder hand the restriction of the homomorphism $C^\infty(\mathcal{J}^\ell T\,M)\longrightarrow C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})}$ to $\beta^*C^\infty(M)$ is the natural homomorphism, this homomorphism from $C^\infty(\mathcal{J}^\ell T\,M)$ onto $C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})}$ is a point of $\mathcal{J}^r_{\beta(\mathfrak{p}^{\ell,0})}(\mathcal{J}^\ell T\,M)$; this fact will be useful in the sequel.

Proposition 2.2. *The vertical space $V_{\mathfrak{p}^{\ell,r}}\mathcal{J}^r(I^\ell(M))$ is isomorphic to $\mathcal{J}^r_{\mathfrak{p}^{\ell,r}}(VI^\ell(M))$. Moreover two elements of this space coincide if and only if their specializations to $C^\infty(\mathcal{J}^\ell T\,M)$ coincide, that is to say, they give the same point of $\mathcal{J}^r(\mathcal{J}^\ell T\,M)$.*

Proof. The first assertion is obtained by applying the previous lemma to the source projection $\alpha: I^\ell(M)\longrightarrow M$. On the other hand, if an element of $V_{\mathfrak{p}^{\ell,r}}\mathcal{J}^r(I^\ell(M))$ is considered as a homomorphism from $C^\infty(VI^\ell(M))$ onto $C^\infty(M)/\mathfrak{m}^{r+1}_{\alpha(\mathfrak{p}^{\ell,0})}$, its restriction to $C^\infty(I^\ell(M))$ is $\mathfrak{p}^{\ell,r}$, hence it is completely determined by the image of the elements of $C^\infty(\mathcal{J}^\ell TM)$, because $I^\ell(M)\times_M \mathcal{J}^\ell TM\approx VI^\ell(M)$, and the ring $C^\infty(I^\ell(M) \times_M \mathcal{J}^\ell T M)$ is the completed tensor product, over $C^\infty(M)$, of $C^\infty(I^\ell(M))$ and $C^\infty(\mathcal{J}^\ell T M)$. □

3. Symbol of a Lie equation

Definition 3.1. A smooth submanifold $\mathcal{R}^\ell$ of $I^\ell(M)$ will be called *a nonlinear Lie equation* if

1. $\alpha : \mathcal{R}^\ell \longrightarrow M$ is a fibred manifold.
2. For each $p \in M$, $i^\ell(p) \in \mathcal{R}^\ell$.
3. For each $\mathfrak{p}^\ell \in \mathcal{R}^\ell$, the translation $R_{\mathfrak{p}^\ell}$ transforms $\mathcal{R}^\ell_{\alpha(\mathfrak{p}^\ell)}$ into $\mathcal{R}^\ell_{\beta(\mathfrak{p}^\ell)}$,

where $\mathcal{R}^\ell_{\alpha(\mathfrak{p}^\ell)}$ and $\mathcal{R}^\ell_{\beta(\mathfrak{p}^\ell)}$ are the fibres of $\alpha : \mathcal{R}^\ell \longrightarrow M$ at $\alpha(\mathfrak{p}^\ell)$ and $\beta(\mathfrak{p}^\ell)$ respectively.

The set of all local transformations of M solutions of a nonlinear Lie equation $\mathcal{R}^\ell$ form a pseudogroup, which is a *Lie pseudogroup* if $\mathcal{R}^\ell$ is completely integrable. The vertical tangent space to $\mathcal{R}^\ell$ at $i^\ell(p)$, $H^\ell_p = V_{i^\ell(p)}\mathcal{R}^\ell$, is a space of derivations, canonically isomorphic to a subspace of $\mathcal{J}^\ell_p TM$; the disjoint union $H^\ell = \bigcup_{p\in M} H^\ell_p$ is a vector subbundle of $\mathcal{J}^\ell T M \longrightarrow M$.

The global sections of the linear system of partial differential equations H^ℓ is a Lie subalgebra of the Lie algebra of all global sections of $\mathcal{J}^\ell TM \longrightarrow M$. The infinitesimal transformations of the nonlinear Lie equation $\mathcal{R}^\ell$ are the solutions of the linear system of partial differential equations H^ℓ. From the first Lie theorem the vertical bundle of the nonlinear Lie equation is isomorphic to the pullback of the attached linear equation by the target projection [3].

Proposition 3.2. $V\mathcal{R}^\ell \approx \mathcal{R}^\ell \times_M H^\ell$

The *symbol* at $\mathfrak{p}^\ell$ of the manifold $I^\ell(M)$, the most general nonlinear Lie equation, is $Q_{\mathfrak{p}^\ell} I^\ell(M)$, the kernel of $V_{\mathfrak{p}^\ell} I^\ell(M) \longrightarrow V_{\mathfrak{p}^{\ell-1}} I^{\ell-1}(M)$, then the *symbol* of $\mathcal{R}^\ell$ at $\mathfrak{p}^\ell$ is the intersection $G^\ell_{\mathfrak{p}^\ell} = V_{\mathfrak{p}^\ell}\mathcal{R}^\ell \cap Q_{\mathfrak{p}^\ell} I^\ell(M)$.

The linear Lie equation attached to $I^\ell(M)$ is $\mathcal{J}^\ell TM$, and its symbol at any point of the fibre $\mathcal{J}^\ell_{\beta(\mathfrak{p}^\ell)}$ is the kernel $Q_{\beta(\mathfrak{p}^\ell)}\mathcal{J}^\ell TM$ of the natural projection $\mathcal{J}^\ell_{\beta(\mathfrak{p}^\ell)} TM \longrightarrow \mathcal{J}^{\ell-1}_{\beta(\mathfrak{p}^\ell)}(M)$, then $H^\ell_{\beta(\mathfrak{p}^\ell)} \cap Q_{\beta(\mathfrak{p}^\ell)}\mathcal{J}^\ell T M$ is the *symbol* of H^ℓ at any point of $H^\ell_{\beta(\mathfrak{p}^\ell)}$.

Since the mapping

$$\begin{aligned}(R_{\mathfrak{p}^\ell})_*^{-1} : Q_{\beta(\mathfrak{p}^\ell)}\mathcal{J}^\ell T M &\longrightarrow Q_{\mathfrak{p}^\ell} I^\ell(M)\\ D_{i^\ell\circ\beta(\mathfrak{p}^\ell)} &\longrightarrow \mathfrak{p}^\ell \circ D_{i^\ell\circ\beta(\mathfrak{p}^\ell)}\end{aligned}$$

is an isomorphism, and $V_{\mathfrak{p}^\ell}\mathcal{R}^\ell \approx (R^{-1}_{\mathfrak{p}^\ell})_* H^\ell_{\beta(\mathfrak{p}^\ell)}$ we have:

Proposition 3.3. *Let $\mathcal{R}^\ell$ be a nonlinear Lie equation and H^ℓ its associated linear equation. For each $\mathfrak{p}^\ell \in \mathcal{R}^\ell$ the translation $R^{-1}_{\mathfrak{p}^\ell}$ establishes an isomorphism between the symbol of $\mathcal{R}^\ell$ at $\mathfrak{p}^\ell$ and the symbol of H^ℓ at any point of $H^\ell_{\beta(\mathfrak{p}^\ell)}$.*

The analogue of the proposition 2.2 for linear Lie equations is the following

Proposition 3.4. *Let H^ℓ be a linear Lie equation on a connected smooth manifold M and let $\mathcal{R}^\ell$ be the attached nonlinear Lie equation. For each $\mathfrak{p}^{\ell,r} \in \mathcal{J}^r(\mathcal{R}^\ell)$ there is an isomorphism $V_{\mathfrak{p}^{\ell,r}}\mathcal{J}^r(\mathcal{R}^\ell) \approx \mathcal{J}^r_{\mathfrak{p}^{\ell,r}}(V\mathcal{R}^\ell)$ and two elements of this space coincide if and only if their specializations to $C^\infty(H^\ell)$ coincide, that is to say, they determine the same point of $\mathcal{J}^r(H^\ell)$.*

Proof. The first statement is obtained when lemma 2.1 is applied to the fibred manifold $\alpha : \mathcal{R}^\ell \longrightarrow M$. The remainder follows from a commutative diagram analogue to the one preceding proposition 2.2

$$\begin{array}{ccccc} \beta^* C^\infty(M) & \longrightarrow & C^\infty(H^\ell) & \longrightarrow & C^\infty(V\mathcal{R}^\ell) \\ {\scriptstyle \text{natural hom.}}\downarrow & & \downarrow & & \downarrow{\scriptstyle D_{\mathfrak{p}^{\ell,r}}} \\ C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})} & = & C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell,0})} & \xrightarrow{\mathfrak{p}^{0,r}} & C^\infty(M)/\mathfrak{m}^{r+1}_{\alpha(\mathfrak{p}^{\ell,0})} \end{array}$$

and the isomorphism of fibre bundles $\mathcal{R}^\ell \times_M H^\ell \approx V\mathcal{R}^\ell$ given by proposition 3.2. □

We denote by $\mathcal{R}^{\ell+r} = \mathcal{J}^r(\mathcal{R}^\ell) \cap I^{\ell+r}(M)$ the r-th prolongation of the nonlinear Lie equation $\mathcal{R}^\ell$, and by $H^{\ell+r} = J^r(H^\ell) \cap \mathcal{J}^{\ell+r}(TM)$ the r-th prolongation of its attached linear Lie equation H^ℓ. In general, these r-th prolongations are not manifolds, but we can define the vertical tangent space, $V_{\mathfrak{p}^{\ell+r}}\mathcal{R}^{\ell+r}$, to the r-th prolongation of $\mathcal{R}^\ell$ at $\mathfrak{p}^{\ell+r}$ as

$$V_{\mathfrak{p}^{\ell+r}}\mathcal{R}^{\ell+r} = V_{\mathfrak{p}^{\ell+r}}\mathcal{J}^r(\mathcal{R}^\ell) \cap V_{\mathfrak{p}^{\ell+r}} I^{\ell+r}(M).$$

This definition can be obtained as a result [3] from the general theory of systems of partial differential equations.

Proposition 3.5. *Let $\mathcal{R}^\ell$ a nonlinear Lie equation and H^ℓ its attached linear Lie equation. If $H^{\ell+r}$ and $\mathcal{R}^{\ell+r}$ are their respective r-prolongations, then*

$$V_{\mathfrak{p}^{\ell+r}}\mathcal{R}^{\ell+r} = (R_{\mathfrak{p}^{\ell+r}})_*^{-1} H^{\ell+r}_{\beta(\mathfrak{p}^{\ell+r})}$$

for each point $\mathfrak{p}^{\ell+r} \in \mathcal{R}^{\ell+r}$.

Proof. Each vector $D_{\mathfrak{p}^{\ell+r}}$ belonging to $V_{\mathfrak{p}^{\ell+r}}\mathcal{J}^r(\mathcal{R}^\ell) \cap V_{\mathfrak{p}^{\ell+r}} I^{\ell+r}(M)$ gives, by composing it with the inverse of $\mathfrak{p}^{\ell+r}$, a vector $D_{i^{\ell+r}\circ\beta(\mathfrak{p}^{\ell+r})} = (R_{\mathfrak{p}^{\ell+r}})_* D_{\mathfrak{p}^{\ell+r}}$ of the fibre at $\beta(\mathfrak{p}^{\ell+r})$ of $\mathcal{J}^{\ell+r}TM$. Moreover, if we consider the vector $D_{\mathfrak{p}^{\ell+r}}$ as an algebra homomorphism from $C^\infty(V\mathcal{R}^\ell)$ onto $C^\infty(M)/\mathfrak{m}^{r+1}_{\beta(\mathfrak{p}^{\ell+r})}$ its restriction to $C^\infty(H^\ell)$ is the composition of a vector of $\mathcal{J}^r(H^\ell)$ and $\mathfrak{p}^r$, then the statement is a consequence of the fact that the vectors of $V_{\mathfrak{p}^{\ell+r}}\mathcal{R}^{\ell+r}$ are determined by $\mathfrak{p}^{\ell+r}$ and the vectors of $\mathcal{J}^r_{\beta(\mathfrak{p}^{\ell+r})}(H^\ell) \cap \mathcal{J}^{\ell+r}_{\beta(\mathfrak{p}^{\ell+r})}TM = H^{\ell+r}_{\beta(\mathfrak{p}^{\ell+r})}$. □

If we use proposition 3.5, the following result may be proved in much the same way as proposition 3.3.

Corollary 3.6. *Under the above assumptions, for each $\mathfrak{p}^{\ell+r} \in \mathcal{R}^{\ell+r}$ the symbol of $\mathcal{R}^{\ell+r}$ at $\mathfrak{p}^{\ell+r}$ is isomorphic to the symbol of $H^{\ell+r}$ at each point of $H^{\ell+r}_{\beta(\mathfrak{p}^{\ell+r})}$.*

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