DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 239–246

AFFINORS AND CONNECTIONS IN HIGHER ORDER GEOMETRY

MIROSLAV KUREŠ

ABSTRACT. Non-holonomic bundles of higher order velocities are studied by means of colored simplices. All natural affinors on these bundles are described. Furthermore, connections and their torsions are investigated.

0. Introduction

We study some geometrical objects as affinors, connections and their torsions on some higher order bundles. This research is motivated by mechanics, especially by Lagrangian dynamics and it is based on the theory of natural operations in differential geometry, [3].

In the present paper we start from the description of the non-holonomic bundle of k–dimensional velocities of order r including semi-holonomic and holonomic cases. Our approach is based on an original use of so-called colored simplices. This idea is a natural generalization of well-known using of simplices for iterated tangent bundles, [9]. Further, we provide coordinate expressions of bundle morphisms, which are necessary for practical computations and which are not explicitly known in this general form. The indisputable preference of our choice of local coordinates is in the fact that expressions of generators of all natural affinors are very simple without coefficients. This is advantageous with respect to the extensive applicability of affinors in differential geometry and mechanics. It is not difficult to extend the classification of all natural affinors to higher order non-holonomic frame bundles.

The important use of affinors is the concept of a general torsion of connections defined in [4] as the Frölicher–Nijenhuis bracket of the associated horizontal projection and an arbitrary natural affinor on a given bundle. We describe weak torsions of r_k–linear connections as a significant special case.

1991 *Mathematics Subject Classification*. 53C05, 58A20.
Key words and phrases. Higher order velocities, affinor, connection, non-holonomic jet.
Supported by GA CR, grant No. 201/96/0079.
This paper is in final form and no version of it will be submitted for publication elsewhere.

1. Higher order bundles

Let M, N are smooth manifolds, $\dim M = m$, $\dim N = n$. As usually, we denote by $J^r_a(M,N)$ the set of all r–jets from M to N with the source $a \in M$. $J^r_a(M,N)$ is a smooth manifold, the dimension of which is $\frac{n(r+m)!}{r!m!}$.

The manifold $T^r_k M = J^r_0(\mathbb{R}^k, M)$ is called the *bundle of k–dimensional velocities of order r on M*. The bundle $T^r M = T^r_1 M$ of one–dimensional velocities of order r is also called the *tangent bundle of order r on M*. Moreover, we obtain $\underbrace{J^1_0(\mathbb{R}^k, J^1_0(\dots(J^1_0}_{r-\text{ times}}(\mathbb{R}^k, M))\dots))$ by the iteration of the functor J^1. There is natural equivalence between this bundle and the *non-holonomic bundle of k–dimensional velocities of order r* $\tilde{T}^r_k M = \tilde{J}^r_0(\mathbb{R}^k, M)$, see [5]. An element $Z \in \tilde{T}^r_k M$ is called the *semi-holonomic r–jet* if for any s, $1 \le s \le r$, $\pi^r_s(Z) = j^1_{z_i}(\pi^{i-1}_{s-1}) \circ \pi^r_i(Z)$ is satisfied, whenever $i = s+1, \dots, r$ and $\pi^r_s : \tilde{T}^r_k M \to \tilde{T}^s_k M$ is the canonical projection and $z_i = \pi^r_{i-1}(Z)$. We denote by $\bar{T}^r_k M$ the subset of semi-holonomic jets. Of course, $\tilde{T}^r_k M$ and $\bar{T}^r_k M$ are smooth fibered manifolds.

There are natural inclusions such that the three-dimensional diagram

(1)
$$
\begin{tikzcd}
 & & \tilde{T}^1_k M \arrow[r] & \cdots \arrow[r] & \cdots \arrow[rrr] & & & \tilde{T}^s_k M \arrow[r] & \cdots \arrow[r] & \cdots \arrow[rrr] & & & \tilde{T}^r_k M \\
 & \bar{T}^1_k M \arrow[ur] & \vdots \arrow[u] & & & & \bar{T}^s_k M \arrow[ur] & \vdots \arrow[u] & & & & \bar{T}^r_k M \arrow[ur] & \vdots \arrow[u] \\
T^1_k M \arrow[ur] & & \vdots \arrow[u] & & & T^s_k M \arrow[ur] & & \vdots \arrow[u] & & & T^r_k M \arrow[ur] & & \vdots \arrow[u] \\
 & & \tilde{T}^1_l M \arrow[u] \arrow[r] & \cdots \arrow[r] & \cdots \arrow[rrr] & & & \tilde{T}^s_l M \arrow[u] \arrow[r] & \cdots \arrow[r] & \cdots \arrow[rrr] & & & \tilde{T}^r_l M \arrow[u] \\
 & \bar{T}^1_l M \arrow[ur] & \vdots \arrow[u] & & & & \bar{T}^s_l M \arrow[ur] & \vdots \arrow[u] & & & & \bar{T}^r_l M \arrow[ur] & \vdots \arrow[u] \\
T^1_l M \arrow[ur] & & \vdots \arrow[u] & & & T^s_l M \arrow[ur] & & \vdots \arrow[u] & & & T^r_l M \arrow[ur] & & \vdots \arrow[u] \\
 & & \tilde{T}^1_1 M \arrow[u] \arrow[r] & \cdots \arrow[r] & \cdots \arrow[rrr] & & & \tilde{T}^s_1 M \arrow[u] \arrow[r] & \cdots \arrow[r] & \cdots \arrow[rrr] & & & \tilde{T}^r_1 M \arrow[u] \\
 & \bar{T}^1_1 M \arrow[ur] & & & & & \bar{T}^s_1 M \arrow[ur] & & & & & \bar{T}^r_1 M \arrow[ur] & \\
T^1_1 M \arrow[ur] & & & & & T^s_1 M \arrow[ur] & & & & & T^r_1 M \arrow[ur] & &
\end{tikzcd}
$$

is commutative. (We omit the arrows in the holonomic plane and in the semi-holonomic one.) Before the description of inclusions in question we introduce the useful concept of colored simplices, which enable us a good view on bundle structure. We also demonstrate local forms of inclusions.

A *k–colored $(r-1)$–simplex* is the collection of all non-empty subsets of the set $\{1,\dots,r\}$, where *vertices* $1,\dots,r$ can acquire all values (*colors*) from the set $\{1,\dots,k\}$. An *h–face* γ_h of the simplex, $0 \le h \le r-1$, is a subset with $h+1$

fixedly colored elements. We shall designate γ_h as a (k/r)–matrix $\begin{pmatrix} a_{11} \dots a_{1r} \\ \dots\dots\dots \\ a_{k1} \dots a_{kr} \end{pmatrix}$ with $a_{jp} \in \{0,1\}$ and $h+1$ of the elements equals to unity. This corresponds to the subset containing each s with color l such that $a_{ls} = 1$. The zero matrix is excluded. We write the *composition* of two disjoint faces γ_h, $\bar{\gamma}_{\bar{h}}$ as $\gamma_h + \bar{\gamma}_{\bar{h}}$.

Given some local coordinates x^i on M and t^j on $\mathbb{R}^k$, the iterated differentiation of $x^i(t^1, \dots, t^k)$ determines the induced coordinates

$$x^i, y^i_{\gamma_h} = \frac{\partial}{\partial t^{j_r}}(\dots(\frac{\partial}{\partial t^{j_2}}(\frac{\partial x^i}{\partial t^{j_1}}))\dots), \quad h = 0, \dots, r-1 \tag{2}$$

on $\tilde{T}^r_k M$, $j_p \in \{0, 1, \dots, k\}$, when $\frac{\partial}{\partial t^0}$ represents no differentiation. In the matrix γ_h, the columns represent the order of the differentiation and the rows correspond with coordinates $t^1, \dots, t^k$. Thus, $a_{ls} = 1$ means the differentiation $\frac{\partial}{\partial t^l}$ at the s–th iteration.

Velocity–inclusions

Firstly, we describe inclusions ${}^V i^{\bar{s}}_s : \tilde{T}^{\bar{s}}_l M \to \tilde{T}^s_l M$, $1 \le \bar{s} < s \le r$, properties of which are observed also for the cases $\bar{T}^{\bar{s}}_l M \to \bar{T}^s_l M$ and $T^{\bar{s}}_l M \to T^s_l M$. h–faces related to $\tilde{T}^{\bar{s}}_l M$ map to the same h–faces related to $\tilde{T}^s_l M$. Thus, the image is represented by faces with a lesser number of vertices which can acquire all l colors. In local coordinates, it deals with elements with matrix indices having zeros in last $s - \bar{s}$ columns.

Dimension–inclusions

Secondly, we describe inclusions ${}^D i^{\bar{l}}_l : \tilde{T}^s_{\bar{l}} M \to \tilde{T}^s_l M$, $1 \le \bar{l} < l \le k$, properties of which are observed also for the cases $\bar{T}^s_{\bar{l}} M \to \bar{T}^s_l M$ and $T^s_{\bar{l}} M \to T^s_l M$. j–colored h–faces, $j = 1, \dots \bar{l}$, related to $\tilde{T}^s_{\bar{l}} M$ map to the same h–faces related to $\tilde{T}^s_l M$. Thus, the image is represented by faces with a lesser number of colors. In local coordinates, it deals with elements with matrix indices having zeros in last $l - \bar{l}$ rows.

First holonomy–inclusions

In fact, inclusions ${}^1 i^s_l : \bar{T}^s_l M \to \tilde{T}^s_l M$ are determined by the above introduced definition of semi-holonomic jets. h–faces, $h = 1, \dots, s-2$, vertices of them are colored in order by colors $C_1, \dots, C_{h+1}$, identify in the semi-holonomic case. However, it is necessary to respect the order of colors. In local coordinates, it deals with leaving zero columns out.

Second holonomy–inclusions

We introduce new local coordinates on $T^r_k M$ before the description of inclusions ${}^2 i^s_l : T^s_l M \to \bar{T}^s_l M$. Given some local coordinates x^i on M and t^j on $\mathbb{R}^k$, the expansion of $x^i(t^1,\dots,t^k)$ determines the induced coordinates

$$x^i, y^{i,\mathbf{h}} = \frac{1}{(h+1)!}\frac{\partial^{h+1}x^i}{(\partial t^1)^{u_1}\dots(\partial t^k)^{u_k}}, \quad h=0,\dots,r-1$$

on $T^r_k M$, where by the bold letter $\mathbf{h}$ we denote the multiindex $\mathbf{h} = u_1\dots u_k$, $|\mathbf{h}| = u_1+\dots+u_k \le r$ (we set $\mathbf{h}! = u_1!\dots u_k!$ and $\mathbf{h}+\bar{\mathbf{h}} = u_1+\bar u_1\dots u_k+\bar u_k$). Equally colored h–faces (without any respecting of the order of colors) identify in the holonomic case. In local coordinates,

$$y^i_{\gamma_h} \mapsto \mathbf{h}! y^{i,\mathbf{h}}, \quad u_j = \sum_{p=1}^{r} a_{jp}, \quad j=1,\dots,k. \tag{3}$$

2. Higher order morphisms

Let x^i, $y^i_{\gamma_h}$ and $\bar x^{\bar i}$, $\bar y^{\bar i}_{\gamma_h}$ be some local coordinates on $\tilde T^r_k M$ and $\tilde T^r_k N$, respectively. Let $\bar x^{\bar i} = f^{\bar i}(x^i)$ be the coordinate expression of an arbitrary smooth map $f: M\to N$. We express the induced fiber bundle morphism $\tilde T^r_k f: \tilde T^r_k M \to \tilde T^r_k N$ as

$$\bar y^{\bar i}_{\gamma_h} = \sum_{q=1}^{h+1}\sum_{\rho\in\mathbb{P}(\gamma_h,q)} \alpha^{\bar i}_{i_1\dots i_q} y^{i_1}_{\gamma_{h_1}}\dots y^{i_q}_{\gamma_{h_q}}, \quad h=0,\dots,r-1, \tag{4}$$

where $\alpha^{\bar i}_{i_1\dots i_q} = \frac{\partial^q f^{\bar i}(x)}{\partial x^{i_1}\dots\partial x^{i_q}}$, $\mathbb{P}(\gamma_h,q)$ is the set of all decompositions of h–face to q faces $\gamma_{h_1},\dots,\gamma_{h_q}$, $h_1+\dots+h_q+q-1=h$ (i.e. we sum as to all such decompositions $\rho\in\mathbb{P}(\gamma_h,q)$). This formula is obtained by a direct evaluation and its special case for $k=1$ is in the book [9].

In the theory of holonomic jets we need another special case of (4) which corresponds with an expression of the jet composition: the morphism $T^r_k f$. We also write up it for its importance here:

$$\bar y^{\bar i,\mathbf{h}} = \sum_{q=1}^{|\mathbf{h}|}\sum_{\pi\in\mathbb{P}(\mathbf{h},q)} k_\pi a^{\bar i}_{i_1\dots i_q} y^{i_1,\mathbf{h}_1}\dots y^{i_q,\mathbf{h}_q}, \quad |\mathbf{h}|=1,\dots,r, \tag{5}$$

where $a^{\bar i}_{i_1\dots i_q} = \frac{1}{q!}\frac{\partial^q f^{\bar i}(x)}{\partial x^{i_1}\dots\partial x^{i_q}}$, $\mathbb{P}(\mathbf{h},q)$ is the set of all decompositions of multiindex $\mathbf{h}$ to q multiindeces $\mathbf{h}_1,\dots,\mathbf{h}_q$ and k_π is the number of permutations of the set of components of a decomposition π.

Sketch of calculation. The translation of (4) to the language of coordinates on $T^r_k M$ means the multiplication by $\frac{q!\mathbf{h}_1!\dots\mathbf{h}_q!}{\mathbf{h}!}$, see (3). Further, we multiply by $\frac{\mathbf{h}!}{\mathbf{h}_1!\dots\mathbf{h}_q!}$, which represents taking of all different selections of multiindices $\mathbf{h}_1,\dots,\mathbf{h}_q$ from the multiindex $\mathbf{h}$. In addition, we divide by the number of indistinguishable permutations which equals $\frac{q!}{k_\pi}$. Summarily, we have the stated multiplication by k_π. $\square$

3. All natural affinors

In general, by an *affinor* A on a manifold M we mean a (1,1)–tensor field, i.e. a linear morphism $A : TM \to TM$ over id_M. A *natural affinor* on a natural bundle F over m–dimensional manifolds is a system of (1,1)–tensor fields $A_M : TFM \to TFM$ for every m–dimensional manifold M satisfying $TFf \circ A_M = A_N \circ TFf$ for every local diffeomorphism $f : M \to N$.

Proposition 1. *All natural affinors on $\tilde{T}^r_k M$ constitute a $(1+k)^r$– parameter family linearly generated by the identical affinor A_0 and by affinors A_{γ_c}, where γ_c represent all c–faces, $0 \le c \le r-1$, which work in this way:*

$$\begin{aligned} dy^i_{\gamma_c} &= dx^i \\ dy^i_{\gamma_c+\gamma_h} &= dy^i_{\gamma_h}, \quad h \le r-c-2 \end{aligned} \tag{6}$$

Proof. All natural affinors on an arbitrary Weil bundle correspond to the multiplication by the elements of the relevant Weil algebra, [4]. The Weil algebra associated with the functor $\tilde{T}^r_k$ is $\mathbb{D}^r_k \approx \underbrace{\mathbb{D}^1_k \otimes \cdots \otimes \mathbb{D}^1_k}_{r\text{–times}}$, where $\mathbb{D}^1_k$ is the sum of k copies of the algebra of dual numbers $\mathbb{D}$. We obtain the generators of natural affinors by a direct evaluation of elements $\mathbb{D}^r_k$ and their products. □

The special case of the formula (6) for $r = 2$, $k = 1$ is in the paper [6].

Proposition 2. *All natural affinors on $T^r_k M$ constitute a $\frac{(r+k)!}{r!k!}$– parameter family linearly generated by the identical affinor A_0 and by affinors $A^{\mathbf{c}}$, where $\mathbf{c}$ represent all multiindices $\mathbf{c} = c_1 \dots c_k$, $|\mathbf{c}| \le r$. The affinors $A^{\mathbf{c}}$ are sums of affinors A_{γ_c} multiplied by $\mathbf{c}!$ in which γ_c are equally colored c–faces without any respecting of the order of colors. They work in this way:*

$$\begin{aligned} dy^{i,\mathbf{c}} &= dx^i \\ dy^{i,\mathbf{c}+\mathbf{h}} &= dy^{i,\mathbf{h}}, \quad |\mathbf{h}| \le r-|\mathbf{c}|-1 \end{aligned} \tag{7}$$

Proof. There are $\frac{r!}{\mathbf{c}!(r-|\mathbf{c}|)!}$ additional affinors in the introduced sum affinor. We apply this sum affinor to coordinates with $\frac{r!}{\mathbf{c}!(r-|\mathbf{c}|)!}\frac{(r-|\mathbf{c}|)!}{\mathbf{h}!(r-|\mathbf{c}|-|\mathbf{h}|)!}$ different matrix indices. The role of the image play coordinates with $\frac{r!}{(\mathbf{c}+\mathbf{h})!(r-|\mathbf{c}+\mathbf{h}|)!}$ different matrix indices. Furthermore, we multiply the affinor $A^{\mathbf{c}}$ by $\mathbf{c}!$. If we use the coordinates on $T^r_k M$, we obtain

$$(\mathbf{c}+\mathbf{h})!dy^{i,\mathbf{c}+\mathbf{h}} = \frac{(\mathbf{c}+\mathbf{h})!}{\mathbf{c}!\mathbf{h}!}\mathbf{c}!\mathbf{h}!dy^{i,\mathbf{h}}.$$

Of course, it is possible come to the same result by the multiplication of the elements of the Weil algebra $\mathbb{A} = \mathbb{R}[t^1, \dots, t^k]/\langle t^1, \dots, t^k\rangle^{r+1}$. □

The case $k = 1$ is stated in [7]. Further, it is known the case $r = 1$, see [4].

Implied results for higher order frame bundles

We say that $Z \in \tilde{J}^r_a(M,N)_b$ is *invertible* if there exists a $Z^{-1} \in \tilde{J}^r_b(N,M)_a$ such that $Z^{-1} \circ Z = j^r_a \operatorname{id}_M$ and $Z \circ Z^{-1} = j^r_b \operatorname{id}_N$. We denote by $\operatorname{inv} \tilde{J}^r_a(M,N)_b$ the open submanifold of all invertible non-holonomic r–jets and we define the *r–th order non-holonomic frame bundle* as $\tilde{P}^r M = \operatorname{inv} \tilde{J}^r_0(\mathbb{R}^m, M)$. The group $\tilde{G}^r_m = \operatorname{inv} \tilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$ acts smoothly on $\tilde{P}^r M$ on the right by the jet composition. Similarly for the *semi-holonomic* and *holonomic* cases.

Proposition 3. *All natural affinors on $\tilde{P}^r M$ ($\bar{P}^r M$, $P^r M$, respectively) are given by the restrictions of natural affinors on $\tilde{T}^r_m M$ ($\bar{T}^r_m M$, $T^r_m M$, respectively).*

Proof. $\tilde{P}^r M$ is an open submanifold natural with local diffeomorphisms. Hence the assertion is evident and we introduce local fiber coordinates $\phi^i_{\gamma_h}$, $h = 0, \dots, r-1$, on $\tilde{P}^r M$ by the way corresponding with the case of $\tilde{T}^r_m M$. We proceed analogously for semi-holonomic and holonomic cases $\bar{P}^r M$, $P^r M$. For $P^r M$ we prefer the coordinates $\phi^{i,\mathbf{h}}$, $|\mathbf{h}| = 1, \dots, r$, including the factorial numbers as well as on $T^r_m M$. □

Indeed, we obtain the classification of all natural affinors on higher order frame bundles easily by this proposition. The holonomic case was studied in [4] ($r = 1$) and [7] (general r).

4. Connections and their torsions

The *general connection* Γ on an arbitrary fibered manifold $Y \to B$ is defined as a section $\Gamma : Y \to J^1 Y$ of the first jet prolongation $J^1 Y \to Y$. Evidently, the bundle $\tilde{T}^r_k M$ disposes of various bundle structures. We consider the fundamental case $Y = \tilde{T}^r_k M$ and $B = M$. The *general torsion* of Γ is defined as the Frölicher–Nijenhuis bracket $[\Gamma, A]$, where we see Γ as the associated horizontal projection and A is a non-identical affinor on $Y \to B$. We recall the classification of all natural affinors on $\tilde{T}^r_k M \to M$ (Proposition 1). Hence we can evaluate all general torsions of Γ on $\tilde{T}^r_k M$ as $\tau_{\gamma_c} = [\Gamma, A_{\gamma_c}]$ and their linear combinations. Especially, we confine ourselves to the simplest class of connections on $\tilde{T}^r_k M$ defined by the property that the flows of the corresponding horizontal lifts of all vector fields on M are constituted by fiber morphisms (4). Such a connection ∇ is called the *r_k–linear connection* (for $k = 1$ *r–linear connection* and for $r = k = 1$ *linear connection*) and its coordinate expression is

$$dy^i_{\gamma_h} = \Big(\sum_{q=1}^{h+1} \sum_{\rho \in \mathbb{P}(\gamma_h, q)} \nabla^i_{i_1 \dots i_q j} y^{i_1}_{\gamma_{h_1}} \dots y^{i_q}_{\gamma_{h_q}}\Big) dx^j, \quad h = 0, \dots, r-1, \tag{8}$$

where ∇'s are symmetric in subscripts $i_1, \dots, i_q$.

Further, the bundle $\tilde{T}^r_k M$ is associated with $\tilde{P}^r M$ with standard fiber $\tilde{L}^r_{k,m} = \tilde{J}^r_0(\mathbb{R}^k, \mathbb{R}^m)_0$. A *principal connection* Γ on $\tilde{P}^r M$ is a $\tilde{G}^r_m$– invariant section of $J^1 \tilde{P}^r M$ and Γ is said to be *totally projectable*, if it determines the same connections on $\tilde{P}^s M$ with respect to all projections $\tilde{P}^r M \to \tilde{P}^s M$ for every $s = 1, \dots, r-1$.

Proposition 4. *The totally projectable principal connections on $\tilde{P}^r M$ are in bijection with the r_k–linear connection on $\tilde{T}^r_k M$.*

Proof. $\tilde{P}^r M$ can be locally identified with the trivial principal bundle $U \times \tilde{G}^r_m$, $U \subset M$. If $A, B \in \tilde{G}^r_m$, then the coordinate expression of the composition $C = B \circ A$ of non-holonomic jets in the sense of Ehresmann is

$$C^i_{\gamma_h} = \sum_{q=1}^{h+1} \sum_{\rho \in \mathbb{P}(\gamma_h, q)} B^i_{i_1 \dots i_q(\rho)} A^{i_1}_{\gamma_{h_1}} \dots A^{i_q}_{\gamma_{h_q}}, \quad h = 0, \dots, r-1,$$

where $i_1 \dots i_q(\rho)$ represents the $(q-1)$–face with units only in the rows $i_1, \dots, i_q$ and in the same columns as in $\gamma_{h_1}, \dots, \gamma_{h_q}$, respectively. In general, we must distinguish between all particular decompositions and write ρ in brackets here. We denote by $s_A : x \mapsto (x, A)$, $s_B : x \mapsto (x, B)$, $s_C : x \mapsto (x, C)$ the sections $M \to \tilde{P}^r M$ and by σ's with the same indices the 1–jets of these sections. We concerned with the commutative property of these 1–jets, i.e. $\sigma_C = \sigma_{B \circ A} = \sigma_B \circ \sigma_A$. We identify $A^i_{\gamma_h}$ with fiber coordinates on $\tilde{P}^r M$. Then the differentiation $\frac{\partial}{\partial x^i}$ provides the local expression of the principal connection as

$$\psi^i_{\gamma_h j} = \sum_{q=1}^{h+1} \sum_{\rho \in \mathbb{P}(\gamma_h, q)} \Gamma^i_{i_1 \dots i_q(\rho) j} \phi^{i_1}_{\gamma_{h_1}} \dots \phi^{i_q}_{\gamma_{h_q}}, \quad h = 0, \dots, r-1,$$

where $\psi^i_{\gamma_h j} = \frac{\partial C^i_{\gamma_h}}{\partial x^j}$ are coordinates on $J^1 \tilde{P}^r M$, $\Gamma^i_{i_1 \dots i_q(\rho) j} = \frac{\partial B^i_{i_1 \dots i_q(\rho)}}{\partial x^j}$. However, we identify $\Gamma^i_{i_1 \dots i_q(\rho) j}$ under the condition of the total projectability and write simply $\Gamma^i_{i_1 \dots i_q j}$. In the horizontal projection form we have

$$d\phi^i_{\gamma_h} = \Big(\sum_{q=1}^{h+1} \sum_{\rho \in \mathbb{P}(\gamma_h, q)} \Gamma^i_{i_1 \dots i_q j} \phi^{i_1}_{\gamma_{h_1}} \dots \phi^{i_q}_{\gamma_{h_q}}\Big) dx^j, \quad h = 0, \dots, r-1, \tag{9}$$

and we have obtained the form corresponding to (8). Certainly, the correspondation between Chistoffels $\Gamma^i_{i_1 \dots i_q j}, \nabla^i_{i_1 \dots i_q j}$ is guaranteed, because decompositions of faces do not depend on the number of colors. □

Remark. Now it is clear that the r_k–linear connections on $\tilde{T}^r_k M$ are in bijection the r_l–linear connections on $\tilde{T}^r_l M$.

Let us occupy ourselves with another possible approaches to connections on $\tilde{T}^r_k M$.

Grifone, [2], gave a new definition of the connection as a certain special $(1,1)$–tensor field on TM. De Andrés, de León and Rodrigues, [1], generalized Grifone's results to tangent bundles of higher order $T^r_1 M$. But we show in the discussion so-called *Grifone's connections* in [6] that a generalization of such a construction is impracticable for the case $TTM = \tilde{T}^2_1 M$.

Virsik, [8], summarizes some results dealing with higher order connections. The $\tilde{G}^r_m$–invariant section of $\tilde{J}^R\tilde{P}^r M \to \tilde{P}^r M$ is called in our terminology the *principal R–th order (non-holonomic) connection* on $\tilde{P}^r M$. It also induces naturally a higher order connection on an arbitrary associated bundle.

The special class of general torsions are *weak torsions*: just the affinors $A_{\gamma_{r-1}}$ are putting into the Frölicher–Nijenhuis bracket. So, the number of generators of weak torsions is k^r. Closing this paper, we describe weak torsions of r_k–linear connections.

Proposition 5. *Weak torsions of r_k–linear connections have coordinate expressions*

$$\nabla^i_{jk}\frac{\partial}{\partial y^i_{\gamma_{r-1}}}\otimes dx^j\wedge dx^k$$

for all $(r-1)$–faces. Geometrically, they represent pullbacks of the classical torsion of the classical linear connection on M determined by the projection of ∇.

Proof. We obtained the formula by a direct evaluation of the Frölicher–Nijenhuis bracket in local coordinates. The idea of geometrization is clear. □

References

[1] de Andrés, L. C., de León, M., Rodrigues, P. R., *Connections on tangent bundles of higher order*, Demonstr. Math. XXII **3** (1989), 189–216.

[2] Grifone, J., *Structure presque-tangente et connexions, I.*, Ann. Inst. Fourier **22** (1972), 287–334.

[3] Kolář, I., Michor, P. W., Slovák, J., *Natural Operations in Differential Geometry*, Springer Verlag, 1993.

[4] Kolář, I., Modugno, M., *Torsions of connections on some natural bundles*, Diff. Geom. and its Appl. **2** (1992), 1–16.

[5] Kolář, I., *Bundle functors of the jet type* (to appear).

[6] Kureš, M., *Some properties of connections on iterated tangent bundles*, Arch. Mech. **50** (1998), 459–466.

[7] ———, *Torsions of connections on tangent bundles of higher order*, Suppl. Rend. Circ. Mat. Palermo **54** (1998), 65–73.

[8] Virsik, G., *Higher order connections: Introduction and survey*, Seminar text, Budapest (1997).

[9] White, J. E., *The Method of Iterated Tangents with Applications in Local Riemannian Geometry*, Pitman, London, 1982.

Department of Mathematics, Technical University of Brno
Technická 2, 616 69 Brno, CZECH REPUBLIC
E-mail: kures@mat.fme.vutbr.cz