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BUNDLE FUNCTORS OF THE JET TYPE

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ABSTRACT. We introduce the general concept of jet functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$. We describe the extensions of jet functors to the category of fibered manifolds with m-dimensional bases and fiber preserving maps with local diffeomorphism as base map.

Let $\mathcal{M}f$ be the category of all manifolds and all smooth maps, $\mathcal{M}f_m$ be the category of m-dimensional manifolds and their local diffeomorphisms and $\mathcal{FM}_m$ be the category of fibered manifolds over m-dimensional bases and their morphisms with the base map in $\mathcal{M}f_m$. In [4], W. Mikulski and the author described all fiber product preserving bundle functors on the category $\mathcal{FM}_m$ in terms of Weil algebras and their homomorphisms. In the present paper, we apply this point of view to various subbundles of the non-holonomic jet bundles by C. Ehresmann, [1], [5]. In §1 we introduce the general concept of jet functor on the category $\mathcal{M}f_m \times \mathcal{M}f$ and we characterize it by means of Weil algebras.

Then we discuss jet functors on the category $\mathcal{FM}_m$. The construction of product fibered manifolds and product morphisms defines an injection $i : \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$. In general, a bundle functor E on $\mathcal{FM}_m$ is said to be extension of a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$, if $E \circ i = F$. In §2 we point out that every jet functor on $\mathcal{M}f_m \times \mathcal{M}f$ determines two extensions, the horizontal one and the vertical one. Using the general description of constructing extensions from [4], we deduce in §3 that these two extensions are the only ones in the case of the functor of holonomic r-jets, $r \geq 2$. In §4, we determine all extensions of the functor of non-holonomic r-jets. In the last paragraph we discuss an appropriate generalization of semiholonomic jets.

All manifolds and maps are assumed to be infinitely differentiable. Unless otherwise specified, we use the terminology and notation from [3].

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1. Jet functors on $\mathcal{M}f_m \times \mathcal{M}f$. Every two manifolds M and N determine the bundle $J^r(M,N)$ of all r-jets of M into N. In [3] we pointed out that this construction is to be interpreted as a functor on the product category $\mathcal{M}f_m \times \mathcal{M}f$, $m = \dim M$. Indeed, every local diffeomorphism $f : M \to \overline{M}$ and every map $g : N \to \overline{N}$ induce a map

$$J^r(f,g) : J^r(M,N) \to J^r(\overline{M},\overline{N})$$

by the jet composition

$$J^r(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}\,, \qquad X \in J^r_x(M,N)_y\,. \tag{1}$$

In [1], C. Ehresmann introduced the bundle $\widetilde{J}^r(M,N) \supset J^r(M,N)$ of non-holonomic r-jets of M into N and defined the composition

$$Y \circ X \in \widetilde{J}^r_x(M,P)_z \tag{2}$$

for every $X \in \widetilde{J}^r_x(M,N)_y$ and $Y \in \widetilde{J}^r_y(N,P)_z$. If X and Y are the classical holonomic r-jets, then (2) coincides with the classical composition. Hence $\widetilde{J}^r$ can also be interpreted as a functor on $\mathcal{M}f_m \times \mathcal{M}f$, if one sets

$$\widetilde{J}^r(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}\,, \qquad X \in \widetilde{J}^r_x(M,N)_y \tag{3}$$

with the composition of non-holonomic jets.

There are several special classes of non-holonomic jets, see e.g. [5]. That's why it seems to be reasonable to formulate the general concept of jet functor on $\mathcal{M}f_m \times \mathcal{M}f$. The following definition is based on our results from [4].

Definition 1. A bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ is called an r-th order jet functor, if $J^r \subset F \subset \widetilde{J}^r$ and F preserves products in the second factor, i.e.

$$F(M, N \times P) = F(M,N) \times_M F(M,P)\,. \tag{4}$$

Thus we have $J^r(M,N) \subset F(M,N)$. The inclusion $F \subset \widetilde{J}^r$ implies

$$F(f,g)(X) = (j^r_y g) \circ X \circ (j^r_x f)^{-1}\,, \qquad X \in F_x(M,N)_y \tag{5}$$

with the composition of non-holonomic jets.

We need to review some results from [4]. Every bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$ induces a bundle functor G^F on $\mathcal{M}f$ by

$$G^F M = F_0(\mathbb{R}^m, N)\,, \quad G^F f = F_0(\mathrm{id}_{\mathbb{R}^m} \times f) : G^F N \to G^F P$$

for every manifold N and every smooth map $f : N \to P$. We say that F is of order r in the first factor, if $j^r_x f_1 = j^r_x f_2$ implies that the restricted maps to the fiber of $F(M,N)$ over $x \in M$ coincide, i.e.

$$F_x(f_1, g) = F_x(f_2, g) : F_x(M,N) \to F_y(\overline{M},\overline{N})\,, \qquad y = f_1(x) = f_2(x)$$

for every two local diffeomorphisms $f_1, f_2 : M \to N$ and every smooth map $g : N \to \overline{N}$. Such a functor induces a smooth group homomorphism $H^F : G^r_m \to \mathcal{NE}(G^F)$, where G^r_m is the r-th jet group in dimension m and $\mathcal{NE}(G^F)$ is the group of all natural equivalences of the functor G^F into itself. Conversely, for every bundle functor G on $\mathcal{M}f$ and every smooth group homomorphism $H : G^r_m \to \mathcal{NE}(G)$ one constructs a bundle functor (G, H) on $\mathcal{M}f_m \times \mathcal{M}f$ by the associated bundle construction

$$(G,H)(M,N) = P^r M[GN, H]\,, \qquad (G,H)(f,g) = P^r f[Gg]\,,$$

where $P^r M$ is the r-th order frame bundle of M, [3]. In this way, we reconstruct $F = (G^F, H^F)$. If G preserves products, then $G = T^A$ is a Weil functor. Hence $\mathcal{NE}(G)$ is the Lie group $\operatorname{Aut} A$ of all algebra automorphisms of A and $H : G^r_m \to \operatorname{Aut} A$ is a Lie group homomorphism.

In the case of $J^r, G^{J^r} = T^r_m$ is the functor of (m,r)-velocities and H is the canonical action of G^r_m on $T^r_m N$ by the jet composition on the right. In the case of $\widetilde{J}^r$, $G^{\widetilde{J}^r} = \widetilde{T}^r_m$ is the functor of non-holonomic m^r-velocities. The action of G^r_m on $\widetilde{T}^r_m N = \widetilde{J}^r_0(\mathbb{R}^m, N)$ is given by the jet composition on the right.

The Weil algebra of T^r_m is $\mathbb{D}^r_m = \mathbb{R}[x_1, \dots, x_m]/\langle x_1, \dots, x_m\rangle^{r+1}$. The relation $\operatorname{Aut} \mathbb{D}^r_m = G^r_m$ is well known, [4]. Write $\widetilde{\mathbb{D}}^r_m$ for the Weil algebra of $\widetilde{T}^r_m$. Since every element of G^r_m determines a natural equivalence $\widetilde{T}^r_m \to \widetilde{T}^r_m$ by the jet composition on the right, we have $G^r_m \subset \operatorname{Aut} \widetilde{\mathbb{D}}^r_m$. There is a natural equivalence $\varepsilon_r : \widetilde{T}^r_m \to T^1_m \circ \dots \circ T^1_m$ defined by the following induction. For $r = 1$, ε_1 is the identity. Let t_u denote the translation on $\mathbb{R}^m$ transforming 0 into u. If $\sigma : \mathbb{R}^m \to \widetilde{J}^{r-1}(\mathbb{R}^m, N)$ is a section, then $\sigma(u) \circ j^{r-1}_0(t_u) \in \widetilde{T}^{r-1}_m N$. Hence $\varepsilon_{r-1}(\sigma(u) \circ j^{r-1}_0 t_u) \in \underbrace{T^1_m \dots T^1_m}_{\text{(r-1)-times}} N$ and we define $\varepsilon_r(j^1_0\sigma) = j^1_0 \varepsilon_{r-1}(\sigma(u) \circ j^{r-1}_0 t_u)$.
By the general theory, ε_r corresponds to an algebra isomorphism

$$\widetilde{\mathbb{D}}^r_m \approx \mathbb{D}^1_m \otimes \dots \otimes \mathbb{D}^1_m\,. \tag{6}$$

Since an r-th order jet functor F on $\mathcal{M}f_m \times \mathcal{M}f$ preserves products in the second factor, G^F is a Weil functor T^A. The natural injections $J^r \subset F$ and $F \subset \widetilde{J}^r$ define two injective algebra homomorphisms $i : \mathbb{D}^r_m \to A$ and $j : A \to \widetilde{\mathbb{D}}^r_m$. Moreover, j defines a group homomorphism $G^r_m \to \operatorname{Aut} A$.

Proposition 1. *The jet functors on $\mathcal{M}f_m \times \mathcal{M}f$ are in bijection with quadruples (A, H, i, j), where A is a Weil algebra, $H : G^r_m \to \operatorname{Aut} A$ is a group homomorphism and $i : \mathbb{D}^r_m \to A$ and $j : A \to \widetilde{\mathbb{D}}^r_m$ are two G^r_m-equivariant injective algebra homomorphisms.*

Proof. Both i and j are equivariant by Proposition 5 of [4]. Conversely, if we have an algebra A with a group homomorphisms $H : G^r_m \to \operatorname{Aut} A$ and two equivariant injective algebra homomorphisms $i : \mathbb{D}^r_m \to A$ and $j : A \to \widetilde{\mathbb{D}}^r_m$, then $F = (T^A, H)$ is a bundle functor on $\mathcal{M}f_m \times \mathcal{M}f$ and i or j induces a natural inclusion $J^r \subset F$ or $F \subset \widetilde{J}^r$, respectively. □

2. Horizontal and vertical extensions. Unless otherwise specified, $M \times N$ means the product fibered manifold $M \times N \to M$. The canonical injection $i : \mathcal{M}f_m \times \mathcal{M}f \to \mathcal{FM}_m$ is defined by $i(M,N) = M \times N$ and $i(f,g) = f \times g$.

Definition 2. A bundle functor E on $\mathcal{FM}_m$ is called an extension of a bundle functor F on $\mathcal{M}f_m \times \mathcal{M}f$, if $F = E \circ i$.

From now on we assume that the bundle functor G on $\mathcal{M}f$ has the so-called point property, i.e. $G(pt) = pt$, where pt denotes one-element set. (This is always true if G preserves products.) In [4] we clarified that the extensions of (G,H) are in bijection with the invariant diagonals in $(G,H)(M,M)$. For a fibered manifold $p : Y \to M$, the value of the extension $D(G,H)$ determined by an invariant diagonal D_M is given by

$$D(G,H)(Y) = \{X \in (G,H)(M,Y);\ (G,H)(\mathrm{id}_M, p)(X) \in D_M\}\,.$$

The effect of $D(G,H)$ on an $\mathcal{FM}_m$-morphism f is defined by the canonical restriction and corestriction of $(G,H)(\underline{f}, f)$, where $\underline{f}$ is the base map of f, [4].

The construction of r-th jet prolongations of fibered manifolds and $\mathcal{FM}_m$-morphisms is a bundle functor on $\mathcal{FM}_m$, which is usually also denoted by J^r. In this theoretical research, we shall use the symbol J^r_h for the latter functor on $\mathcal{FM}_m$ to distinguish it from the former functor J^r on $\mathcal{M}f_m \times \mathcal{M}f$.

Definition 3. For every r-th order jet functor F on $\mathcal{M}f_m \times \mathcal{M}f$, we define its horizontal extension F_h by

$$F_h Y = \{X \in F(M,Y);\ F(\mathrm{id}_M, p)(X) = j^r_x \mathrm{id}_M\}$$

and its vertical extension F_v by

$$F_v Y = \{X \in F(M,Y);\ F(\mathrm{id}_M, p)(X) = j^r_x \widehat{x}\}\,,$$

where $\widehat{x}$ denotes the constant map of M into $x \in M$. The effect of both F_h and F_v on $\mathcal{FM}_m$-morphisms is defined by the canonical restriction and corestriction.

By a standard property of non-holonomic jets, $F_v Y$ can also be expressed in the form

$$F_v Y = \bigcup_{x \in M} F_x(M, Y_x)\,.$$

3. Extensions of J^r. We deduce a complete description of all extensions of the holonomic r-th jet functor J^r.

Proposition 2. *For $r \geq 2$, J^r_h and J^r_v are the only two extensions of J^r.*

Proof. By invariancy, it is sufficient to find all invariant elements in $L^r_{m,m} = J^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$. So we are looking for all $Y \in L^r_{m,m}$ satisfying

$$Y = X^{-1} \circ Y \circ X \qquad \text{for all} \quad X \in G^r_m\,.$$

Let $Y = (y^i_j, y^i_{j_1 j_2}, \dots, y^i_{j_1 \dots j_r})$. There is a canonical injection of $GL(m, \mathbb{R}) = G^1_m$ into G^r_m. The action of $GL(m, \mathbb{R})$ on each component of Y is the classical tensorial one, so that each $y^i_{j_1 \dots j_k}$ is an invariant tensor. By the invariant tensor theorem, [3], p. 214, $y^i_{j_1 \dots j_k} = 0$ for $k \geq 2$ and

$$y^i_j = c\delta^i_j\,, \qquad c \in \mathbb{R}\,. \tag{7}$$

Consider the underlying 2-jet $Y_2 = (c\delta^i_j, 0)$. An element X of the kernel of the canonical projection $G^2_m \to G^1_m$ is of the form $X = (\delta^i_j, b^i_{jk})$. Then $X^{-1} = (\delta^i_j, -b^i_{jk})$ and the jet composition yields

$$X^{-1} \circ Y_2 \circ X = (c\delta^i_j, -c^2 b^i_{jk} + c b^i_{jk})\,.$$

The second term implies

$$c = c^2\,, \qquad \text{i.e.} \quad c = 0 \text{ or } c = 1\,. \tag{8}$$

One verifies directly that $c = 0$ corresponds to J^r_v and $c = 1$ corresponds to J^r_h. □

For $r = 1$, we have to stop at (7). Geometrically, $J^1(M, N) = T^*M \otimes TN$ is a vector bundle over $M \times N$. In particular, we can define an invariant diagonal

$$D^1_c M = \{c j^1_x id_M, x \in M\}\,, \qquad 0 \neq c \in \mathbb{R}\,,$$

and the corresponding extension

$$J^1_{h,c} Y = \{X \in J^1(M, Y)\,, J^1(\mathrm{id}_M, p)(X) \in D^1_c M\}\,, \qquad c \neq 0\,.$$

(The case $c = 0$ corresponds to J^1_v.) Thus, we have proved

Proposition 3. *All extensions of J^1 are J^1_v and $J^1_{h,c}$, $0 \neq c \in \mathbb{R}$.*

4. Extensions of $\widetilde{J}^r$. Write $\widetilde{L}^r_{m,m} = \widetilde{J}^r_0(\mathbb{R}^m, \mathbb{R}^m)_0$. The canonical coordinates of an element $Y \in \widetilde{L}^r_{m,m}$ are

$$y^i_{k_1 \dots k_r}\,, \qquad k_1, \dots, k_r = 0, 1, \dots, m$$

with at least one non-zero subscript, [2], [6]. Each coordinate with s non-zero subscripts is called an s-th order coordinate, $1 \leq s \leq r$. By [6] or [2], we have r geometric projections $p_s : \widetilde{J}^r(M, N) \to J^1(M, N)$, whose restrictions to $\widetilde{L}^r_{m,m}$ are

$$p_s(Y) = (y^i_{0 \dots j_s \dots 0})\,, \qquad j_s = 1, \dots, m\,.$$

Moreover, there are $\binom{r}{2}$ geometric projections $q_{st} : \widetilde{J}^r(M, N) \to \widetilde{J}^2(M, N)$, $s < t$, whose restrictions to $\widetilde{L}^r_{m,m}$ are

$$q_{st}(Y) = (y^i_{0 \dots j_s \dots 0}, y^i_{0 \dots j_t \dots 0}, y^i_{0 \dots j_s \dots j_t \dots 0})\,.$$

Lemma 1. *The invariant elements of* $\widetilde{L}^r_{m,m}$ *are of the form*

$$y^i_{j\dots 0} = c_1\delta^i_j, \dots, y^i_{0\dots j} = c_r\delta^i_j\,, \qquad c_1,\dots,c_r \in \mathbb{R} \tag{9}$$

satisfying

$$c_s c_t = c_s\,, \qquad s < t\,, \tag{10}$$

and with all zero coordinates in every order $r \geq 2$.

Proof. The invariancy condition for $Y \in \widetilde{L}^r_{m,m}$ reads

$$Y = X^{-1} \circ Y \circ X \qquad \text{for all} \quad X \in G^r_m\,.$$

Analogously to the proof of Proposition 2, if we consider the image of $GL(m,\mathbb{R})$ into G^r_m, then all coordinates of the order ≥ 2 vanish by the invariant tensor theorem and (9) holds. If we consider the invariancy of the underlying non-holonomic 2-jet $q_{st}(Y)$, we find the condition $-c_s c_t b^i_{jk} + c_s b^i_{jk} = 0$, $s < t$. This is equivalent to (10). □

For $r = 2$, Lemma 1 yields the following assertion (in which $\widetilde{J}^2_h$ has the form $J^1_h \circ J^1_{h,1}$).

Proposition 4. *All extensions of* $\widetilde{J}^2$ *are*

$$\widetilde{J}^2_v\,, \quad J^1_{h,c} \circ J^1_v\,, \quad J^1_h \circ J^1_{h,c}\,, \qquad 0 \neq c \in \mathbb{R}\,. \tag{11}$$

Proof. The condition $c_1c_2 = c_1$ is fulfiled for $c_1 \neq 0$, $c_2 = 1$ or $c_1 = 0$, c_2 arbitrary. The type of the extension is determined by the diagonal in question. In this way we find that for $c_1 = 0$, $c_2 = 0$ we obtain $\widetilde{J}^2_v$. The case $c_1 = 0$, $c_2 \neq 0$ corresponds to $J^1_{h,c_2} \circ J^1_v$. For $c_1 \neq 0$, $c_2 = 1$ we obtain $J^1_h \circ J^1_{h,c_1}$. □

For arbitrary r, we deduce a recurrence formula.

Proposition 5. *All extensions of* $\widetilde{J}^r$ *are*

$$I \circ J^1_v\,, \ \widetilde{J}^{r-1}_h \circ J^1_{h,c}\,, \qquad 0 \neq c \in \mathbb{R}\,, \tag{12}$$

where I *is any extension of* $\widetilde{J}^{r-1}$.

Proof. We have to discuss all conditions (10). If $c_1 \neq 0$, then $c_2 = \dots = c_r = 1$. Such a diagonal corresponds to $\widetilde{J}^{r-1}_h \circ J^1_{h,c}$. If $c_1 = 0$, then $c_2,\dots,c_r$ satisfy the conditions from the order $r-1$. This yields $I \circ J^1_v$. □

5. One-semiholonomic jets. We introduce an appropriate generalization of the concept of semiholonomic r-jet, [1].

Definition 4. An r-th order jet functor F on $\mathcal{M}f_m \times \mathcal{M}f$, $r \geq 2$, is called one-semiholonomic, if

$$p_1(X) = \cdots = p_r(X) \in J^1(M, N)$$

for every $X \in F(M, N)$.

In particular, the classical functor $\overline{J}^r$ of semiholonomic r-jets is one-semiholonomic. In the second order, $p_1(X) = p_2(X)$ is the standard semiholonomicity condition.

Proposition 6. *The only two extensions of a one-semiholonomic jet functor F are F_h and F_v.*

Proof. By one-semiholonomicity, we have $c_1 = \cdots = c_r$ in (10). Hence 0 and 1 are the only two possibilities. □

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