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DEFORMATIONS OF HYPERSURFACES IN $\boldsymbol{R}^4$

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Abstract. This short expository note deals with local rigidity problems for hypersurfaces of type number two in R^4. The main theorem proves the existence of hypersurfaces with exactly one nontrivial isometric deformation. Such hypersurfaces were considered by E. Cartan in 1916, but their actual existence has not been proved until recently.

Let us recall definition of the type number.

Definition. Let M be a hypersurface embedded into R^{n+1}. The type number at a point $p \in M$ is the rank of the shape operator S in T_pM; it is denoted $t(p)$.

R. Beez and W. Killing proved independently the following:

Theorem. *Let $M \subset R^{n+1}$ be a hypersurface whose type number is greater than or equal to three everywhere. Then M is locally rigid.*

Moreover it is known: If $t(p) \leq 1$ everywhere, then the induced metric on M is locally flat. In the complete case, if $t(p) = 0$ everywhere, then M is a hyperplane, and if $t(p) = 1$ everywhere, then M is a generalized cylinder. Further we have ([3]):

Theorem. *Let $(M, g) \subset R^{n+1}$ be a hypersurface with the induced Riemannian metric. Then for any $p \in M$,*

1. *$t(p) = 0$ or 1 if and only if $R_p = 0$;*
2. *if $t(p) \geq 2$ then $t(p) = n - dim\, T^*_p$, where*
   $T^*_p = \{X \in T_pM;\ R_p(X, Y) = 0\ \forall\ Y \in T_pM\}$.

(R denotes the Riemannian curvature tensor.)

Especially if $t(p) = 2$ everywhere, then (M, g) is a Riemannian manifold of conullity 2. This means that every tangent space T_pM can be decomposed in the

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form $T_pM = V_p^{(0)} + V_p^{(1)}$, where $\dim V_p^{(0)} = n-2$, $\dim V_p^{(1)} = 2$ for all $p \in M$ and $V_p^{(0)} = \{X \in T_pM;\ R_p(X,Y) = 0\ \forall\ Y \in T_pM\}$ is the nullity space of the curvature tensor at p.

The most interesting and difficult case is that when $n \geq 3$ and $t(p) = 2$ everywhere. We will speak shortly about a hypersurface of type number two. The local situation was analyzed by E. Cartan in 1916. He proved that just the following three cases can occur:

a) M is locally continuously deformable;
b) M is locally rigid;
c) M is locally deformable in a unique way:
to every sufficiently small domain $\Sigma \subset M$ there is exactly one hypersurface $\overline{\Sigma} \subset R^{n+1}$ which is isometric to Σ and which can not be obtained from Σ by rigid motions or reflections.

In the recent time much attention has been paid to the global study of hypersurfaces of type number two. As concerns the local theory, many explicit examples were known for the Cartan's type a), but no example for the types b) and c).

We take into consideration foliated semi-symmetric spaces to show the existence of a family of hypersurfaces in R^4 for the most interesting Cartan's type c).

Definition. A semi-symmetric space is a Riemannian manifold (M,g) satisfying the identity

$$R(X,Y) \cdot R = 0 \text{ for all } X,\, Y \in T_pM,\ \ p \in M.$$

It is known, that every Riemannian manifold of conullity two is a semi-symmetric space called a foliated semi-symmetric space. A general existence theorem for foliated semi-symmetric spaces was proved first by Z. Szabó ([5]).

Now we always suppose $n = \dim M = 3$, (M,g) is a foliated semi-symmetric space. In this case the nullity spaces $V_p^{(0)}$ are 1-dimensional and their curves are geodesics, so called principal geodesics.

In 1991 O. Kowalski ([4]) gave explicit formulas for these spaces in dimension 3 and made a classification of 3-dimensional foliated semi-symmetric spaces according to the number of so called asymptotic foliations. Each asymptotic foliation consists of "asymptotic leaves".

Definition. An asymptotic leaf is a surface $N \subset M$ with the following properties:

1. N is generated by a system of principal geodesics;
2. the tangent planes of N are parallel along each principal geodesic with respect to the Riemannian connection of (M,g).

For 3-dimensional foliated semi-symmetric spaces the following cases occur:

elliptic case (there is no real asymptotic foliation);
hyperbolic case (there are two different asymptotic foliations)
- *orthogonal* subcase (asymptotic foliations are everywhere orthogonal to each other)
- *non-orthogonal* subcase (asymptotic foliations are everywhere non-orthogonal)

parabolic case (there is one asymptotic foliation);
planar case (there are infinitely many asymptotic foliations).

Let us recall that, in the non-elliptic cases, we can obtain explicit classifications using a special system of local coordinates $(w,\, x,\, y)$ which are adapted to an asymptotic foliation (see [4]).

A space of hyperbolic or parabolic type is said to be generic if the asymptotic foliations are never totally geodesic.

In particular, we have the following ([2]):

Theorem. *(M, g) is a generic orthogonally hyperbolic semi-symmetric space if and only if the metric is given locally by an orthogonal coframe of the form*

$$\omega^1 = (ty+u)dw,$$
$$\omega^2 = (py+q)dx,$$
$$\omega^3 = dy,$$

where p and q are arbitrary functions of two variables w and x satisfying

$$\left(\frac{p}{q}\right)'_w \neq 0 \quad \textit{and} \quad \left(\frac{p'_w}{q'_w}\right)'_x \neq 0,$$

t and u are calculated from p and q by

$$u = \frac{\Phi q'_w}{pq'_w - qp'_w}, \quad t = \frac{\Phi p'_w}{pq'_w - qp'_w},$$

where

$$|\Phi| = \exp\left(\int \frac{q'_w p'_x - p'_w q'_x}{pq'_w - qp'_w} dx\right).$$

Now we ask a question when this kind of space can be locally isometrically embedded as a hypesurface into R^4. Let us recall the fundamental theorem on hypersurfaces ([3]):

Theorem. *Let (M, g) be an n-dimensional Riemannian manifold and S a tensor field of type (1,1) on M, which defines a symmetric transformation of each tangent space T_pM and which satisfies the Gauss and Codazzi equations. Let p_0 be a point of M and let $\{X_1, \dots, X_n\}$ be an orthonormal basis of $T_{p_0}M$. Also, let y_0 be a point of R^{n+1} and let $\{e_1, \dots, e_n, u\}$ be an orthonormal frame at y_0. Then there exists an isometric embedding f of a neighbourhood U of p_0 into R^{n+1} such that $f(p_0) = y_0$, $f_*(X_i) = e_i$, $i = 1, \dots, n$; u is normal to $f(U)$ at y_0 and such that S is the symmetric operator corresponding to the shape operator of the embedding f. Moreover, any two such isometric embeddings coincide in a neighbourhood of p_0.*

Having in mind this theorem one should look for a tensor field S of type (1,1) on M satisfying the Gauss and Codazzi equations. If such tensor field S exists, then an embedding into R^4 is possible, namely with S as the shape operator and the corresponding hypersurface is of type number two. When we solve this problem, we find a system of partial differential equations for the components of S and we apply the Cauchy-Kowalewski theorem, so we consider the real-analytic case. We have proved the following main result (see [1], Chapter 6 and Chapter 10):

Theorem. *The generic 3-dimensional orthogonally hyperbolic spaces which can be locally realised as hypersurfaces in R^4 depend on one arbitrary function of two variables, six arbitrary functions of one variable and five real parameters. In general, these spaces can be realised as hypersurfaces in R^4 in exactly two qualitative different ways (up to an isometry of R^4).*

Hence these spaces are examples of the Cartan's type c).

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