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BASIC COHOMOLOGY FOR RIEMANNIAN FOLIATIONS

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Abstract. We present results on basic forms and basic cohomology for singular Riemannian foliations. Finally, we develop a basis for the theory of the basic intersection cohomology for these foliations.

A foliation on a manifold is a partition of this manifold into submanifolds, called leaves, which are "nicely" distributed. We distinguish regular foliations, the leaves are of the same dimension, and singular ones when we allow the dimension of leaves to vary. Several types of problems are particular to foliated manifolds. All of them are based on the interplay between the properties of the partition and the ambient manifold. A very important role is played by the space of leaves of a foliation. It can have a very sofisticated topology. Its topological properties can tell us a lot about the way in which the leaves are distributed in the ambient manifold. It is not surprising that at a very early stage of the development of the theory of foliation B. Reinhardt, cf. [17], introduced a cohomology theory for the space of leaves using special differential forms, called basic, defined on the ambient manifold. The cohomology of the complex of basic differential forms is called the basic cohomology of the foliated manifold. For a foliation defined by a submersion with connected fibres, basic forms correspond to differential forms on the base and the basic cohomology is isomorphic to the cohomology of the base. In general, for regular foliations the basic cohomology can be infinite dimensional. However, as a nice cohomology theory should be, it is zero for degrees greater than the codimension of the foliation. Moreover, for regular Riemannian foliations of a compact manifold the basic cohomology is finite dimensional. It was conjectured that for such foliations the basic cohomology satisfies the Poincare duality. In fact, it is not true as a Carriere's example well illustrates, cf. [4, 15]; the top dimensional basic cohomology group of his foliation is zero. Within several years the problem was solved and the geometric meaning of this condition is well understood, see Section 2 of our note. These results concerning the properties of the basic cohomology of

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regular Riemannian foliations are the starting point of our study of cohomological properties of singular Riemannian foliations.

The closures of leaves form a singular foliations for which a geodesic orthogonal to it at one point remains orthogonal to it at any point of the domain. Using this property P. Molino demonstrated that the sets of points of leaves whose closures are of the same dimension form a stratification of the manifold. This remark introduces elements of the singularity theory into Riemannian foliations.

The main aim of our research is to study singular Riemannian foliations using cohomological tools. First, we generalize well-known theorems on basic cohomology of regular Riemannian foliations to the singular case. Secondly, using the natural stratification of the manifold we introduce the basic intersection cohomology and study its properties. In this note we report on the progress of our research. The details will be presented in the forthcoming papers.

1. Regular Riemannian foliations

In this section, for reader's convenience, we gather some basic definitions and properties of regular Riemannian foliations.

Let $\mathcal{F}$ be a foliation on a compact smooth manifold M. Let g be a Riemannian metric on M.

Then the following conditions are equivalent:

i) The Riemannian metric is bundle–like for the foliation $\mathcal{F}$, i.e. for any adapted chart (U, φ), $\varphi = (x_1, ...x_p, y_1, ...y_q)$

$$g|U = \Sigma_{i,j=1}^{p} g_{ij}(x,y) dx_i \otimes dx_j + \Sigma_{\alpha,\beta=1}^{q} g_{\alpha,\beta}(y) v_\alpha \otimes v_\beta$$

ii) any geodesic of g orthogonal to the foliation $\mathcal{F}$ at one point remains orthogonal to the foliation at any point of its domain.

Examples

1. Any smooth action of a Lie group by isometries of a Riemannian manifold defines a Riemannian foliation provided that the orbits are of the same dimension.
2. A suspension of a finitely generated group of isometries of a compact manifold.

On the total space of the bundle of transverse orthonormal frames $B(M, O(q); \mathcal{F})$ there exists a TP foliation $\mathcal{F}_1$ of the same dimension as $\mathcal{F}$ whose leaves are holonomy coverings of leaves of $\mathcal{F}$. On the manifold M there is a locally constant sheaf of germs of vector fields whose "transverse" orbits are the closures of leaves of $\mathcal{F}$. Therefore the closures of leaves are submanifolds of M and form a partition.

2. Basic forms

Our main tool in the study of Riemannian foliations is the complex of basic forms and its cohomology. We start with classical basic differential forms.

A differential form ω is called basic for the foliation $\mathcal{F}$ if for any vector X tangent to the foliation $i_X\omega = i_X d\omega = 0$. In an adapted chart (U, φ), $\varphi = (x_1, ...x_p, y_1, ...y_q)$ a basic k–form can be written as follows:

$$\omega = \Sigma_{i_1 < i_2, \dots < i_k} \omega_{i_1 i_2, \dots i_k}(y) dy_{i_1} \wedge dy_{i_2} \wedge \dots dy_{i_k}.$$

Basic forms of a foliated manifold $(M, \mathcal{F})$ form a differential complex $A(M, \mathcal{F})$ whose cohomology we denote $H(M, \mathcal{F})$ and call the basic cohomology of the foliated manifold $(M, \mathcal{F})$.

Examples

1. The fibres of a fibration with connected fibre: $F \to M \to B$ form a foliation of M. Basic forms for this foliation are in one–to–one correspondence with differential forms on the base space B.

2. In the case of an action of a connected (compact) Lie group basic forms are precisely invariant horizontal forms.

Let the foliation $\mathcal{F}$ be given by a cocycle $\mathcal{U} = \{U_i, f_i, g_{ij}\}$ modelled on a manifold N_0, i.e.

i) $\{U_i\}$ is an open covering of M,

ii) $f_i : U_i \longrightarrow N_0$ are submersions with connected fibres defining $\mathcal{F}$,

iii) g_{ij} are local diffeomorphisms of N_0 and $g_{ij} \circ f_j = f_i$ on $U_i \cap U_j$.

The manifold $N = \coprod f_i(U_i)$ we call the transverse manifold of $\mathcal{F}$ associated to the cocycle $\mathcal{U}$ and the pseudogroup H generated by g_{ij} the holonomy pseudogroup (representative) on the transverse manifold N.

Basic forms on $(M, \mathcal{F})$ are in one–to–one correspondence with H–invariant forms on the transverse manifold N.

The basic cohomology of a nice foliation can be infinite dimensional as an example due to E. Ghys well illustrates, cf. [9].

Let $T^2 = R^2/Z^2$ be the 2-dimensional torus. We suspend the diffeomorphim $\phi : T^2 \to T^2$ $\phi(x, y) = (x, x + y)$ and get a 3–manifold T^3_ϕ with 1-dimensional foliation $\mathcal{F}_\phi$. Basic forms correspond to ϕ–invariant forms on T^2. Then the following lemma shows that the basic cohomology of this foliation is infinite dimensional.

Lemma 1. i) Let ω be a 1–form on T^2 which is ϕ–invariant. Then ω is a closed form and $\omega = f(x)dx$,

ii) ϕ–invariant 2–forms on T^2 are exact and of the form: $g(x)dx \wedge dy$.

We recall main properties of the basic cohomology of a Riemannian foliation, cf. [6, 7, 8, 11, 12].

Theorem 1. (El Kacimi, Hector, Sergiescu) Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. Then the basic cohomology $H^*(M, \mathcal{F})$ is a finite dimensional vector space over R.

Theorem 2. (El Kacimi, Hector, Kamber, Tondeur) Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. Then the top dimensional space $H^q(M,\mathcal{F})$ is isomorphic to 0 or R. In the second case the Poincaré duality holds in the basic cohomology ring.

Theorem 3. (El Kacimi, Nicolau) The basic cohomology of a Riemannian foliation of a compact manifold is a topological invariant.

Theorem 4. (Masa) Let $\mathcal{F}$ be a Riemannian foliation on a compact manifold M. Then the top dimensional space $H^q(M,\mathcal{F})$ is isomorphic to R, then the foliation $\mathcal{F}$ is minimalisable.

3. Singular Riemannian foliations

Studying (regular) Riemannian foliations Pierre Molino noticed that the closures of leaves of a Riemannian foliation form a singular foliation with one additional property:

Any geodesic of the bundle–like metric orthogonal to the closures at one point is orthogonal to the closures at any point of its domain.

This remark led Pierre Molino to propose the following definition. Let M be a manifold with a Riemannian metric g and a singular foliation $\mathcal{F}$.

The metric g is adapted to $\mathcal{F}$ if any geodesic of g orthogonal to the foliation at one point is orthogonal to the foliation at any point of its domain.

Definition 1. A singular foliation $\mathcal{F}$ of a manifold M is called Riemannian (SRF) if it admits an adapted Riemannian metric.

Let L be a leaf of the foliation and $N(L)$ its normal bundle. The adapted Riemannian metric induces a Riemannian metric in the bundle $N(L)$. Local vector fields tangent to the foliation act on the bundle $N(L)$. This action preserves the induced Riemannian metric. This property characterizes adapted metrics.

Compact Lie group actions define a class of SRF which is best known. In this case we have a very nice description of basic forms which is due to P. Michor, cf. [13, 14].

Theorem 5. Let $M \times G \to M$ be a proper isometric action of a Lie group G on a smooth Riemannian manifold M, which admits a section Σ. Then the restriction of differential forms induces an isomorphism

$$A^p_{hor}(M)^G \to A^p(\Sigma)^{W(\Sigma)}$$

between the space of horizontal G–invariant differential forms on M and the space of all differential forms on Σ which are invariant under the action of the generalized Weyl group $W(\Sigma)$ of the section Σ.

In the paper [19] we have proved the following two theorems for singular foliations.

Theorem 6. Let $\mathcal{F}$ be a singular Riemannian foliation on a compact manifold M. Then the basic cohomology $H^*(M/\mathcal{F})$ is a finite dimensional vector space over R.

Theorem 7. The basic cohomology of a singular Riemannian foliation of a compact manifold is a topological invariant.

We have been able to generalized Michor's result to the class of transversely integrable singular Riemannian foliations. The main reference for transversely integrable singular Riemannian foliations is H. Boualem's paper, cf. [2]. We are going to recall some results of his.

Theorem 8. (Boualem) Let $\mathcal{F}$ be a transversely integrable SRF with leaves of compact closure. Through any point of M passes an immersed connected submanifold which meets orthogonally all leaves of $\mathcal{F}$.

Boualem takes a leaf of the horizontal distribution (on the regular part). Owing to a result of Blumenthal–Hebda, cf. [1], this leaf meets any "regular" leaf of $\mathcal{F}$ orthogonally. The idea is to extend this leaf over the singular set of the foliation $\mathcal{F}$. To achieve this Boualem studies a tubular neighbourhood of a singular leaf. It turns out that any connected component of the trace of such a leaf is contained in a fibre of a suitably chosen tubular neighbourhood and moreover, it is a part of the trace of a linear subspace of the fibre. Therefore by taking the whole trace of this linear subspace he was able to extend the leaf. However, it could happen that the extension would intersect itself (it might consist of the trace of several linear subspaces). Therefore the extension of a leaf (of the horizontal distribution) is only an immersed submanifold and it can contain several leaves of the horizontal distribution.

In the regular part any $\mathcal{F}$–leaf curve defines a local isometry between the leaf of $\mathcal{F}^\perp$ passing through the starting point of the curve and the leaf passing through the end point of the curve, cf. [1]. Moreover, Boualem proved that any such a local isometry in a suitable small (tubular) neighbouhood of a singular leaf can be extended to a local isometry of generalized sections, cf. Prop.2.1.1 of [2]. The result is formulated for a very special singular leaf (a point), but obviously it is true for any leaf.

Another important result of Boualem's paper is the following theorem about a singular Riemannian foliation $\mathcal{F}$ in a ball B in R^q.

Theorem 9. If S is a section of $\mathcal{F}$ in B, then there exists a group W(S) of isometries of S having the following properties:

(1) for any $x \in SW(S)x = L_x \cap S$;
(2) if $\bar{W}(S)$ is the closure of $W(S)$ in the group of all isometries of S, then for any $x \in S$ $\bar{W}(S)x = \bar{L}_x \cap S$.
(3) the component $\bar{W}(S)_0$ of the 1 in $\bar{W}(S)$ is a torus.

Our main result on basic forms for transversely integrable SRF is contained in the following theorem cf. [20].

Theorem 10. Let $\mathcal{F}$ be a transversely integrable SRF with compact leaves on a compact Riemannian manifold M. Let S be a generalized section with the Weyl pseudogroup $\mathcal{W}(S)$. Then the restriction mapping defines an isomorphism

$$\Omega(M;\mathcal{F}) \to \Omega(S)^{\mathcal{W}(S)}$$

where $\Omega(M;\mathcal{F})$ is the algebra of basic forms of the foliated manifold $(M,\mathcal{F})$ and $\Omega(S)^{\mathcal{W}(S)}$ is the algebra of $\mathcal{W}(S)$–invariant forms on the generalized section S.

In [16] P. Molino described in detail a new class of SRF called orbit–like foliations. The main characteristic feature of this foliations is the fact that any point x has an adapted neighbourhood in which the foliation is the product of an open neighbourhood of x in the leaf L_x passing through this point and the foliation defined by an action of a group of isometries on a suitable vector space (the fibre N_x of the normal bundle $N(L_x) = TL_x^\perp$ at the point x). This foliation is called the infinitesimal transverse model of $\mathcal{F}$ at x. For a suitable small tubular neighbourhood the holonomy transformation defined by a leaf curve begining and ending in N_x can be extended to a global transformation of N_x. In our case when leaves are compact the traces of leaves of $\mathcal{F}$ on a suitable small ball in N_x are the orbits of a compact group of isometries G_x, whose action generates all holonomy transformations. The restriction of the action of G_x to a section generates its Weyl (pseudo)group.

Proof of Theorem 10

A) The restriction to the chosen section is injective

The foliation $\mathcal{F}$ is regular on an open and dense subset M_0 of M, cf. [15]. The orthogonal distribution $\mathcal{F}^\perp$ on M_0 is integrable and its leaves are connected components of the trace of generalized sections on M_0. Any such a leaf intersects all leaves of $\mathcal{F}$ in M_0. Thus any two basic forms whose rectrictions to a leaf of $\mathcal{F}^\perp$ (in M_0) are equal are themselves equal on M_0. By continuity they are equal on the whole manifold M.

B) The restriction to the chosen section is surjective.

First let us look at the local problem. Let us choose a point x and let L_x be the leaf passing through the point x. Let $N(L_x)$ be a suitably chosen small tubular neighbourhood of L_x. Any (generalized) section passing through the point x is of the form $exp_x(W)$ where W is a vector subspace of $T_x\mathcal{F}^\perp$ and the trace of the foliation $\mathcal{F}$ on $exp_x(T_x\mathcal{F}^\perp) = N_x$ is given by an action of a Lie group G_x.

Basic forms on $N(L)$ are in one–to–one correspondence with G_x–invariant horizontal forms on N_x. For any section S_x of the foliation $\mathcal{F}$ passing through x in $N(L)$, is a section of the action of the group G_x. Therefore, according to Michor, basic forms on $N(L)$ are in one-to-one correspondence with $W(S_x)$–invariant forms on S_x.

The main difficulty in passing from the local case to the global one is the fact that a generalized section can pass many times through a given neighbourhood.

Let S be our generalized section. The trace $M_0 \cap S$ of S on the regular part M_0 consists of leaves $\{S_\alpha\}$, $\alpha \in A$, of the foliation $\mathcal{F}^\perp$. The Weyl pseudogroup $\mathcal{W}_\alpha$ of any S_α consists of local isometries and it is the holonomy psudogroup $\mathcal{H}_\alpha$ of the complete transverse manifold S_α (of $(M_0, \mathcal{F})$. Therefore any $\mathcal{W}_\alpha$–invariant form ω can be extended to a basic form ω_0 on M_0. Moreover, the restriction of this form to $\bigcup S_\alpha = M_0 \cap S$ must be $\mathcal{W}(S)$– invariant – directly from the fact that the form ω_0 is basic and that the pseudogroup $\mathcal{W}(S)|M_0$ is generated by holonomy mapping along leaf curves.

It remains to extend the form ω_0 over the singular set Σ Let L be a leaf in Σ and let us choose a component S_L of the trace of the generalized section S in a sufficiently small tubular neighbourhood $N(L)$ of L. Denote by $\mathcal{W}(L)$ the restriction of the Weyl pseudogroup to S_L. The form $\omega|S_L$ is $\mathcal{W}(L)$–invariant. So it can be extended to a basic form ω_L on $N(L)$. On $N(L) \cap M_0$ the basic forms ω_0 and ω_L coincide, so $\omega_0 \cup \omega_L$ is a smooth basic extension of the form ω. After a finite number of such steps we obtain a global basic form $\tilde{\omega}$ extending the $\mathcal{W}(S)$–invariant form ω. We have just proved the surjectivity of our restriction mapping and thus have completed the proof of the theorem. □

4. Basic intersection cohomology

The space of orbits of an action of a compact group is a stratified space, cf. [5]. So the study of its intersection cohomology provides us with some useful information about its topology and the structure of the stratification. As in the case of differential forms, intersection forms on the space of orbits correspond to basic intersection forms on the manifold, cf. [18]. We have decided to begin the study of basic intersection forms (BIF) for SRF. Before concentrating on the BIF we have to study properties of the induced stratification in more detail.

Filtrating a singular foliated manifold $(M, \mathcal{F})$ by the dimension of leaves one finds a stratification. When this stratification is conical enough, we are able to define the basic intersection cohomology. We introduce the notion of a conical foliation whose main example are SRF.

Let $\mathcal{F}$ be a singular foliation of a manifold M. Classifying the points of M following the dimension of leaves passing through them one gets a stratification $\mathcal{S}_\mathcal{F}$ of M. The foliation $\mathcal{F}$ is regular when this stratification has just one stratum $\{M\}$.

Examples

1. In any open subset $U \subset M$ we have the singular foliation $\mathcal{F}_U = \{$connected components of $L \cap U : L \in \mathcal{F}\}$. The associated stratification is $\mathcal{S}_{\mathcal{F}_U} = \{$connected *componnets of* $S \cap U : S \in \mathcal{S}_\mathcal{F}\}$.

2. Consider $(N, \mathcal{H})$ a regular foliated manifold. In the product $N \times M$ we have the singular foliation $\mathcal{H} \times \mathcal{F} = \{L_1 \times L_2 : L_1 \in \mathcal{H},\ L_2 \in \mathcal{F}\}$. The associated stratification is $\mathcal{S}_{\mathcal{H}\times\mathcal{F}} = \{N \times S : S \in \mathcal{S}_\mathcal{F}\}$.

3. Consider the sphere S^n of dimension n endowed with a singular foliation $\mathcal{G}$ without 0-dimensional leaves. Identify the disk D^{k+1} with the cone $cS^n = S^n \times [0, 1[/S^n \times \{0\}$ by $x \mapsto [x/|x|, |x|]$ where $[u, t]$ is a generic element of the cone. We shall consider on D^{n+1} the singular foliation

$$c\mathcal{G} = \{F \times \{t\} : F \in \mathcal{G}, t \in]0,1[\} \cup \{\theta\},$$

where θ is the vertex $[u,0]$ of the cone. The induced stratification is

$$\mathcal{S}_{c\mathcal{G}} = \{S \times \{t\} : S \in S_{c\mathcal{G}}, t \in]0,1[\} \cup \{\theta\}.$$

For technical reasons we allow n to take the value -1; in this case $S^n =$ and $cS^n = \{\theta\}$.

The stratification $\mathcal{S}_{\mathcal{F}}$ can be very wild, too much to support an intersection cohomology structure. For that purpose a certain amount of conicalicity is needed. (see [10] for the case of stratified pseudomanifolds). This leads us to introduce the following definition.

Definition 2. A singular foliation $(M, \mathcal{F})$ is said to be a conical foliation if for any point $x \in M$ we can find

i) a neighbourhood U of x,

ii) a regular foliation $(R^{m-n-1}, \mathcal{H})$,

iii) a conical foliation $(S^n, \mathcal{G})$ without 0-dimensional leaves and

iv) a foliated diffeomorphism $\varphi\colon (R^{m-n-1} \times cS^n, \mathcal{H} \times c\mathcal{G}) \to (U, \mathcal{F}_U)$ sending $(0,\theta)$ on x.

This definition is made by induction on the dimension of M.

Examples

1. Any regular foliation is a conical foliation.

2. Using the description of a foliated neighbourhood of a closed submanifold foliated by a regular foliation, cf. [3], it is not difficult to demonstrate that a SRF is a conical foliation.

For conical foliations, and for SRF in particular we can define basic intersection cohomology. The precise definition is a little too long to present in this short note. We conjecture that we can prove the Poincare duality theorem under some natural assumptions on the induced foliations on the sphere bundles over the closures of singular strata.

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