DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 149–159

THE MODULI SPACE OF LOCALLY HOMOGENEOUS SPACES AND LOCALLY HOMOGENEOUS SPACES WHICH ARE NOT LOCALLY ISOMETRIC TO GLOBALLY HOMOGENEOUS SPACES

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Abstract. Contrary to a natural expectation, there exist locally homogeneous spaces which are not locally isometric to any (globally) homogeneous spaces. Such examples were found by Kowalski. In this paper we try to understand these mysterious examples well. We denote by $\mathcal{LH}(n)$ the set of local isometry classes of n-dimensional locally homogeneous spaces. We can introduce the topology on $\mathcal{LH}(n)$ by imbedding it in the space of abstract curvature tensors and covariant derivatives. We show that the set of local isometry classes of n-dimensional locally homogeneous spaces which are locally isometric to homogeneous spaces are dense in $\mathcal{LH}(n)$.

1. Introduction

A Riemannian manifold M is said to be *(globally) homogeneous* if for any two points $p, q \in M$ there exists an isometry of M which maps p to q. On the other hand, it is called to be *locally homogeneous* if for any two points $p, q \in M$ there exist a neighbourhood U of p and a neighbourhood V of q and an isometry of U onto V which maps p to q. It is natural to expect that for a locally homogeneous space M, there exists a homogeneous space $\tilde{M}$ to which M is locally isometric. It is well-known that if we change a locally homogeneous space to a locally symmetric space, the statement holds. Contrary to this, O. Kowalski ([5]) found the examples of locally homogeneous spaces which are not locally isometric to any homogeneous spaces. This phenomenon is interesting and mysterious. If collecting pieces of a locally homogeneous space and attaching them, we can extend it to a complete one, then its universal covering space is homogeneous owing to a theorem of I.M.Singer (Theorem 2.1'). But the examples found by Kowalski cannot be extended. Since Kowalski found these examples, several authors have tried to clarify this phenomenon (cf. Kowalski [6], A. Spiro [11], [12], and F. Tricerri [13] etc).

1991 *Mathematics Subject Classification.* 53C30, 53B20.
Key words and phrases. Locally homogeneous spaces, infinitesimally homogeneous spaces.
This paper is in final form and no version of it will be submitted for publication elsewhere.

In this paper we will investigate the structure of the pair of Lie algebras associated with a locally homogeneous space which is not locally isometric to any homogeneous space (section 4). In section 3, slightly rewriting the theory by Singer [10] and Nicolodi-Tricerri [9], we will imbed the set $\mathcal{LH}(n)$ of local isometry classes of n-dimensional locally homogeneous spaces in the space of abstract curvature tensors and covariant derivatives. Thus we can introduce the topology on $\mathcal{LH}(n)$. Our main result is the following.

Theorem 4.1. *The set of local isometry classes of n-dimensional locally homogeneous spaces which are locally isometric to homogeneous spaces are dense in $\mathcal{LH}(n)$.*

Moreover, we will show the curvature properties of a locally homogeneous space which is not locally isometric to any homogeneous space (Proposition 4.4) and show that a five-dimensional locally homogeneous space which is not locally isometric to any homogeneous space is locally isometric to one of the examples found by Kowalski in [5] and [6] (Proposition 4.7).

Finally the author would like to express his thanks to the referee for his useful comments.

2. Preliminaries

At first we recall the theory of infinitesimally homogeneous spaces by I. M. Singer [10]. It gives a sufficient condition of a Riemannian manifold to be homogeneous or locally homogeneous.

Given a Riemmanian manifold M, we consider the following condition:

$P(l)$: for every $p, q \in M$ there exists a linear isometry $\phi : T_pM \to T_qM$ such that

$$\phi^*(\nabla^i R)_q = (\nabla^i R)_p \quad \text{for } i = 0, 1, ..., l.$$

If M is locally homogeneous, then M obviously satisfies $P(l)$ for any l. It is enough to set $\phi =$ the differential of a local isometry which maps p to q.

We denote by $\mathfrak{so}(T_pM)$ the Lie algebra of the skew-symmetric endomorphisms of T_pM. For a non-negative integer l, we define a Lie subalgebra $\mathfrak{g}_l(p)$ of $\mathfrak{so}(T_pM)$ by

$$\mathfrak{g}_l(p) = \{A \in \mathfrak{so}(T_pM) \mid A \cdot (\nabla^i R)_p = 0, \quad i = 0, 1, ..., l \},$$

where A acts as a derivation on the tensor algebra on T_pM. Since $\mathfrak{g}_l(p) \supseteq \mathfrak{g}_{l+1}(p)$, there exists a first integer $k(p)$ such that $\mathfrak{g}_{k(p)}(p) = \mathfrak{g}_{k(p)+1}(p)$. Namely, we have

$$\mathfrak{so}(T_pM) \supseteq \mathfrak{g}_0(p) \supsetneq \mathfrak{g}_1(p) \supsetneq \mathfrak{g}_2(p) \supsetneq \cdots \supsetneq \mathfrak{g}_{k(p)}(p) = \mathfrak{g}_{k(p)+1}(p).$$

Following Singer, we say that M is *infinitesimally homogeneous* if M satisfies $P(k(p)+1)$ for some point $p \in M$. If M satisfies $P(l)$, then the linear isometry ϕ induces a Lie algebra isomorphism of $\mathfrak{g}_i(p)$ to $\mathfrak{g}_i(q)$ for $i = 0, 1, ..., l$. Therefore if M is infinitesimally homogeneous, $k(q)$ does not depend on $q \in M$. We put $k_M = k(p)$ and call it *the Singer invariant* of an infinitesimally homogeneous space M. If M is locally homogeneous, then M obviously satisfies $P(l)$ for any l and in particular M is infinitesimally homogeneous. Singer proved the converse.

Theorem 2.1. *A connected infinitesimally homogeneous space is locally homogeneous.*

The global version of the theorem above is the following:

Theorem 2.1'. *A connected, simply connected, complete infinitesimally homogeneous space is homogeneous.*

Singer's original statement is Theorem 2.1'. However by his proof it immediately follows that Theorem 2.1 holds. L. Nicolodi and F. Tricerri reproved Theorem 2.1 by a more direct approach ([9] Theorem 2.1). Moreover, the proof of Theorem 2.1 implies the following:

Theorem 2.2. *Let M and M' be two locally homogeneous spaces and $p \in M$ and $p' \in M'$ be their points. Suppose that there exists a linear isometry $\phi : T_pM \to T_{p'}M'$ such that*

$$\phi^*(\nabla^i R')_{p'} = (\nabla^i R)_p \quad \textit{for } i = 0, 1, \ldots, k_M + 1\,.$$

Then there exists a local isometry φ of a neighborhood of p onto a neighborhood of p' which satisfies $\varphi(p) = p'$ and $\varphi_{*p} = \phi$.

For a detailed argument, see [9] Theorem 2.5.

Secondly we will show a method of the construction of locally homogeneous spaces. We consider a Lie algebra $\mathfrak{k}$ and its Lie subalgebra $\mathfrak{h}$. $\mathfrak{m}$ is a complement of $\mathfrak{h}$ in $\mathfrak{k}$ which is invariant by the adjoint representation ad restricted to $\mathfrak{h}$. Namely we have

$$\begin{aligned} &\mathfrak{k} = \mathfrak{h} + \mathfrak{m} \quad \text{(a direct sum as a vector space)},\\ &ad\mathfrak{h}(\mathfrak{m}) \subset \mathfrak{m}\,. \end{aligned}$$

We denote by $ad_{\mathfrak{m}}\mathfrak{h}$ the representation of $\mathfrak{h}$ on $\mathfrak{m}$. $\langle , \rangle$ is an inner product on $\mathfrak{m}$ which is invariant by $ad_{\mathfrak{m}}\mathfrak{h}$. That is,

$$\langle ad_{\mathfrak{m}}X(Y), Z\rangle + \langle Y, ad_{\mathfrak{m}}X(Z)\rangle = 0 \quad \text{for } X \in \mathfrak{h}, Y, Z \in \mathfrak{m}\,.$$

We call $(\mathfrak{k}, \mathfrak{h}, \mathfrak{m}, \langle , \rangle)$ in the above an *infinitesimal homogeneous pair*. In addition, if an ideal of $\mathfrak{k}$ which is contained in $\mathfrak{h}$ is $\{0\}$, we say that an infinitesimal homogeneous pair is *effective*. To each effective infinitesimal homogeneous pair $(\mathfrak{k}, \mathfrak{h}, \mathfrak{m}, \langle , \rangle)$ corresponds a locally homogeneous space M (see the proof of Theorem 4.1 in F. G. Lastaria and F. Tricerri [7]). Moreover, it is uniquely determined up to local isometries. Similarly to the usual theory of a reductive homogeneous space with an invariant Riemannian metric (see S. Kobayashi and K. Nomizu [4] Chapter 10), we can express the Riemannian connection and the curvature tensor of M in the terms of Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$. We identify the tangent space T_oM at the origin $o \in M$ with $\mathfrak{m}$. For $X \in \mathfrak{k}$, we denote by $X_{\mathfrak{h}}$ and $X_{\mathfrak{m}}$ the $\mathfrak{h}$-component and the $\mathfrak{m}$-component of X, respectively. Then the Riemannian connection is given by

$$\Lambda_{\mathfrak{m}}(X)Y = \frac{1}{2}[X, Y]_{\mathfrak{m}} + U(X, Y) \quad \text{for } X, Y \in \mathfrak{m}\,,$$

where $U(X,Y)$ is the symmetric bilinear mapping of $\mathfrak{m} \times \mathfrak{m}$ into $\mathfrak{m}$ defined by

$$2\langle U(X,Y), Z\rangle = \langle [Z,X]_{\mathfrak{m}}, Y\rangle + \langle X, [Z,Y]_{\mathfrak{m}}\rangle \quad \text{for} \quad X,Y,Z \in \mathfrak{m}\,.$$

For the curvature tensor R and its i-th covariant derivative $\nabla^i R$, we have for $X, Y \in \mathfrak{m}$

$$\begin{aligned} R(X,Y) &= -ad_{\mathfrak{m}}[X,Y]_{\mathfrak{h}} + [\Lambda_{\mathfrak{m}}(X), \Lambda_{\mathfrak{m}}(Y)] - \Lambda_{\mathfrak{m}}([X,Y]_{\mathfrak{m}}), \\ i(X)\nabla^i R &= \Lambda_{\mathfrak{m}}(X) \cdot \nabla^{i-1} R\,, \end{aligned}$$

where $i(X)$ is the inner product and $\Lambda_{\mathfrak{m}}(X)$ acts as a derivation.

Let K be a connected and simply connected Lie group whose Lie algebra is $\mathfrak{k}$ and H be its connected Lie subgroup corresponding to $\mathfrak{h}$. If H is closed in K, by the usual way we obtain a reductive homogeneous space K/H equipped with a K-invariant Riemannian metric induced by the inner product $\langle , \rangle$ on $\mathfrak{m}$ and the locally homogeneous space M is locally isometric to K/H. Conversely, given a locally homogeneous space M, we can associate an effective infinitesimal homogeneous pair $(\mathfrak{k}, \mathfrak{h}, \mathfrak{m}, \langle , \rangle)$ to it by the canonical way (see A. Spiro [11], F. Tricerri [13]). We denote by K a connected and simply connected Lie group whose Lie algebra is $\mathfrak{k}$ and by H its connected Lie subgroup corresponding to $\mathfrak{h}$. Then we have

Theorem 2.3. *A locally homogeneous space M is locally isometric to a homogeneous space if and only if H is closed in K.*

For the proof, see [11] and [13].

3. The moduli space of locally homogeneous spaces

Slightly rewriting the theory by Singer [10] and Nicolodi-Tricerri [9], we will express the set of local isometry classes of n-dimensional locally homogeneous spaces using their curvature tensors and covariant derivatives.

We denote by $\mathcal{LH}(n)$ the set of local isometry classes of n-dimensional locally homogeneous spaces. We explain more precisely. We consider a pair (M,p) of an n-dimensional locally homogeneous space M and its point p. If for such pairs (M,p) and (M',p'), there exists an isometry f of a neighborhood of p onto a neighborhood of p' such that $f(p) = p'$, we put $(M,p) \sim (M',p')$ and introduce an equivalence relation $\sim$. We fix the dimension n and denote by $\mathcal{LH}(n)$ the set of all equivalence classes of such pairs. To such a pair (M,p), we add a linear isometry $u : \mathbb{R}^n \to T_pM$ (i.e., an orthonormal frame at p). For such triples, we define $(M,p,u) \sim (M',p',u')$ by the existence of a local isometry f which satisfies $f(p) = p'$ and $f_* u = u'$. We denote by $\mathcal{FLH}(n)$ the set of all equivalence classes of such triples. Corresponding (M,p,u) to (M,p), we define the natural projection $\pi : \mathcal{FLH}(n) \longrightarrow \mathcal{LH}(n)$. Moreover,the orthogonal group $O(n)$ naturally acts on the right on $\mathcal{FLH}(n)$ as follows:

$$[(M,p,u)]a = [(M,p,ua)]$$

for $[(M,p,u)] \in \mathcal{FLH}(n)$ and $a \in O(n)$. Then $\mathcal{LH}(n)$ is the orbit space of this action and π is just the corresponding projection.

We introduce the space of abstract curvature tensors and their covariant derivatives. Let $\mathbb{R}^n$ be an n-dimensional vector space with a usual inner product $\langle , \rangle$ and T^1_l be the space of tensors of type $(1,l)$ on $\mathbb{R}^n$. $O(n)$ acts on T^1_l on the left in the usual manner. We denote by $\mathcal{R}^0$ the space of curvature tensors on $\mathbb{R}^n$, i.e., tensors R of type $(1,3)$ which satisfy the following identities:

$$\begin{aligned} R(x,y)z &= -R(y,x)z, \\ \langle R(x,y)z, w\rangle &= -\langle R(x,y)w, z\rangle, \\ R(x,y)z + R(y,z)x &+ R(z,x)y = 0\,. \end{aligned}$$

Similarly we define $\mathcal{R}^1$ by the space of tensors of type $(1,4)$ which satisfy

$$\begin{gathered} R(v,\cdots) \in \mathcal{R}^0 \quad \text{for any } v \in \mathbb{R}^n\,, \\ R(x,y,z;w) + R(y,z,x;w) + R(z,x,y;w) = 0\,. \end{gathered}$$

Further, we inductively define $\mathcal{R}^i$ $(i \geq 2)$ as follows :

$$\mathcal{R}^i = \{R \in T^1_{3+i} \mid R(v,\cdots) \in \mathcal{R}^{i-1} \quad \text{for any } v \in \mathbb{R}^n\}\,.$$

Then $\mathcal{R}^i$ is an $O(n)$-invariant subspace of T^1_{3+i}.

For a sufficiently large integer d, we define the map Φ as follows:

$$\begin{aligned} \Phi :&\mathcal{FLH}(n) \longrightarrow \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^d \\ &[(M,p,u)] \longmapsto \Phi([(M,p,u)]) = (u^* R_p, u^*(\nabla R)_p, \cdots, u^*(\nabla^d R)_p)\,. \end{aligned}$$

If $(M,p,u) \sim (M',p',u')$, then we have $u'^*(\nabla^i R')_{p'} = u^*(\nabla^i R)_p$. Therefore Φ is well-defined. It is easily seen that Φ is an $O(n)$-equivariant map, which means that $\Phi([(M,p,u)]a) = a^{-1}\Phi([(M,p,u)])$. Hence we have the following diagram:

$$\begin{array}{ccc} \mathcal{FLH}(n) & \xrightarrow{\Phi} & \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^d \\ \downarrow & & \downarrow \\ \mathcal{LH}(n) & \xrightarrow[\tilde{\Phi}]{} & \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^d / O(n) \end{array}$$

How large should we take the integer d? We consider the longest decreasing sequence of Lie subalgebras $\mathfrak{g}_i$ of $\mathfrak{so}(n)$ which may occur:

$$\mathfrak{so}(n) \supseteq \mathfrak{g}_0 \supsetneq \mathfrak{g}_1 \supsetneq \cdots \supsetneq \mathfrak{g}_k\,.$$

Then we put $k(n) = k$. This number $k(n)$ is important in Singer's theory of infinitesimally homogeneous spaces. Therefore it is an important and basic problem to determine or estimate $k(n)$ for each n. M.Gromov says that $k(n) < \frac{3}{2}n - 1$ in his book ([3] p.165). We take $d \geq k(n)+1$. Then by Theorem 2.2, Φ is injective and hence $\widetilde{\Phi}$ is injective. On $\mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^d$, the usual topology is defined. Therefore we can introduce the topology on $\mathcal{LH}(n)$ through $\widetilde{\Phi}$.

We take $d \geq k(n)+2$. Then the image of Φ is expressed by algebraic conditions owing to the theorem which was announced without proof by Singer [10] and was completely proved by Nicolodi and Tricerri [9]. We prepare the notation. Given $(R^0, R^1, \ldots, R^d) \in \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^d$, we define the maps μ and ν as follows:

$$\begin{aligned}
\mu :\mathfrak{so}(n) &\longrightarrow \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^{d-1} \\
A &\longmapsto \mu(A) = (A \cdot R^0, A \cdot R^1, \cdots, A \cdot R^{d-1}), \\
\nu :\mathbb{R}^n &\longrightarrow \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^{d-1} \\
X &\longmapsto \nu(X) = (R^1(X, \cdots), R^2(X, \cdots), \cdots, R^d(X, \cdots)).
\end{aligned}$$

Theorem 3.1 (Singer-Nicolodi-Tricerri). *For $(R^0, R^1, \ldots, R^d) \in \mathcal{R}^0 \oplus \mathcal{R}^1 \oplus \cdots \oplus \mathcal{R}^d$ $(d \geq k(n)+2)$, $(R^0, R^1, \ldots, R^d)$ is contained in the image of Φ if and only if the following conditions hold:*

(i) $R^{i+2}(x, y; \cdots) - R^{i+2}(y, x; \cdots) = R^0(x, y) \cdot R^i \quad (i = 0, 1, \ldots, d-2)$,

(ii) $\nu(\mathbb{R}^n) \subset \mu(\mathfrak{so}(n))$.

We remark that the conditions (i) and (ii) are algebraic ones. The condition (i) is a system of quadratic equations with respect to $(R^0, R^1, \ldots, R^d)$. The condition (ii) means the existence of a solution $A \in \mathfrak{so}(n)$ for a system of the following linear equations defined for each $X \in \mathbb{R}^n$:

$$\begin{aligned}
A \cdot R^0 &= i(X)R^1 \\
A \cdot R^1 &= i(X)R^2 \\
&\cdots \\
A \cdot R^{d-1} &= i(X)R^d .
\end{aligned}$$

It will be an interesting problem to investigate the geometric description of the image of Φ or the image of $\widetilde{\Phi}$.

4. Locally homogeneous spaces which are not locally isometric to homogeneous spaces

We consider the following problem. How many locally homogeneous spaces exist which are not locally isometric to homogeneous spaces in $\mathcal{LH}(n)$? An answer we obtained is the following:

Theorem 4.1. *The set of local isometry classes of n-dimensional locally homogeneous spaces which are locally isometric to homogeneous spaces are dense in $\mathcal{LH}(n)$.*

Let M be an n-dimensional locally homogeneous space which is not locally isometric to any homogeneous space. We will show that there exist locally homogeneous spaces which are locally isometric to homogeneous spaces arbitrarily near M. As stated in section 2, we associate an effective infinitesimal homogeneous pair $(\mathfrak{k}, \mathfrak{h}, \mathfrak{m}, \langle , \rangle)$ to M. Let K be a connected and simply connected Lie group whose Lie algebra is $\mathfrak{k}$ and H be its connected Lie subgroup corresponding to $\mathfrak{h}$. By Theorem 2.3, H is not closed in K. Using this fact, we investigate the structure of Lie algebras $\mathfrak{k}$ and $\mathfrak{h}$. We denote by $\bar{H}$ the closure of H in K. Then $\bar{H}$ is a connected closed subgroup of K. We denote by $\bar{\mathfrak{h}}$ the Lie algebra of $\bar{H}$. By our assumption, we have $\mathfrak{h} \subsetneq \bar{\mathfrak{h}}$. It is known that $[\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] = [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}$. We put $\mathfrak{m}_1 = \bar{\mathfrak{h}} \cap \mathfrak{m}$. Then $\dim \mathfrak{m}_1 \geq 1$ and $\bar{\mathfrak{h}} = \mathfrak{h} + \mathfrak{m}_1$ (a direct sum). We denote by $\mathfrak{m}_2$ the orthogonal complement of $\mathfrak{m}_1$ in $\mathfrak{m}$ with respect to $\langle , \rangle$ and denote by $X_{\mathfrak{h}}$, $X_{\mathfrak{m}_1}$ and $X_{\mathfrak{m}_2}$, the $\mathfrak{h}$-component, the $\mathfrak{m}_1$-component and the $\mathfrak{m}_2$-component of X, respectively.

Lemma 4.2. *The following relations hold:*

(1) $[\mathfrak{h}, \mathfrak{m}_1] = \{0\}$, $[\mathfrak{h}, \mathfrak{m}_2] \subset \mathfrak{m}_2$.

(2) $[\mathfrak{m}_1, \mathfrak{m}_1] = \{0\}$.

(3) $[\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2$, *and moreover, we have*

$$\langle adX(Y), Z\rangle + \langle Y, adX(Z)\rangle = 0 \quad \textit{for } X \in \mathfrak{m}_1, Y, Z \in \mathfrak{m}_2 .$$

(4) $[\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{m}_1} = \mathfrak{m}_1$.

Proof of Lemma 4.2. (1). Evidently, $[\mathfrak{h}, \mathfrak{m}_1] \subset [\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$. On the other hand, $[\mathfrak{h}, \mathfrak{m}_1] \subset [\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subset \mathfrak{h}$. Therefore we have $[\mathfrak{h}, \mathfrak{m}_1] = \{0\}$. This implies that $[\mathfrak{h}, \mathfrak{m}_2] \subset \mathfrak{m}_2$.

(2), (3). We define a subset J of K by

$$J = \{a \in K \mid Ad(a)(\mathfrak{m}) \subset \mathfrak{m}, \langle Ad(a)X, Ad(a)Y\rangle = \langle X, Y\rangle \quad \text{for } X, Y \in \mathfrak{m} \ \} .$$

Then J is a closed subgroup of K. J contains H and hence $\bar{H}$, too. Therefore we have $[\bar{\mathfrak{h}}, \mathfrak{m}] \subset \mathfrak{m}$ and the representation $ad_{\mathfrak{m}}\bar{\mathfrak{h}}$ leaves the inner product $\langle , \rangle$ on $\mathfrak{m}$ invariant. Since $[\mathfrak{m}_1, \mathfrak{m}_1] \subset [\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subset \mathfrak{h}$ and $[\mathfrak{m}_1, \mathfrak{m}_1] \subset [\bar{\mathfrak{h}}, \mathfrak{m}] \subset \mathfrak{m}$, $[\mathfrak{m}_1, \mathfrak{m}_1] = \{0\}$. For $X \in \mathfrak{m}_1$, $ad_{\mathfrak{m}}X$ leaves the inner product $\langle , \rangle$ on $\mathfrak{m}$ invariant and hence (3) holds.

(4) We assume that $[\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{m}_1} \neq \mathfrak{m}_1$. We put $\mathfrak{k}' = \mathfrak{h} + [\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{m}_1} + \mathfrak{m}_2$. Then $\mathfrak{k}'$ is an ideal of $\mathfrak{k}$. Let K' be a connected Lie subgroup of K which corresponds to $\mathfrak{k}'$. Since K' is a normal subgroup of K, K' is closed in K. K' contains H and hence $\bar{H}$, too. Therefore $\mathfrak{k}' \supset \bar{\mathfrak{h}} = \mathfrak{h} + \mathfrak{m}_1$ and in particular $\mathfrak{m}_1 \subset [\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{m}_1}$. This is contrary to our assumption.

From Lemma 4.2 (1), we immediately obtain the following.

Corollary 4.3. *Let $(\mathfrak{k}, \mathfrak{h}, \mathfrak{m}, \langle , \rangle)$ be an effective infinitesimal homogeneous pair such that the representation of $ad_{\mathfrak{m}}\mathfrak{h}$ is irreducible. Then a locally homogeneous space which corresponds to it is locally isometric to a homogeneous space.*

We introduce a symmetric bilinear form $(,)$ on $\mathfrak{h}$ as follows:

$$(A, B) = -tr(ad_{\mathfrak{m}}A ad_{\mathfrak{m}}B) \quad \text{for } A, B \in \mathfrak{h} .$$

Because of the effectiveness, the homomorphism $ad_{\mathfrak{m}} : \mathfrak{h} \to \mathfrak{so}(\mathfrak{m})$ is injective. Therefore $(,)$ is an $ad\mathfrak{h}$-invariant (positive-definite) inner product. Let $\mathfrak{z}$ be the center of $\mathfrak{h}$ and $\mathfrak{s}$ be an orthogonal complement of $\mathfrak{z}$ in $\mathfrak{h}$ with respect to $(,)$. Then $\mathfrak{s}$ is an ideal of $\mathfrak{h}$ and a compact semi-simple Lie algebra. We denote by S the connected Lie subgroup of K corresponding to $\mathfrak{s}$. S is a compact subgroup and hence closed in K. In particular $\dim \mathfrak{z} \geq 1$. S is a normal subgroup of H and a factor Lie group of H/S is a connected Abelian group. By Lemma 4.2, $\mathfrak{s}$ is an ideal of $\bar{\mathfrak{h}}$ and hence S is a normal subgroup of $\bar{H}$. Moreover, a factor group $\bar{H}/S$ is a connected Abelian group. By a natural inclusion, H/S is a connected Lie subgroup of $\bar{H}/S$ and the closure of H/S in $\bar{H}/S$ coincides with $\bar{H}/S$.

We denote by $\mathfrak{a}$ and $\bar{\mathfrak{a}}$ the Lie algebras of H/S and $\bar{H}/S$, respectively and by $\exp : \bar{\mathfrak{a}} \to \bar{H}/S$ the exponential map. It is easily seen that there exists a Lie subalgebra $\mathfrak{a}'$ of $\bar{\mathfrak{a}}$ arbitrarily near $\mathfrak{a}$ such that $\exp(\mathfrak{a}')$ is a closed subgroup of $\bar{H}/S$. We denote by $\pi : \bar{\mathfrak{h}} \to \bar{\mathfrak{a}}$ the homomorphism of Lie algebras which corresponds to a natural projection $\pi : \bar{H} \to \bar{H}/S$. We put $\mathfrak{h}' = \pi^{-1}(\mathfrak{a}')$. Then $\mathfrak{h}'$ is a Lie subalgebra of $\bar{\mathfrak{h}}$. A connected Lie subgroup H' of $\bar{H}$ which corresponds to $\mathfrak{h}'$ is closed in $\bar{H}$ and hence in K. Since $\mathfrak{h}'$ is sufficiently near $\mathfrak{h}$ in $\bar{\mathfrak{h}}$, $\mathfrak{h}' \cap \mathfrak{m} = \{0\}$. Since $[\bar{\mathfrak{h}}, \mathfrak{m}] \subset \mathfrak{m}$, we see that $[\mathfrak{h}', \mathfrak{m}] \subset \mathfrak{m}$ and $ad_{\mathfrak{m}}\mathfrak{h}'$ leaves the inner product $\langle , \rangle$ invariant. Moreover, $ad_{\mathfrak{m}} : \mathfrak{h}' \to \mathfrak{so}(\mathfrak{m})$ is injective. Thus we obtain an effective infinitesimal homogeneous pair $(\mathfrak{k}, \mathfrak{h}', \mathfrak{m}, \langle , \rangle)$.

A locally homogeneous space M' which corresponds to $(\mathfrak{k}, \mathfrak{h}', \mathfrak{m}, \langle , \rangle)$ is locally isometric to a homogeneous space, since H' is closed in K. The curvature tensor R' and its i-th covariant derivative $\nabla^i R'$ of M' are sufficiently near R and $\nabla^i R$ of M, respectively. We will explain this assertion, below. Let X and Y be vectors of $\mathfrak{m}$. We have the decompositions

$$\begin{aligned} [X, Y] &= [X, Y]_{\mathfrak{h}} + [X, Y]_{\mathfrak{m}} \\ &= [X, Y]_{\mathfrak{h}'} + [X, Y]'_{\mathfrak{m}} \end{aligned}$$

according to the decompositions $\mathfrak{k} = \mathfrak{h} + \mathfrak{m}$ and $\mathfrak{k} = \mathfrak{h}' + \mathfrak{m}$, respectively. Since $\mathfrak{h}'$ is sufficiently near $\mathfrak{h}$, $[X, Y]_{\mathfrak{h}'}$ is sufficiently near $[X, Y]_{\mathfrak{h}}$ in $\bar{\mathfrak{h}}$ and $[X, Y]'_{\mathfrak{m}}$ is sufficiently near $[X, Y]_{\mathfrak{m}}$ in $\mathfrak{m}$. Noticing the formulas of the Riemannian connection, the curvature tensor, and its covariant derivatives stated in section 2, we see that our assertion holds.

Thus Theorem 4.1 has been proved.

Next, we will investigate the curvature properties of a locally homogeneous space which is not locally isometric to any homogeneous space. In addition to the assumption in the proof of Theorem 4.1, we assume that $Ad_{\mathfrak{m}}(H)$ is closed in $SO(\mathfrak{m})$. This assumption is satisfied by the effective infinitesimal homogeneous pair associated with a locally homogeneous space by the canonical way. Then we have $Ad_{\mathfrak{m}}(H) = Ad_{\mathfrak{m}}(\bar{H})$. From this it follows that $ad_{\mathfrak{m}}(\mathfrak{h}) = ad_{\mathfrak{m}}(\bar{\mathfrak{h}})$. We put $\mathfrak{n} =$ the kernel of $ad_{\mathfrak{m}}$ in $\bar{\mathfrak{h}}$. Because of the effectiveness, $\mathfrak{h} \cap \mathfrak{n} = \{0\}$.Therefore we have the direct sum decomposition $\bar{\mathfrak{h}} = \mathfrak{h} + \mathfrak{n}$ as a vector space. Since $\mathfrak{n}$ is an ideal of $\bar{\mathfrak{h}}$, $[\mathfrak{h}, \mathfrak{n}] \subset \mathfrak{n}$. On the other hand, $[\mathfrak{h}, \mathfrak{n}] \subset [\bar{\mathfrak{h}}, \bar{\mathfrak{h}}] \subset \mathfrak{h}$. Hence $[\mathfrak{h}, \mathfrak{n}] = \{0\}$. In particular

$\mathfrak{n}$ is contained in the center of $\mathfrak{k}$. It is easily seen that we may replace $\mathfrak{m}_1$ by $\mathfrak{n}$ in Lemma 4.2. That is, we may assume that $\mathfrak{m} = \mathfrak{n} + \mathfrak{m}_2$ (an orthogonal direct sum). Then the Riemannian connection $\Lambda_{\mathfrak{m}}$ stated in section 2 satisfies that

$$\begin{aligned} \Lambda_{\mathfrak{m}}(X_1)X_2 &= 0 \quad \text{for } X_1, X_2 \in \mathfrak{n}\,, \\ \Lambda_{\mathfrak{m}}(X)Y &= \Lambda_{\mathfrak{m}}(Y)X \quad \text{for } X \in \mathfrak{n}, Y \in \mathfrak{m}_2\,, \\ \langle \Lambda_{\mathfrak{m}}(X)Y, Z\rangle &= -\frac{1}{2}\langle [Y,Z]_{\mathfrak{m}}, X\rangle \quad \text{for } X \in \mathfrak{n}, Y, Z \in \mathfrak{m}_2\,. \end{aligned}$$

Moreover for a non-zero vector $X \in \mathfrak{n}$, $\Lambda_{\mathfrak{m}}(X) \neq 0$. In fact, if $\langle \Lambda_{\mathfrak{m}}(X)Y, Z\rangle = -\frac{1}{2}\langle [Y,Z]_{\mathfrak{m}}, X\rangle = 0$ for any $Y, Z \in \mathfrak{m}_2$, we have $[\mathfrak{m}_2, \mathfrak{m}_2]_{\mathfrak{n}} \subsetneq \mathfrak{n}$. It is contrary to Lemma 4.2 (4). For $X \in \mathfrak{m}$, we define the Jacobi operator R_X by the map $Y \mapsto R(Y,X)X$. The Jacobi operator R_X is a symmetric linear operator of $\mathfrak{m}$. Straightforward computing the formula of the curvature tensor given in section 2, we have the following.

Proposition 4.4. *Let X be a non-zero vector of $\mathfrak{n}$. Then*

$$\begin{aligned} R_X(X') &= 0 \quad \textit{for } X' \in \mathfrak{n}\,, \\ R_X(Y) &= -\Lambda_{\mathfrak{m}}(X)^2 Y \quad \textit{for } Y \in \mathfrak{m}_2\,. \end{aligned}$$

In particular, all the eigenvalues of R_X are non-negative and at least one eigenvalue is positive. Moreover, the Ricci curvature $\rho(X,X)$ is positive.

From the proposition above, we immediately obtain the following.

Corollary 4.5 (Spiro [12]). *If M is a locally homogeneous space whose Ricci curvature tensor is non-positive, then it is locally isometric to a homogeneous space.*

Corollary 4.6. *If M is an even dimensional locally homogeneous space which has positive sectional curvature, then it is locally isometric to a homogeneous space.*

Proof of Corollary 4.6. Suppose that M is not locally isometric to any homogeneous space. By Proposition 4.4, $\dim \mathfrak{n} = 1$. Let X be a nono-zero vector of $\mathfrak{n}$. Since the dimension of $\mathfrak{m}_2$ is odd, there exists a non-zero vector Y in $\mathfrak{m}_2$ such that $\Lambda_{\mathfrak{m}}(X)Y = 0$. The sectional curvature of the plane spanned by X and Y is zero. It is contrary to our assumption.

Finally we will determine five-dimensional locally homogeneous spaces which are not locally isometric to globally homogeneous spaces. Here we remark that a locally homogeneous space whose dimension is not greater than four is locally isometric to a homogeneous space (G. D. Mostow [8]).

Proposition 4.7. *Let M be a five-dimensional locally homogeneous space which is not locally isometric to a homogeneous space. Then M is locally isometric to one of the examples found by Kowalski in [5] (see also [6]).*

Proof. Let $(\mathfrak{k}, \mathfrak{h}, \mathfrak{m}, \langle\,,\rangle)$ be an effective infinitesimal homogeneous pair associated to M. We denote by K a connected and simply connected Lie group whose Lie algebra is $\mathfrak{k}$ and by H its connected Lie subgroup corresponding to $\mathfrak{h}$. Let S be the compact semi-simple Lie subgroup of H defined in the proof of Theorem 4.1. Then applying Theorem 1 in M. Goto [2], we can prove the following lemma. We omit its detailed argument.

Lemma 4.8. *K has a torus subgroup of dimension at least $2+rank(S)$.*

As before we assume that $Ad_{\mathfrak{m}}(H)$ is closed in $SO(\mathfrak{m})$ and have the decompositions: $\bar{\mathfrak{h}} = \mathfrak{h} + \mathfrak{n}$, $\mathfrak{m} = \mathfrak{n} + \mathfrak{m}_2$. Since $\mathfrak{n}$ is contained in the center of $\mathfrak{k}$, $\mathfrak{n}$ is an ideal of $\mathfrak{k}$. We denote by N a connected Lie subgroup of K corresponding to $\mathfrak{n}$. Since N is a normal subgroup of K, it is closed in K and simply connected.

Lemma 4.9. *The center of $\mathfrak{h}$ is of dimension at least $1 + \dim \mathfrak{n}$.*

Proof of Lemma 4.9. We recall the proof of Theorem 4.1. A factor group $\bar{H}/S$ is isomorphic to a product group $\mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$ of a d_1-dimensional torus $\mathbb{T}^{d_1}$ and a d_2-dimensional vector group $\mathbb{R}^{d_2}$. N is contained in $\bar{H}$ and isomorphic to $\mathbb{R}^l$. The natural projection $\bar{H} \to \bar{H}/S$ restricted to N is injective. Therefore $d_2 \geq \dim \mathfrak{n}$. Since the closure of H/S coincides with $\bar{H}/S \cong \mathbb{T}^{d_1} \times \mathbb{R}^{d_2}$, we have $\dim(H/S \cap \mathbb{T}^{d_1}) \geq 1$ and the natural projection $\bar{H}/S \to \mathbb{R}^{d_2}$ restricted to H/S is surjective. In particular dim of the center of $\mathfrak{h} = \dim H/S \geq 1 + d_2 \geq 1 + \dim \mathfrak{n}$.

We denote by K_1 and H_1 the factor groups K/N and $\bar{H}/N$, respectively and by $p : K \to K_1$ the natural projection. $\mathfrak{k}_1$ and $\mathfrak{h}_1$ are the Lie algebras of K_1 and H_1, respectively. Then $dp : \mathfrak{k} \to \mathfrak{k}_1$ is a Lie algebra homomorphism. $dp|_{\mathfrak{h}}$ is an isomorphism of $\mathfrak{h}$ onto $\mathfrak{h}_1$. We put $dp(\mathfrak{m}_2) = \mathfrak{m}$ and define an inner product $\langle , \rangle$ on $\mathfrak{m}$ such that $dp|_{\mathfrak{m}_2} : \mathfrak{m}_2 \to \mathfrak{m}$ is a linear isometry. Then we have an effective infinitesimal homogeneous pair $(\mathfrak{k}_1, \mathfrak{h}_1, \mathfrak{m}, \langle , \rangle)$. Let $\tilde{\mathbb{T}}$ be a torus subgroup of K. Since N is isomorphic to $\mathbb{R}^l$, $\tilde{\mathbb{T}} \cap N = \{e\}$. Therefore p restricted to $\tilde{\mathbb{T}}$ is injective. This, together with Lemma 4.8, implies that K_1 has a torus subgroup of dimension at least $2 + \mathrm{rank}(S)$.

From now on we assume that $\dim M = 5$. We will show that $\dim \mathfrak{n} = 1$. Since $\Lambda_{\mathfrak{m}} : \mathfrak{n} \to \mathfrak{so}(\mathfrak{m}_2)$ is injective, $\dim \mathfrak{n} \leq 2$. Suppose that $\dim \mathfrak{n} = 2$. Then by Lemma 4.9, dim of the center of $\mathfrak{h} \geq 1 + \dim \mathfrak{n} = 3$. On the other hand, $ad : \mathfrak{h} \to \mathfrak{so}(\mathfrak{m}_2) \cong \mathfrak{so}(3)$ is injective. This does not occur. Thus we see that $\dim \mathfrak{n} = 1$. In particular K_1/H_1 is a four-dimensional homogeneous space. In the classification list of four-dimensional homogeneous spaces (cf. L. Bérard Bergery [1]), we find the ones which satisfy the following two conditions:

(i) $\mathfrak{h}_1$ has the center of dimension at least $1 + \dim \mathfrak{n} = 2$,

(ii) K_1 has a torus subgroup of dimension at least $2+$rank of $\mathfrak{s}$, where $\mathfrak{s}$ denotes a semi-simple part of $\mathfrak{h}_1$.

Then we have $\mathfrak{k}_1 = \mathfrak{su}(2) + \mathfrak{su}(2)$, $\mathfrak{h}_1 = \mathfrak{so}(2) + \mathfrak{so}(2)$. Consequently, $\mathfrak{k} = \mathfrak{su}(2) + \mathfrak{su}(2) + \mathbb{R}$ and $\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{k}$ such that $dp|_{\mathfrak{h}}$ is an isomorphism of $\mathfrak{h}$ onto $\mathfrak{so}(2) + \mathfrak{so}(2)$. Therefore there exists a linear map $f : \mathfrak{so}(2) + \mathfrak{so}(2) \to \mathbb{R}$ such that

$$\mathfrak{h} = \{X + f(X) \mid X \in \mathfrak{so}(2) + \mathfrak{so}(2) \subset \mathfrak{su}(2) + \mathfrak{su}(2) \} .$$

It is sufficient to find a map f such that the connected Lie subgroup H of K which corresponds to $\mathfrak{h}$ is not closed. The effective infinitesimal homogeneous pair

$(\mathfrak{k}, \mathfrak{h}, \mathfrak{m}, \langle , \rangle)$ obtained in the above is associated to one of the examples found by Kowalski.

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