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GEOMETRY OF COMPACT SYMMETRIC SPACES

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ABSTRACT. We determined g-signatures of compact symmetric spaces. Relating with this, we also determined the self-intersection of subspaces in compact symmetric spaces in several cases by using double fibrations.

1. Introduction

This is a joint work with Prof. Tadashi Nagano. We will discuss and determine the signature and g-signature of every compact oriented symmetric space M of dim $M = 4n$ and for a orientation-preserving involution g of M. We will also discuss the self-intersection of a subspace of M in several cases. Our method is basically to apply the Atiyah-Singer theory [1], especially the g-signature theorem below to our geometric results ([3] and others). Informations on g-signature and on the self-intersection are reciprocal to some extent. The self-intersection is realized by a symmetric subspace of M and we uses double fibrations, $G/K \cap H \to G/K$ and $G/K \cap H \to G/H$, to determine the self-intersections in some cases. Based on the found value of the self-intersection, we will state a geometric application concerning the geometric structures of exceptional symmetric spaces FII, $EIII$, EVI and $EVIII$.

2. Preliminaries

Let M be a compact orientable $4k$ - dimensional Riemannian manifold. The signature $\tau(M)$ is the index of the quadratic form on $H^{2k}(M : \boldsymbol{R})$ defined with the cup product and the orientation. It is reasonable to define $\tau(M)$ to be zero if $dim M$ is not a multiple of 4. $\tau(M)$ and $g - \tau(M)$ are also defined as the indices of certain elliptic complexes. It is known that $g - \tau(M)$ is entirely determined by the action of g on the the real cohomology of M; hence it depends on the homotopy class of g only.

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Proposition 2.1 ([1]). *The g-signature $g-\tau(M)$ equals the signature $\tau(M)$ if g is homotopic with 1_M, the identity map, or, more generally, if g acts on $H^{2k}(M:\boldsymbol{R})$ as the identity.*

Proposition 2.2. *$\tau(M)$ is a certain Pontryagin number of M (the Hirzebruch index theorem). Hence (1) an m-fold covering space of M has the signature $= m\tau(M)$. (2) $\tau(M)=0$ if M is a group or M is a covering space of a non-orientable manifold. (3) $\tau(M\times N)=\tau(M)\times\tau(N)$; actually this product formula is valid in a much more general setting including the case where the dimension of M or N is not divisible by 4 and hence its signature equals by definition.*

Proposition 2.3 ([1]). *Let X be a compact oriented manifold of dimension $4k$, and let g be an oriented preserving involution of X. Then one has*

$$g-\tau(X)=\tau(SI(F(g,X);X)),$$

where $F(g,X)$ is the fixed point set of g acting on X and $SI(F(g,X);X)$ means the self-intersection of $F(g,X)$ in X.

Proposition 2.4 ([2]). *If the circle group $U(1)$ acts on a compact oriented manifold X, then the signature $\tau(X)$ equals the sum of $\tau(F_j)$ for the connected components F_j of the fixed point set $F(U(1),X)$ such that $codim F_j\equiv 0 \mod 4$; F_j is oriented in a certain way by means of the action of $U(1)$ as in [1]. The sum of $\tau(F_k)$ for F_k with $codim F_k\equiv 2 \mod 4$ equals 0.*

Proposition 2.5 ([4]). *The signature of a quaternionic Kähler manifold X of dimension $4n$ is equal to b_{2n}, the Betti number of X at the half dimension $2n$, up to sign.*

Proposition 2.6. *The signature of a symmetric Kähler manifold M is equal to $P(i)$, the Poincaré polynomial $P(t)$ with the imaginary unit i substituted.*

We will work on the compact connected simple symmetric spaces unless otherwise mentioned. Such a space M is a homogeneous space G/K of a compact connected Lie group G which acts on M as isometries; the action is locally effective usually. M admits the point symmetry s_o for every point o of M, which is isometry and acts on the tangent space T_oM as -1 times the identity.

Proposition 2.7. *s_o is homotopic to the identity 1_M if and only if (1) the Euler number $\chi(M)$ is positive, or equivalently, (2) the rank $r(M)$ equals $r(G)$ for $M=G/K$.*

Corollary 2.8. *Let $s\in G$ have a finite order. Then one has the signature $\tau(M)=\sum\tau(F_j)$ for the connected components F_j of $F(s,M)$ if F_j is oriented in an appropriate way.*

Proof. s is then a member of some subgroup $T\cong U(1)$ of G. Since T acts on $F(s,M)$ and $F(T,M)$ is its subspace, one obtains the conclusion by using Proposition 1.4 twice; $\tau(M)$ equals the sum of the signatures of the components of $F(T,M)$ as well as $\tau(F(s,M))$. □

We call each component of $F(s_o, M)$ *a polar of o in M*; the polars in M are concretely described in [3].

Corollary 2.9. *(1) The signature $\tau(M)$ equals the sum of those of the polars of a point, if $\chi(M)$ is positive. (2) One has $\tau(M) = 0$ otherwise.*

Proof. (1) is immediate from Corollary 1.8. In the case of (2), one has $r(K) < r(G)$ and hence G contains a circle group T which fixes no point; and Proposition 1.4 gives $\tau(M) = 0$. □

Remark 2.10. The polars in M are thus closely related to M in terms of topology. For another example, M is orientable if and only if every polars has an even dimension. Also the Lefschetz number $Lef(s_o)$ equals the sum of their Euler numbers; hence, in case $\chi(M)$ is positive, the sum equals $\chi(M)$.

Proposition 2.11. *$s_o - \tau(M) = \tau(M)$ for the point symmetry s_o, if $dim M \equiv 0$ mod 4.*

Proof. This equality obtains if s_o acts on $H^{2k}(M : \boldsymbol{R})$ trivially, where $dim M = 4k$, by the generalized Lefschetz fixed-point theorem of Atiyah-Singer. s_o acts trivially on $H^{2k}(M : \boldsymbol{R})$ which consists of the G-invariant (or, equivalently, harmonic) p-forms on M, provided p is even. □

3. Determination of the signature

We will determine the signature $\tau(M)$ for every compact symmetric space M, which we may assume is simple and simply connected with $\tau(M) \geq 0$.

Proposition 3.1. *$\tau(M) = 0$ if M is a group.*

Proposition 3.2. *$\tau(AI(n)) = \tau(AII(n)) = \tau(DIII(n)) = \tau(CI(n)) = \tau(FI)) = \tau(EI) = \tau(EIV) = \tau(EV) = \tau(EVII) = 0$.*

Proof. $\chi(M) = 0$ for $M = AI(n), AII(n), EI$ and EIV. $DIII(n)$ admits an involution whose fixed point set is $DIII(2) \times DIII(n-2)$ and $DIII(2) \cong S^2$. Also $CI(n)$ admits an involution whose fixed point set is $CI(1) \times CI(n-1)$ and $CI(1) \cong S^2$. $\tau(S^2) = 0$ implies $\tau(DIII(n)) = \tau(CI(n)) = 0$. There is a space $EV^{\%}$ (resp. $EVII^{\%}$) which is doubly covered by EV (resp. $EVII$) and is not oriented. It implies $\tau(EV) = \tau(EVII) = 0$. □

Proposition 3.3. *$\tau(G_p(\boldsymbol{C}^n)) = \tau(G_p(\boldsymbol{H}^n)) = \chi(G_p(\boldsymbol{R}^n))$.*

Proof. One may use the involution $s = I_1 \in U(1) \subset U(n)$, such that $F(s, G_p(\boldsymbol{C}^n)) = G_p(\boldsymbol{C}^{n-1}) \amalg G_{p-1}(\boldsymbol{C}^{n-1})$. Hence $\tau(G_p(\boldsymbol{C}^n)) = I_1 - \tau(G_p(\boldsymbol{C}^n)) = \tau(G_p(\boldsymbol{C}^{n-2})) + \tau(G_{p-2}(\boldsymbol{C}^{n-2}) = \chi(G_p(\boldsymbol{R}^n))$. $G_p(\boldsymbol{H}^n)$ admits an involution whose fixed point set is $G_p(\boldsymbol{C}^n)$, so we conclude $\tau(G_p(\boldsymbol{H}^n)) = \tau(G_p(\boldsymbol{C}^n))$ □

Proposition 3.4. *(1) $\tau(G_r(\boldsymbol{R}^n)) = 0$ if n or $r(n-r)$ is odd. (2) If $r = 2p$ and $n = 2m$, then $\tau(G_{2p}(\boldsymbol{R}^{2m})) = \chi(G_p(\boldsymbol{R}^n))$.*

Proof. (1) is the case when $G_r(\boldsymbol{R}^n)$ is the covering space of a non-orientable manifold. For (2), $G_{2p}(\boldsymbol{R}^{2m})$ has an involution whose fixed point set is $G_p(\boldsymbol{C}^m)$, so we have $\tau(G_{2p}(\boldsymbol{R}^{2m})) = \tau(G_p(\boldsymbol{C}^m)) = \chi(G_p(\boldsymbol{R}^n))$. $\square$

Proposition 3.5. $\tau(GI) = \tau(FII) = 1$.

Proof. The polar of GI is $\{o\} \amalg S^2 \cdot S^2$. $\tau(\{o\}) = 1$ and $\tau(S^2 \cdot S^2) = 0$ implies $\tau(GI) = 1$. The polar of FII is $\{o\} \amalg S^8$. $\tau(\{o\}) = 1$ and $\tau(S^8) = 0$ implies $\tau(FII) = 1$. $\square$

Proposition 3.6. $\tau(EII) = 4$.

Proof. EII has an involution whose fixed point set is $G_4^o(\boldsymbol{R}^{10}) \amalg DIII(5)$. $\tau(G_4^o(\boldsymbol{R}^{10})) = 4$ and $\tau(DIII(5)) = 0$ implies $\tau(EII) = 4$. $\square$

Proposition 3.7. $\tau(EIII) = 3$.

Proof. $EIII$ has an involution whose fixed point set is $G_2(\boldsymbol{C}^6) \amalg S^2 \times G_1(\boldsymbol{C}^6)$. $\tau(G_2(\boldsymbol{C}^6)) = 3$ and $\tau(S^2 \times G_1(\boldsymbol{C}^6)) = 0$ implies $\tau(EIII) = 3$. $\square$

Proposition 3.8. $\tau(EVI) = 7$.

Proof. EVI has involutions whose fixed point sets are $\{o\} \amalg G_4^o(\boldsymbol{R}^{12}) \amalg S^2 \cdot DIII(6)$ and $EII \amalg EIII$ respectively. So $\tau(EVI) = 1 \pm 6 + 0 = \pm 4 \pm 3$ must be 7. $\square$

Proposition 3.9. $\tau(EVIII) = 7$.

Proof. EVI has an involution whose fixed point set is $EVI \amalg S^2 \cdot EV$. So $\tau(EVI) = 0 \pm 7 = 7$. $\square$

Proposition 3.10. $\tau(EIX) = 8$.

Proof. EIX has an involution whose fixed point set is $G_4^o(\boldsymbol{R}^{16}) \amalg DIII(8)^{\%}$. So $\tau(EIX) = \pm 8 + 0 = 8$. $\square$

Corollary 3.11. *One has*

$$3\tau(M) \leq \chi(M)$$

if dimM > 0 *and this is sharp. In particular, the equality* $\tau(M) = \chi(M)$ *implies* $\chi(M) = 0 = \tau(M)$.

4. The g-signature of symmetric spaces

In this section the proofs are omitted and refer to [5] for the proofs.

Proposition 4.1. *Let* X *be an R-space in a Kählerian symmetric space* M, $F(t, M) = X$. *Then one has (1) the normal bundle* NX *is isomorphic with* TX, *and hence* NX *has the Euler number* $\chi(X)$; *(2)* X *is connected; (3)* $SI(X; M) = \chi(X)$; *and finally (4)* $t - \tau(M) = \chi(X)$.

In most cases, one has $t - \tau(M) = \tau(M)$ in (4) above. Precisely, this is valid for $(X, M) = (G_r(\boldsymbol{R}^{2r}), G_r(\boldsymbol{C}^{2r}))$, $(G_r(\boldsymbol{H}^n), G_r(\boldsymbol{C}^n))$, $(SO(n), DIII(n))$, $(T \cdot AII(n), DIII(2n))$, $(T \cdot AI(n), CI(n))$, $(Sp(n), CI(n))$, $(G_2(\boldsymbol{H}^4)^{\%}, EIII)$, $(FII, EIII)$, $(T \cdot EIV, EVII)$, and $(AII(4)/Z_2, EVII)$. The other cases are $(U(r), G_r(\boldsymbol{C}^{2r}))$, $(S^p \cdot S^q, G_2^o(\boldsymbol{R}^{p+q+2}))$, $(X, X \times X^-)$ where X is Kählerian and the diagonal of the product of X and its "complex conjugate" X^-. In every case, one has $t - \tau(M) = Lef(t)$.

Case of $M = G_p(\boldsymbol{C}^n)$.

(1) $g = \kappa$, the complex conjugation. $F(g, M) = G_p(\boldsymbol{R}^n)$. $g - \tau(M) = \chi(G_p(\boldsymbol{R}^n))$.

(2) $g = s^{II}$ and p is even, $g \cong \kappa$. $F(g, M) = G_{p/2}(\boldsymbol{H}^{n/2})$. $g - \tau(M) = \chi(G_p(\boldsymbol{R}^n))$.

(3) $g = s_o \circ \kappa \circ ad(J_p)$, $n = 2p$. $F(g, M) = OIII(p)$. $g - \tau(M) = 0$.

(4) $g = \kappa \circ ad(J_p)$, $n = 2p$. $F(g, M) = CI(p)$. $g - \tau(M) = 0$.

(5) $g = ad(J_p)$, $n = 2p$. $F(g, M) = U(p)$. $g - \tau(M) = 0$.

Case of $M = G_p^o(\boldsymbol{R}^n)$.

(1) $g = I_1 \simeq I_h$, $h =$ odd. $F(g, M) = G_p^o(\boldsymbol{R}^{n-1})$. $g - \tau(M) = \tau(G_p^o(\boldsymbol{R}^{n-2}))$.

(2) $g = ad(J_p)$, $n = 2p$. $F(g, M) = UI(p)$. $g - \tau(M) = \tau(M)$.

(3) $g = s_o \circ ad(J_p)$, $n = 2p$. $F(g, M) = O(p)$. $g - \tau(M) = 0$.

Case of $M = G_p(\boldsymbol{H}^n)$.

(1) $g = ad(J_p)$, $n = 2p$. $F(g, M) = UII(p)$. $g - \tau(M) = 0$.

(2) $g = s_o \circ ad(J_p)$, $n = 2p$, $g \cong ad(J_p)$. $F(g, M) = Sp(p)$. $g - \tau(M) = 0$.

Case of $M = CI(n)$.

(1) $g = ad(I_n)$; g preserves the orientation iff $n \equiv 0$ or $3 \bmod 4$. $F(g, M) = UI(n)$. $g - \tau(M) = 0$.

(2) $g = ad((1_2 \otimes J_p)K_n)$, $n = 2p$. $F(g, M) = Sp(p)$. $g - \tau(M) = 0$.

Case of $M = DIII(n)$.

(1) $g = ad(I_n)$; g preserves the orientation iff $n \equiv 0 \bmod 4$. $F(g, M) = SO(n)$. $g - \tau(M) = 0$.

(2) $g = ad((1_2 \otimes J_p)K_n)$, $n = 2p$. $F(g, M) = UII(p)$. $g - \tau(M) = 0$.

Case of $M = AI(n)$.

(1) $g = I_p$, $p =$ odd. $F(g, M) = T \cdot (AI(p) \times AI(n-p))$. $g - \tau(M) = 0$.

(2) $g = s_o \circ ad(I_1)$. $F(g, M)$ is the disjoint union of $G_k(\boldsymbol{R}^n)$ for $0 < k < n$, $k =$ odd. $g - \tau(M) = 0$.

(3) $g = s_o \circ J$, $n = 2p$. $F(g, M) = CI(p)$. $g - \tau(M) = 0$.

Case of $M = AII(n)$.

(1) $g = \kappa$, $p =$ odd. $F(g, M) = DIII(n)$. $g - \tau(M) = 0$.

(2) $g = s_o \circ (I_1 \oplus I_1)$. $F(g, M)$ is the disjoint union of $G_k(\boldsymbol{H}^n)$ for $0 < k < n$, $k =$ odd. $g - \tau(M) = 0$.

As for the exceptional type spaces, we have the following proposition.

Proposition 4.2. *One has $s_I - \tau(M) = \tau(M)$ for every simply connected space $M = E_6/K$, where s_I is an outer involution of E_6 induced on the spaces EI through EIV; we may assume that s_I is the point symmetry at a point of EI, since the outer involutions of E_6 are homotopic with each other.*

5. Self-intersections of some subspaces

In this section the proofs are omitted and refer to [5] for the proofs.

The following are obvious facts: (1) One has $SI(M;M) = 0$ if $2dimN < dimM$. (2) One has $SI(N;M) = g - \tau(M)$ if N is a component of $F(t,M)$ such that $2dimN = dimM$ and the other components have dimension $< dimN$, where t and g are orientation preserving involutions of M which are homotopic. (3) If the homology class $[N] = 0$ in $H_*(M;\boldsymbol{R})$, then $SI(N;M) = 0$ in $H_*(M;\boldsymbol{R})$. So we have the next proposition.

Proposition 5.1. $SI(G_1(\boldsymbol{H}^3); FI) = 0$,
$SI(S^2 \cdot CI(3); FI) = 0$,
$SI(S^8; FII) = 1$,
$SI(G_1(\boldsymbol{H}^3); FII) = 1$,
$SI(FI; EI) = 0$,
$SI(FII; EIV) = 0$.

Proposition 5.2. *Let s and t be involutions of G for the symmetric space $M = G/K$ and G/H respectively, where K and H are the connected components of $F(s,G)$ and $F(t,G)$. Assume that (1) s commutes with t, (2) G/H has an equal rank to the subspace $K/K\cap H$ and (3) $dimG/K$ is less than $2dimH/K\cap H$. Then one has*

$$SI(H/K\cap H; M) = H(0)/H(0)\cap K$$

and is connected, where $H(0)$ is the centralizer of a maximal torus $A \subset G/H$ in H, $A \ni o = H(o)$.

Corollary 5.3. $SI(CI(p)\times CI(q); CI(p+q)) = CI(1)\times\ldots\times CI(1)\times CI(q-p)$ *if* $p \le q$ *(* $CI(1)\cong S^2$ *),*
$SI(G_4^o(\boldsymbol{R}^9); FI) = G_4^o(\boldsymbol{R}^7)$,
$SI(G_4^o(\boldsymbol{R}^{10}); EII) = G_4^o(\boldsymbol{R}^6)$,
$SI(FI; EII) = SI(EII; EVI) = SI(EVI; EIX) = G_4^o(\boldsymbol{R}^8)$,
$SI(DIII(5); EIII) = G_2(\boldsymbol{C}^4)$,
$SI(CI(4)^{\%}; EII) = 4$.

Corollary 5.4. *Let M be a compact, simply connected and simple Lie group and B be its subspace with the same root system as M. Then one has*

$$SI(B;M) = [\textit{the maximal torus}] = 0\,.$$

By using the result of the self-intersections, we characterize the geometric structure of the exceptional type spaces: $FII \subset EIII \subset EVI \subset EVIII$. Let $M = M_r$ be one of these spaces with rank $r = 2^a, a \in \{0,1,2,3\}$ and $dimM = 16r$. Then we have $SI(M_{2/r}, M_r) = 3$.

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