DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 137–140

A SHORT PROOF OF EIGENVALUE ESTIMATES FOR THE DIRAC OPERATOR ON RIEMANNIAN AND KÄHLER MANIFOLDS

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Abstract. Aim of this note is to present short and natural proofs for the lower bounds of the spectrum of the Dirac operator on compact Riemannian and Kähler manifolds with positive scalar curvature.

1. Introduction

For Riemannian manifolds a lower bound of the spectrum of the Dirac operator was first given by Th. Friedrich ([Fri80]). The corresponding result in the Kähler case is due to K.-D. Kirchberg ([Kir86] and [Kir90]). A modified proof was given by O. Hijazi ([Hij94]). The idea of the proof in this note is to estimate the norm of the covariant derivative of an eigenspinor using the direct sum decomposition of the tensor product of the spinor bundle and cotangent bundle. To achieve this we use the explicit formula for the embedding of the spinor bundle S into the tensor product $S \otimes TM$ (with modifications for Kähler manifolds). The approach presented here was also the starting point for the proof of the estimate in the last remaining case, i.e. for the Dirac on quaternionic Kähler manifolds of positive scalar curvature ([KSW97-1], [KSW97-2], [KSW97-3]).

2. Riemannian manifolds

Let (M^n, g) be a Riemannian spin manifold with positive scalar curvature R. The spinor bundle is denoted by S. The tensor product $S \otimes TM$ splits into two summands: $S \otimes TM \cong S \oplus \ker \mu$, here the Clifford multiplication μ also defines the projection onto the first summand. The Dirac operator is defined as composition: $D = \mu \circ \nabla$, where ∇ is the covariant derivative induced from the Levi–Civita connection. Complementary, the twistor operator T is defined as projection onto

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1991 *Mathematics Subject Classification.* Primary 53A50; Secondary 58G25, 58G30.
Key words and phrases. Dirac operators, eigenvalue, lower bound.
This paper is in final form and no version of it will be submitted for publication elsewhere.

$\ker\mu$. Let $I : S \longrightarrow S \otimes TM$ be the right inverse to μ , i.e. $\mu \circ I = \mathrm{id}$. With respect to an ortho–normal frame $\{e_i\}_{i=1,\dots,n}$ the embedding I is locally given by:

$$I(\phi) \;=\; -\tfrac{1}{n}\sum_{i=1}^{n} e_i \cdot \phi \otimes e_i \,,$$

where ϕ is any spinor and $\cdot$ again denotes the Clifford multiplication.

Lemma 2.1.

$$\begin{aligned} |I(\phi)|^2 \;&=\; \tfrac{1}{n}\,|\phi|^2 \\ \nabla\phi \;&=\; I(D\phi) \;+\; T\phi \end{aligned}$$

Proof. The first equation follows from an easy calculation. For the second we note that $\nabla\phi = I(\psi) + T\phi$ for some spinor ψ. But then $D\phi = \mu\nabla\phi = \mu I(\psi) = \psi$. ☐

This Lemma and the orthogonality of the decomposition of $S \otimes TM \cong S \oplus \ker\mu$ imply

$$|\nabla\phi|^2 \;=\; \tfrac{1}{n}\,|D\phi|^2 \;+\; |T\phi|^2 \,. \tag{1}$$

Let ϕ be any eigenspinor of D for the eigenvalue λ. Integrating (1) and using the Lichnerowicz formula, $D^2 = \nabla^*\nabla + R/4\,\mathrm{id}$, we obtain

$$\lambda^2\|\phi\|^2 \;-\; \|\tfrac{R}{4}\,\phi\|^2 \quad=\quad \tfrac{1}{n}\,\lambda^2\,\|\phi\|^2 \;+\; \|T\phi\|^2 \,.$$

Estimating $\|T\phi\|^2$ by zero and the scalar curvature by its minimum R_0 the lower bound immediately follows, i.e.

$$\lambda^2 \quad\geq\quad \tfrac{n}{n-1}\,\tfrac{R_0}{4}\,.$$

Let ϕ be any eigenspinor for the lowest possible eigenvalue. Then $T\phi = 0$ and $\nabla\phi = \lambda I(\phi)$. Using the definition of I we obtain the Killing equation for ϕ, which characterizes the limiting case, i.e.

$$\nabla_X\phi = -\tfrac{\lambda}{n} X \cdot \phi \,,$$

where X is any tangent vector and $\lambda = \frac{n}{n-1}\frac{R_0}{4}$ is the lower bound of the spectrum.

3. Kähler manifolds

Let $(M^{2m},\, g,\, J)$ be a compact Kähler manifold with positive scalar curvature R. The Kähler form Ω acts via Clifford multiplication on the spinor bundle S and induces the decomposition $S = \oplus_{r=0}^{m} S_r$, where each fibre of the subbundle S_r is an eigenspace of Ω for the eigenvalue $i(m-2r)$. Similar to the Riemannian case we have a splitting $S_r \otimes TM \cong S_{r-1} \oplus S_{r+1} \oplus \ker\mu$. Accordingly, the Dirac operator splits into two components, i.e. $D = D^+ + D^-$ and there are two embeddings $I^\pm$: $S_{r\pm1} \longrightarrow S_r \otimes TM$. Let $\{e_i\}_{i=1,\dots,2m}$ be an ortho–normal frame with $Je_{2i-1} = e_{2i}$. With respect to the complex basis $f_i := \frac{1}{2}\,(e_{2i-1} - ie_{2i})$ the embeddings are locally given by

$$I^+(\phi_{r+1}) = -\tfrac{1}{r+1}\sum_{i=1}^{m} \bar f_i \cdot \phi_{r+1} \otimes f_i; \qquad I^-(\phi_{r-1}) = -\tfrac{1}{m-r+1}\sum_{i=1}^{m} f_i \cdot \phi_{r-1} \otimes \bar f_i \,,$$

where $\phi_{r\pm 1} \in \Gamma(S_{r\pm 1})$. Note, that the Kähler form Ω considered as an element of the Clifford algebra is given as $\Omega = 2i\sum_{i=1}^m f_i \cdot \bar{f}_i + i\,m = -2i\sum_{i=1}^m \bar{f}_i \cdot f_i - i\,m$. Hence, $\sum_{i=1}^m f_i \cdot \bar{f}_i\,\phi_r = -r\phi_r$ and $\sum_{i=1}^m \bar{f}_i \cdot f_i\,\phi_r = (r-m)\phi_r$. As in the Riemannian case we have

Lemma 3.1.

$$\begin{aligned} |I^+(\phi_{r+1})|^2 &= \tfrac{1}{2(r+1)}\,|\phi_{r+1}|^2\,, \\ |I^-(\phi_{r-1})|^2 &= \tfrac{1}{2(m-r+1)}\,|\phi_{r-1}|^2\,, \\ \nabla\phi &= I^+(D^+\phi) + I^-(D^-\phi) + T\phi\,. \end{aligned}$$

Using this Lemma we conclude for any spinor $\phi_r \in \Gamma(S_r)$

$$|\nabla\phi_r|^2 = \tfrac{1}{2(r+1)}\,|D^+\phi_r|^2 + \tfrac{1}{2(m-r+1)}\,|D^-\phi_r|^2 + |T\phi|^2\,. \tag{2}$$

It is easy to see that we can assume an eigenspinor of D to be of the form $\phi = \phi_{r-1} + \phi_r \in \Gamma(S_{r-1} \oplus S_r)$ with $D^+\phi_{r-1} = \lambda\phi_r$, $D^-\phi_r = \lambda\phi_{r-1}$, $D^+\phi_r = 0 = D^-\phi_{r-1}$ and $\|\phi_{r-1}\|^2 = \|\phi_r\|^2$. Applying equation (2) to ϕ_{r-1} and ϕ_r separately yields two inequalities

$$\lambda^2 \geq \tfrac{2r}{2r-1}\,\tfrac{R_0}{4} \quad \text{and} \quad \lambda^2 \geq \tfrac{2(m-r+1)}{2(m-r)+1}\,\tfrac{R_0}{4}\,.$$

Here, we used again the Lichnerowicz formula and estimated R by R_0 and the last summand in (2) by zero. Combining these inequalities we obtain

$$\lambda^2 \geq \max\left(\tfrac{2r}{2r-1}\,,\,\tfrac{2(m-r+1)}{2(m-r)+1}\right)\tfrac{R_0}{4}\,.$$

The expression on the right side is minimal for $r = \frac{m+1}{2}$. Hence, we have two cases: for odd m we can take $r = \frac{m+1}{2}$ and for even m we have to take $r = \frac{m}{2}+1$. The corresponding inequalities are

$$\lambda^2 \geq \tfrac{m+1}{m}\,\tfrac{R_0}{4} \quad \text{for odd m} \qquad \lambda^2 \geq \tfrac{m}{m-1}\,\tfrac{R_0}{4} \quad \text{for even m}\,.$$

Like for Riemannian spin manifolds it is at this point easy to determine the Kählerian Killing equation (c.f.: [Kir93]) characterizing the limit case.

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