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CURVATURES ESTIMATIONS OF LEFT INVARIANT RIEMANNIAN METRICS ON THREE DIMENSIONAL LIE GROUPS

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ABSTRACT. The estimations of the sectional curvature, one-dimensional curvature and the Ricci curvature on three dimensional Lie groups with left invariant Riemannian metrics are given in this paper. Becides this the Skhouten-Weil tensor and conformal flat one-parameter subgroups of three dimensional Lie groups with left invariant Riemannian metrics are investigated. For the proof of general theorems of this paper some formulas of J.Milnor's paper [1] are used. The results of this paper are intersected with the results of D.Alekseevskii-B.Kimelfeld on a classification of conformal flat homogeneous Riemannian manifolds [5] and the results of O.Kowalski-S.Nikcevic on Ricci eigenvalues of locally homogeneous Riemann 3-manifolds [8].

1. Preliminaries

In this section some results and facts that are necessary for the proof of general theorems are presented. The most part of its corresponds to the results of J.Milnor's paper [1]. Let G be a connected 3-dimensional unimodular Lie group with Lie algebra g. Denote by $(\cdot,\cdot)$ an arbitrary scalar product on g. Then there is positively oriented orthonormal basis e_1, e_2, e_3 in g such that [1]

$$[e_1, e_2] = \lambda_3 e_3,\ [e_2, e_3] = \lambda_1 e_1,\ [e_3, e_1] = \lambda_2 e_2. \tag{1.1}$$

The Ricci quadratic form $r(x) = R_{ij}x^i x^j$, where R_{ij} are the components of the Ricci tensor, is diagonalized in this basis and its principal curvatures and trace are equal [1]:

$$\begin{aligned} r_1 &= 2\,(\frac{1}{2}\,\lambda_1 - \frac{1}{2}\,\lambda_2 + \frac{1}{2}\,\lambda_3)\,(\frac{1}{2}\,\lambda_1 + \frac{1}{2}\,\lambda_2 - \frac{1}{2}\,\lambda_3)\\ r_2 &= 2\,(-\frac{1}{2}\,\lambda_1 + \frac{1}{2}\,\lambda_2 + \frac{1}{2}\,\lambda_3)\,(\frac{1}{2}\,\lambda_1 + \frac{1}{2}\,\lambda_2 - \frac{1}{2}\,\lambda_3)\\ r_3 &= 2\,(-\frac{1}{2}\,\lambda_1 + \frac{1}{2}\,\lambda_2 + \frac{1}{2}\,\lambda_3)\,(\frac{1}{2}\,\lambda_1 - \frac{1}{2}\,\lambda_2 + \frac{1}{2}\,\lambda_3) \end{aligned} \tag{1.2}$$

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$$\rho = -\frac{1}{2}\,{\lambda_1}^2 - \frac{1}{2}\,{\lambda_2}^2 + \lambda_2\,\lambda_3 - \frac{1}{2}\,{\lambda_3}^2 + \lambda_1\,\lambda_3 + \lambda_1\,\lambda_2$$

Besides this, there is a connection between structure constants $(\lambda_1, \lambda_2, \lambda_3)$ and non isomorphic Lie algebras in three dimensional unimodular case.

Case	Signs $(\lambda_1, \lambda_2, \lambda_3)$	Associated Lie group	Lie algebra
(a)	$+,+,+$	$SU(2)$ or $SO(3)$	$su(2)$ – compact, simple
(b)	$+,+,-$	$SL(2,R)$ or $O(1,2)$	$sl(2,R)$ – noncompact, simple
(c)	$+,+,0$	$E(2)$	$e(2)$ – solvable
(d)	$+,-,0$	$E(1,1)$	$e(1,1)$ – solvable
(e)	$+,0,0$	$H - Heisenberg\ group$	h – nilpotent
(f)	$0,0,0$	$R \oplus R \oplus R$	R^3 – commutative

Remark 1. We note that each of these 3-dimensional unimodular Lie groups G possesses a discrete subgroup Γ so that the quotient $\Gamma \setminus G$ is a compact Lie group. Therefore we consider not all Lie groups, but only associated.

Let us define in three dimensional Riemannian case one dimensional curvature as a value on unit vector $\vec{\xi}$ of the quadratic form $A_{ij} = \dfrac{1}{n-2}\left(R_{ij} - \dfrac{\rho g_{ij}}{2(n-1)}\right)$, where $n = 3$, R_{ij} – the Ricci tensor, g_{ij} – the metric tensor. Then the Riemannian sectional curvature in two dimensional section $\vec{\xi_1} \wedge \vec{\xi_2}$ is calculated by the formula [6]

$$K(\vec{\xi_1} \wedge \vec{\xi_2}) = (A\vec{\xi_1}, \vec{\xi_1}) + (A\vec{\xi_2}, \vec{\xi_2}). \tag{1.3}$$

It can be note from the above formulas that the quadratic form A has diagonal form in this basis and its principal values are equal

$$
(1.4)\qquad
\begin{aligned}
k_1 &= \frac{5}{8}\,{\lambda_1}^2 - \frac{3}{8}\,{\lambda_2}^2 + \frac{3}{4}\,\lambda_2\,\lambda_3 - \frac{3}{8}\,{\lambda_3}^2 - \frac{1}{4}\,\lambda_1\,\lambda_3 - \frac{1}{4}\,\lambda_1\,\lambda_2 \\
k_2 &= -\frac{3}{8}\,{\lambda_1}^2 + \frac{3}{4}\,\lambda_1\,\lambda_3 + \frac{5}{8}\,{\lambda_2}^2 - \frac{3}{8}\,{\lambda_3}^2 - \frac{1}{4}\,\lambda_2\,\lambda_3 - \frac{1}{4}\,\lambda_1\,\lambda_2 \\
k_3 &= -\frac{3}{8}\,{\lambda_1}^2 + \frac{3}{4}\,\lambda_1\,\lambda_2 - \frac{3}{8}\,{\lambda_2}^2 + \frac{5}{8}\,{\lambda_3}^2 - \frac{1}{4}\,\lambda_2\,\lambda_3 - \frac{1}{4}\,\lambda_1\,\lambda_3
\end{aligned}
$$

Let σ_{12}, σ_{23}, σ_{31}, be two dimensional sectional curvatures of the corresponding two dimensional sections, then we have from the formula (1.3):

$$
(1.5)\qquad
\begin{aligned}
\sigma_{12} &= \frac{1}{4}\,{\lambda_1}^2 + \frac{1}{4}\,{\lambda_2}^2 + \frac{1}{2}\,\lambda_2\,\lambda_3 - \frac{3}{4}\,{\lambda_3}^2 + \frac{1}{2}\,\lambda_1\,\lambda_3 - \frac{1}{2}\,\lambda_1\,\lambda_2 \\
\sigma_{23} &= -\frac{3}{4}\,{\lambda_1}^2 + \frac{1}{2}\,\lambda_1\,\lambda_3 + \frac{1}{4}\,{\lambda_2}^2 + \frac{1}{4}\,{\lambda_3}^2 - \frac{1}{2}\,\lambda_2\,\lambda_3 + \frac{1}{2}\,\lambda_1\,\lambda_2 \\
\sigma_{31} &= \frac{1}{4}\,{\lambda_1}^2 + \frac{1}{2}\,\lambda_1\,\lambda_2 - \frac{3}{4}\,{\lambda_2}^2 + \frac{1}{4}\,{\lambda_3}^2 + \frac{1}{2}\,\lambda_2\,\lambda_3 - \frac{1}{2}\,\lambda_1\,\lambda_3
\end{aligned}
$$

Let G be a connected 3-dimensional nonunimodular Lie group with Lie algebra g. Denote by $(\cdot,\cdot)$ an arbitrary scalar product on g. Then there exists positively oriented orthonormal basis e_1, e_2, e_3 in g such that [1]

$$
(1.6)\qquad [e_1, e_2] = \alpha e_2 + \beta e_3,\ [e_1, e_3] = \gamma e_2 + \delta e_3,\ [e_2, e_3] = 0,
$$

where $\alpha+\delta \neq 0$ and $\alpha\gamma+\beta\delta = 0$. We note that in the case $\alpha+\delta = 2$ the determinant $D = \alpha\delta - \gamma\beta$ determines the Lie algebra g up to an isomorphism. Therefore by analogy with [1] we introduce the numbers

$$
\alpha = 1+\xi,\ \beta = (1+\xi)\eta,\ \gamma = -(1-\xi)\,\eta,\ \delta = 1-\xi,\ \text{ where } \xi \geq 0,\ \eta \geq 0.
$$

Then the Ricci quadratic form is diagonalized in this basis and its principal curvatures and trace are equal correspondingly

$$
(1.7)\qquad
\begin{aligned}
r_1 &= -(1+\xi)^2 - (1-\xi)^2 - \frac{1}{2}\,((1+\xi)\,\eta - (1-\xi)\,\eta)^2 \\
r_2 &= -2 - 2\,\xi + \frac{1}{2}\,(1-\xi)^2\,\eta^2 - \frac{1}{2}\,(1+\xi)^2\,\eta^2 \\
r_3 &= -2 + 2\,\xi + \frac{1}{2}\,(1+\xi)^2\,\eta^2 - \frac{1}{2}\,(1-\xi)^2\,\eta^2, \quad \rho = -6 - 2\,\xi^2 - 2\,\eta^2\,\xi^2
\end{aligned}
$$

The principal values of A are equal

$$
(1.8)\qquad
\begin{aligned}
k_1 &= -\frac{1}{2} - \frac{3}{2}\,\xi^2 - \frac{3}{2}\,\eta^2\,\xi^2 \\
k_2 &= -\frac{1}{2} - 2\,\xi - 2\,\eta^2\,\xi + \frac{1}{2}\,\eta^2\,\xi^2 + \frac{1}{2}\,\xi^2 \\
k_3 &= -\frac{1}{2} + 2\,\xi + 2\,\eta^2\,\xi + \frac{1}{2}\,\eta^2\,\xi^2 + \frac{1}{2}\,\xi^2
\end{aligned}
$$

For the corresponding two dimensional sectional curvatures we have

$$
\begin{aligned}
&\sigma_{12} = -1 - \xi^2 - \eta^2\,\xi^2 - 2\,\xi - 2\,\eta^2\,\xi, \\
&\sigma_{23} = -1 + \eta^2\,\xi^2 + \xi^2, \quad \sigma_{31} = -1 + 2\,\xi + 2\,\eta^2\,\xi - \eta^2\,\xi^2 - \xi^2.
\end{aligned} \tag{1.9}
$$

Let us consider the Schouten-Weil tensor which is an important characteristic of the Riemannian manifolds

$$S_{ijk} = A_{ij,k} - A_{ik,j}, \tag{1.10}$$

where $A_{ij,k}$ are covariant derivatives of A_{ij}. We note that in three dimensional case this tensor is vanishing if and only if the Riemannian metric is conformal flat [4]. Besides this in three dimensional case the Schouten-Weil tensor is conformally invariant tensor of zero weight.

The notion of the conformal development of a curve from Riemannian space in to the pseudo-euclidean Minkowski space was introduced in [6]. As a result the class of conformal flat curves or "circles" was considered. The equation of these curves has the form

$$\frac{D^2\tau_i}{(dt)^2} + A_{ij}\tau^j = C\tau_i, \tag{1.11}$$

where $\dfrac{D}{dt}$ is covariant derivative along the curve, τ is the unit tangent vector to the curve, $C = (A\tau, \tau) - (\chi)^2$, where χ is the geodesical curvature of the curve.

We shall call one-parameter subgroup of G conformally flat if and only if it satisfies to this equation.

2. Curvatures estimations

Let us consider unimodular case.

Lemma 2.1. *The extremal values of the sectional curvature are contained among the numbers*

$$
\begin{aligned}
\sigma_{12} &= \frac{1}{4}\,{\lambda_1}^2 + \frac{1}{4}\,{\lambda_2}^2 + \frac{1}{2}\,\lambda_2\,\lambda_3 - \frac{3}{4}\,{\lambda_3}^2 + \frac{1}{2}\,\lambda_1\,\lambda_3 - \frac{1}{2}\,\lambda_1\,\lambda_2 \\
\sigma_{23} &= -\frac{3}{4}\,{\lambda_1}^2 + \frac{1}{2}\,\lambda_1\,\lambda_3 + \frac{1}{4}\,{\lambda_2}^2 + \frac{1}{4}\,{\lambda_3}^2 - \frac{1}{2}\,\lambda_2\,\lambda_3 + \frac{1}{2}\,\lambda_1\,\lambda_2 \\
\sigma_{31} &= \frac{1}{4}\,{\lambda_1}^2 + \frac{1}{2}\,\lambda_1\,\lambda_2 - \frac{3}{4}\,{\lambda_2}^2 + \frac{1}{4}\,{\lambda_3}^2 + \frac{1}{2}\,\lambda_2\,\lambda_3 - \frac{1}{2}\,\lambda_1\,\lambda_3
\end{aligned} \tag{2.1}
$$

Proof. This fact one can deduce either from the Courant-Fisher minimax theorem [7] or from the formula $(A\vec{\xi_1}, \vec{\xi_1}) + (A\vec{\xi_2}, \vec{\xi_2}) = tr(A) - (A\vec{\xi_3}, \vec{\xi_3})$ where unit vectors $\{\vec{\xi_1}, \vec{\xi_2}, \vec{\xi_3}\}$ are mutually orthogonal, $tr(A)$ is the trace of A. □

The following theorem is an immediate consequence of this lemma.

Theorem 2.1. *Let $K_{(\cdot,\cdot)}(\pi) : G_2 \to R$ be the sectional curvature of the Riemannian manifold $(G, (\cdot,\cdot))$. Then there are the following estimations:*

Case	*Signes* $(\lambda_1, \lambda_2, \lambda_3)$	*Lie algebra*	*Sectional curvature*
(a)	$+, +, +$	$su(2)$	$\sigma_{12} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{23}$, $\sigma_{12} \leq \sigma_{31} \leq \sigma_{23}$, $\sigma_{23} > 0$ *if* $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_1 + \lambda_2$. $\sigma_{12} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{31}$, $\sigma_{12} < \sigma_{23} \leq \sigma_{31}$, $\sigma_{31} > 0$ *if* $0 < \lambda_1 \leq \lambda_2 < \lambda_1 + \lambda_2 \leq \lambda_3$.
(b)	$+, +, -$	$sl(2, R)$	$\sigma_{12} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{23}$, $\sigma_{23} = \sigma_{31}$, $\sigma_{23} > 0$, $\sigma_{12} < 0$ *if* $\lambda_1 = \lambda_2 > 0$. $\sigma_{12} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{23}$, $\sigma_{12} \leq \sigma_{31} < \sigma_{23}$, $\sigma_{12} < 0$, $\sigma_{23} > 0$ *if* $\lambda_1 < \lambda_2 \leq \lambda_1 + \|\lambda_3\|$. $\sigma_{31} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{23}$, $\sigma_{31} < \sigma_{12} < \sigma_{23}$, $\sigma_{31} < 0$, $\sigma_{23} > 0$ *if* $0 < \lambda_1 < \lambda_1 + \|\lambda_3\| < \lambda_2$.
(c)	$+, +, 0$	$e(2)$	$K_{(\cdot,\cdot)}(\pi) = 0$, *if* $0 < \lambda_1 = \lambda_2$. $\sigma_{31} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{23}$, $\sigma_{31} < \sigma_{12} < \sigma_{23}$, $\sigma_{31} < 0$, $\sigma_{23} > 0$ *if* $\lambda_1 < \lambda_2$.
(d)	$+, -, 0$	$e(1, 1)$	$\sigma_{31} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{12}$, $\sigma_{31} \leq \sigma_{23} < \sigma_{12}$, $\sigma_{12} > 0$, $\sigma_{31} < 0$ *if* $0 < \lambda_1 \leq \|\lambda_2\|$. $\sigma_{23} \leq K_{(\cdot,\cdot)}(\pi) \leq \sigma_{12}$, $\sigma_{23} \leq \sigma_{31} < \sigma_{12}$, $\sigma_{23} < 0$, $\sigma_{12} > 0$ *if* $0 < \|\lambda_2\| \leq \lambda_1$.

Case	*Signes* $(\lambda_1, \lambda_2, \lambda_3)$	*Lie algebra*	*Sectional curvature*
(e)	$+, 0, 0$	h	$\sigma_{23} \le K_{(\cdot,\cdot)}(\pi) \le \sigma_{12}$, $\sigma_{23} = -\frac{3\lambda_1^2}{4} < 0$, $\sigma_{12} = \sigma_{31} = \frac{\lambda_1^2}{4} > 0$.
(f)	$0, 0, 0$	R^3	$K_{(\cdot,\cdot)}(\pi) = 0$

Proof. We shall investigate every case consecutively.
(a) Let us consider remainders

$$\sigma_{12} - \sigma_{23} = -(\lambda_3 - \lambda_1)(\lambda_3 - \lambda_2 + \lambda_1)$$
$$\sigma_{23} - \sigma_{31} = -(\lambda_2 - \lambda_1)(-\lambda_2 + \lambda_3 - \lambda_1)$$
$$\sigma_{31} - \sigma_{12} = (-\lambda_2 + \lambda_3)(\lambda_3 + \lambda_2 - \lambda_1)$$

We may assume without loss of generality that $0 < \lambda_1 \le \lambda_2 \le \lambda_3$. The case (a) follows immediately from the above facts.
(b) In this case we may assume that $0 < \lambda_1 \le \lambda_2$, $\lambda_3 = -|\lambda_3|$. Then we have

$$\sigma_{12} - \sigma_{23} = (|\lambda_3| + \lambda_1)(\lambda_1 - \lambda_2 - |\lambda_3|)$$
$$\sigma_{23} - \sigma_{31} = (\lambda_2 - \lambda_1)(\lambda_1 + \lambda_2 + |\lambda_3|)$$
$$\sigma_{31} - \sigma_{12} = (\lambda_2 + |\lambda_3|)(\lambda_1 - \lambda_2 + |\lambda_3|)$$

Let $\lambda_1 = \lambda_2 > 0$, then $\sigma_{12} < \sigma_{23} = \sigma_{31}$,

$$\sigma_{12} = -\frac{1}{2}\lambda_2|\lambda_3| - \frac{3}{4}|\lambda_3|^2 - \frac{1}{2}\lambda_2|\lambda_3| < 0,$$
$$\sigma_{23} = -\frac{3}{4}\lambda_2^2 - \frac{1}{2}\lambda_2|\lambda_3| + \frac{1}{4}\lambda_2^2 + \frac{1}{4}|\lambda_3|^2 + \frac{1}{2}\lambda_2|\lambda_3| + \frac{1}{2}\lambda_2^2 = \frac{1}{4}|\lambda_3|^2 > 0,$$

and the assertion of our Theorem follows from Lemma 2.1

Let $0 < \lambda_1 < \lambda_2$. Then we have: $\sigma_{12} - \sigma_{23} < 0$, $\sigma_{23} - \sigma_{31} > 0$, and the sign of $\sigma_{31} - \sigma_{12}$ depends from the sign of $\lambda_1 - \lambda_2 + |\lambda_3|$. Therefore we shall consider two subcases

$$0 < \lambda_1 < \lambda_2 \le \lambda_1 + |\lambda_3|, \text{ and } 0 < \lambda_1 < \lambda_1 + |\lambda_3| \le \lambda_2.$$

At first subcase we have inequalities $\sigma_{12} \le \sigma_{31} < \sigma_{23}$, in second $\sigma_{31} < \sigma_{12} < \sigma_{23}$. Further for both subcases the following estimations are true

$$\sigma_{23} = \frac{1}{4}(\lambda_1 - |\lambda_3|)^2 - \lambda_1^2 + \frac{1}{4}\lambda_2^2 + \frac{1}{2}\lambda_2|\lambda_3| + \frac{1}{2}\lambda_1\lambda_2 \ge$$
$$\ge \frac{1}{4}(\lambda_1 - |\lambda_3|)^2 - \frac{1}{4}\lambda_1^2 + \frac{2}{4}\lambda_2|\lambda_3| = \frac{1}{4}|\lambda_3|^2 + \frac{1}{2}|\lambda_3|(\lambda_2 - \lambda_1) > 0.$$

Let us prove that $\sigma_{12} < 0$ for the first subcase and $\sigma_{31} < 0$ for the second. Really, if $0 < \lambda_1 < \lambda_2 \le |\lambda_3| + \lambda_1$, then

$$\sigma_{12} < \frac{1}{4}\lambda_1^2 + \frac{1}{4}\lambda_2^2 - \frac{1}{2}\lambda_2|\lambda_3| - \frac{3}{4}|\lambda_3|^2 - \frac{1}{2}\lambda_1|\lambda_3| - \frac{2}{4}\lambda_1^2 =$$
$$= \frac{1}{4}(\lambda_2 - \lambda_1)(\lambda_2 + \lambda_1) - \frac{1}{2}\lambda_2|\lambda_3| - \frac{1}{2}\lambda_1|\lambda_3| - \frac{3}{4}|\lambda_3|^2 <$$
$$< \frac{1}{4}\lambda_2|\lambda_3| + \frac{1}{4}\lambda_1|\lambda_3| - \frac{1}{2}\lambda_2|\lambda_3| - \frac{1}{2}\lambda_1|\lambda_3| - \frac{3}{4}|\lambda_3|^2 < 0.$$

For the case $0 < \lambda_1 < \lambda_1 + |\lambda_3| \le \lambda_2$ we have

$$\sigma_{31} = \frac{1}{4}\lambda_1^2 + \frac{1}{2}\lambda_1\lambda_2 - \frac{1}{4}\lambda_2^2 - \frac{1}{2}\lambda_2^2 + \frac{1}{2}|\lambda_3|\left(\frac{1}{2}|\lambda_3| + \lambda_1 - \lambda_2\right) =$$
$$= (\lambda_1 - \lambda_2)\left(\frac{1}{4}\lambda_1 + \frac{3}{4}\lambda_2\right) + \frac{1}{2}|\lambda_3|\left(\frac{1}{2}|\lambda_3| + \lambda_1 - \lambda_2\right) < 0.$$

Obviously that the case (b) of Theorem follows from the above estimations and Lemma 2.1.
(c) Let $0 < \lambda_1 = \lambda_2,\ \lambda_3 = 0$. Then we see that $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$, hence $K_{(\cdot,\cdot)}(\pi) = 0$.

If $0 < \lambda_1 < \lambda_2$, $\lambda_3 = 0$, then we have

$$\sigma_{12} = \frac{1}{4}\lambda_1^2 + \frac{1}{4}\lambda_2^2 - \frac{1}{2}\lambda_1\lambda_2 = \frac{1}{4}(\lambda_1 - \lambda_2)^2 > 0$$
$$\sigma_{23} = -\frac{3}{4}\lambda_1^2 + \frac{1}{4}\lambda_2^2 + \frac{1}{2}\lambda_1\lambda_2 = \frac{1}{4}(\lambda_1 + \lambda_2)^2 - \lambda_1^2$$
$$\sigma_{31} = \frac{1}{4}\lambda_1^2 + \frac{1}{2}\lambda_1\lambda_2 - \frac{3}{4}\lambda_2^2 = \frac{1}{4}(\lambda_1 + \lambda_2)^2 - \lambda_2^2 < 0$$
$$\sigma_{31} < \sigma_{12} < \sigma_{23}$$

and we use Lemma 2.1.
(d) In this case $0 < \lambda_1$, $\lambda_2 = -|\lambda_2| < 0$, $\lambda_3 = 0$, and we see that

$$\sigma_{12} - \sigma_{23} = \lambda_1(\lambda_1 + |\lambda_2|) > 0$$
$$\sigma_{23} - \sigma_{31} = (\lambda_1 + |\lambda_2|)(|\lambda_2| - \lambda_1)$$
$$\sigma_{31} - \sigma_{12} = -|\lambda_2|(\lambda_1 + |\lambda_2|) < 0$$
$$\sigma_{12} = \frac{1}{4}(\lambda_1 + |\lambda_2|)^2 > 0,\ \sigma_{23} = \frac{1}{4}(\lambda_1 + |\lambda_2|)(|\lambda_2| - 3\lambda_1),$$
$$\sigma_{31} = \frac{1}{4}(\lambda_1 + |\lambda_2|)(\lambda_1 - 3|\lambda_2|).$$

Let $0 < \lambda_1 \le |\lambda_2|$, then $\sigma_{31} \le \sigma_{23} < \sigma_{12}$, $\sigma_{31} < 0$, $\sigma_{12} > 0$. If $0 < |\lambda_2 \le |\lambda_1|$, then $\sigma_{23} \le \sigma_{31} < \sigma_{12}$, $\sigma_{23} < 0$, $\sigma_{12} > 0$.
(e) In this case $0 < \lambda_1,\ \lambda_2 = \lambda_3 = 0$. Therefore we have $\sigma_{23} = -\frac{3}{4}\lambda_1^2$, $\sigma_{12} = \sigma_{31} = \frac{1}{4}\lambda_1^2$. The assertion of our Theorem follows now from these facts and Lemma 2.1.
(f) Since $\lambda_1 = \lambda_2 = \lambda_3 = 0$, then $\sigma_{12} = \sigma_{23} = \sigma_{31} = 0$, and we see from Lemma 2.1 that $K_{(\cdot,\cdot)}(\pi) = 0$. □

Corollary. *In the case (a) of Theorem the set of left invariant Riemannian metrics of positive sectional curvature is defined by the following systems of iqualities*

$$\begin{cases} 0 < \sigma_{12}, \\ 0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \lambda_1 + \lambda_2, \end{cases} \qquad \begin{cases} 0 < \sigma_{12}, \\ 0 < \lambda_1 \le \lambda_2 < \lambda_1 + \lambda_2 \le \lambda_3, \end{cases}$$

with non trivial decision set. Besides this

$$\delta = \left(\min_{G_2} K_{(\cdot,\cdot)}(\pi) \right) \left(\max_{G_2} K_{(\cdot,\cdot)}(\pi) \right)^{-1} \in (0,1],$$

and $\delta = 1$ *if and only if* $\lambda_1 = \lambda_2 = \lambda_3$.

Proof. The first part of this assertion follows immediately from the Theorem. Further the function $\delta = \delta(\lambda_1, \lambda_2, \lambda_3)$ is continious with connected domain of definition, therefore $\delta \in (0,1]$. Finally, the condition $\delta = 1$ is equivalent to the $\sigma_{12} = \sigma_{23} = \sigma_{31}$, or $\lambda_1 = \lambda_2 = \lambda_3$. □

Remark 2. We note that for all cases besides the case (a), the sectional curvature either equal to zero, or negative, or non positive, or oscillates. Its corresponds to the Theorem of N.Wallach about left invariant Riemannian metrics of positive sectional curvature on simply connected Lie groups [2].

Theorem of Wallach. The 3 – sphere group $SU(2)$, consisting of 2×2 unitary matrices of determinant 1, is the only simply connected Lie group which admits a left invariant metric of strictly positiv sectional curvature.

Further, at first the study of left invariant Riemannian metrics with positive δ - pinched sectional curvature on Lie group $Spin(3)$ was done by F.M.Valiev [3].

Let us estimate the one dimensional curvature.

Lemma 2.2. *The following formulas are true*

$$\begin{cases} k_1 = \frac{1}{2}\left(\sigma_{12} + \sigma_{31} - \sigma_{23}\right), \\ k_2 = \frac{1}{2}\left(\sigma_{12} + \sigma_{23} - \sigma_{31}\right), \\ k_3 = \frac{1}{2}\left(\sigma_{31} + \sigma_{23} - \sigma_{12}\right) \end{cases} \qquad \begin{cases} k_1 - k_2 = \sigma_{31} - \sigma_{23}, \\ k_1 - k_3 = \sigma_{12} - \sigma_{23}, \\ k_2 - k_3 = \sigma_{12} - \sigma_{13} \end{cases}$$

Proof. We see that

$$\begin{cases} k_1 + k_2 = \sigma_{12} \\ k_2 + k_3 = \sigma_{23} \\ k_3 + k_1 = \sigma_{31} \end{cases}$$

The proof of our Lemma follows from the decision of this system. □

From Lemma 2.2 and Theorem 2.1 follows

Theorem 2.2. *Let* $A_{(\cdot,\cdot)}(\pi) : G_1 \to R$ *be one dimensional curvature of the Riemannian manifold* $(G, (\cdot,\cdot))$. *Then there are following estimations*

Case	*Signes* $(\lambda_1, \lambda_2, \lambda_3)$	*Lie algebra*	*One dimensional curvature*
(a)	$+,+,+$	$su(2)$	$k_1 \leq A_{(\cdot,\cdot)}(\pi) \leq k_3,$ $k_1 \leq k_2 \leq k_3,\ k_3 > 0$ *if* $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_1 + \lambda_2.$ $k_2 \leq A_{(\cdot,\cdot)}(\pi) \leq k_3,$ $k_2 \leq k_1 < k_3,\ k_3 > 0$ *if* $0 < \lambda_1 \leq \lambda_2 < \lambda_1 + \lambda_2 \leq \lambda_3.$
(b)	$+,+,-$	$sl(2,R)$	$k_1 = k_2 \leq A_{(\cdot,\cdot)}(\pi) \leq k_3,$ $k_1 < 0,\ k_3 > 0$ *if* $\lambda_1 = \lambda_2 > 0.$ $k_1 \leq A_{(\cdot,\cdot)}(\pi) \leq k_3,$ $k_1 < k_2 \leq k_3,\ k_1 < 0,\ k_3 > 0$ *if* $0 < \lambda_1 < \lambda_2 \leq \lambda_1 + \|\lambda_3\|.$ $k_1 \leq A_{(\cdot,\cdot)}(\pi) \leq k_2,$ $k_1 < k_3 < k_2,\ k_1 < 0,\ k_2 > 0$ *if* $0 < \lambda_1 < \lambda_1 + \|\lambda_3\| < \lambda_2.$
(c)	$+,+,0$	$e(2)$	$A_{(\cdot,\cdot)}(\pi) = 0,$ *if* $0 < \lambda_1 = \lambda_2.$ $k_1 \leq A_{(\cdot,\cdot)}(\pi) \leq k_2,$ $k_1 < k_3 < k_2,\ k_1 < 0,\ k_2 > 0$ *if* $0 < \lambda_1 < \lambda_2.$
(d)	$+,-,0$	$e(1,1)$	$k_3 \leq A_{(\cdot,\cdot)}(\pi) \leq k_2,$ $k_3 < k_1 \leq k_2,\ k_3 < 0,\ k_2 > 0$ *if* $0 < \lambda_1 \leq \|\lambda_2\|.$ $k_3 \leq A_{(\cdot,\cdot)}(\pi) \leq k_1,$ $k_3 < k_2 \leq k_1,\ k_3 < 0,\ k_1 > 0$ *if* $0 < \|\lambda_2\| \leq \lambda_1.$

Case	*Signes* $(\lambda_1, \lambda_2, \lambda_3)$	*Lie algebra*	*One dimensional curvature*
(e)	$+, 0, 0$	h	$k_2 \leq A_{(\cdot,\cdot)}(\pi) \leq k_1$, $k_2 = k_3 < k_1$, $k_2 < 0$, $k_1 > 0$.
(f)	$0, 0, 0$	R^3	$A_{(\cdot,\cdot)}(\pi) = 0$

Corollary. *In the case (a) of Theorem the set of left invariant Riemannian metrics of positive one dimentional curvature is defined by the following systems of inequalities*

$$\begin{cases} 0 < \sigma_{12} + \sigma_{31} - \sigma_{23}, \\ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_1 + \lambda_2, \end{cases} \qquad \begin{cases} 0 < \sigma_{12} - \sigma_{31} + \sigma_{23}, \\ 0 < \lambda_1 \leq \lambda_2 < \lambda_1 + \lambda_2 \leq \lambda_3, \end{cases}$$

Besides this $\Delta = \left(\min_{G_1} A_{(\cdot,\cdot)}(\pi)\right)\left(\max_{G_1} A_{(\cdot,\cdot)}(\pi)\right)^{-1} \in (0, 1]$, *and* $\Delta = 1$ *if and only if* $\lambda_1 = \lambda_2 = \lambda_3$.

Lemma 2.3. *The following formulas are true*

$$r_1 - r_2 = (\lambda_1 - \lambda_2)(\lambda_1 + \lambda_2 - \lambda_3)$$
$$r_2 - r_3 = (\lambda_2 - \lambda_3)(\lambda_2 + \lambda_3 - \lambda_1)$$
$$r_3 - r_1 = (\lambda_3 - \lambda_1)(\lambda_3 + \lambda_1 - \lambda_2)$$

The proof of our Lemma follows from formulas (1.2).

From Lemma 2.3, formulas (1.2), Courant-Fisher's Theorem [7] and the results of J. Milnor follows

Theorem 2.3. *Let* $r_{(\cdot,\cdot)}(\pi) : G_1 \to R$ *be the Ricci curvature of the Riemannian manifold* $(G, (\cdot,\cdot))$. *Then there are following estimations*

Case	*Signes* $(\lambda_1, \lambda_2, \lambda_3)$	*Lie algebra*	*Ricci curvature*
(a)	$+, +, +$	$su(2)$	$r_1 \leq r_{(\cdot,\cdot)}(\pi) \leq r_3$, $r_1 \leq r_2 \leq r_3$, $r_3 > 0$, $r_1 \geq 0$ *if* $0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_1 + \lambda_2$. $r_2 \leq r_{(\cdot,\cdot)}(\pi) \leq r_3$, $r_2 \leq r_1 \leq r_3$, $r_3 > 0$, $r_2 \leq 0$ *if* $0 < \lambda_1 \leq \lambda_2 < \lambda_1 + \lambda_2 \leq \lambda_3$.

Case	*Signes* $(\lambda_1, \lambda_2, \lambda_3)$	*Lie algebra*	*Ricci curvature*
(b)	$+, +, -$	$sl(2, R)$	$r_1 \leq r_{(\cdot,\cdot)}(\pi) \leq r_3$, $r_1 = r_2 < r_3$, $r_1 < 0$, $r_3 > 0$ *if* $0 < \lambda_1 = \lambda_2$. $r_1 \leq r_{(\cdot,\cdot)}(\pi) \leq r_3$, $r_1 < r_2 \leq r_3$, $r_1 < 0$, $r_3 \geq 0$ *if* $0 < \lambda_1 < \lambda_2 \leq \lambda_1 + \lvert\lambda_3\rvert$. $r_1 \leq r_{(\cdot,\cdot)}(\pi) \leq r_2$, $r_1 < r_3 < r_2$, $r_1 < 0$, $r_2 > 0$, *if* $0 < \lambda_1 < \lambda_1 + \lvert\lambda_3\rvert < \lambda_2$.
(c)	$+, +, 0$	$e(2)$	$r_{(\cdot,\cdot)}(\pi) = 0$, *if* $0 < \lambda_1 = \lambda_2$. $r_1 \leq r_{(\cdot,\cdot)}(\pi) \leq r_2$, $r_1 < r_3 < r_2$, $r_1 < 0$, $r_2 > 0$ *if* $0 < \lambda_1 < \lambda_2$.
(d)	$+, -, 0$	$e(1, 1)$	$r_3 \leq r_{(\cdot,\cdot)}(\pi) \leq r_1$, $r_1 = r_2 = 0$, $r_3 < 0$ *if* $0 < \lambda_1 = \lvert\lambda_2\rvert$. $r_3 \leq r_{(\cdot,\cdot)}(\pi) \leq r_2$, $r_3 < r_1 < r_2$, $r_3 < 0$, $r_2 > 0$ *if* $0 < \lambda_1 < \lvert\lambda_2\rvert$. $r_3 \leq r_{(\cdot,\cdot)}(\pi) \leq r_1$, $r_3 < r_2 < r_1$, $r_3 < 0$, $r_1 > 0$ *if* $0 < \lvert\lambda_2\rvert < \lambda_1$.
(e)	$+, 0, 0$	h	$r_2 \leq r_{(\cdot,\cdot)}(\pi) \leq r_1$, $k_2 = k_3 = -\frac{1}{2}\lambda_1^2 < 0$, $r_1 = \frac{1}{2}\lambda_1^2 > 0$.
(f)	$0, 0, 0$	R^3	$r_{(\cdot,\cdot)}(\pi) = 0$

Corollary. *Let $(G, (\cdot,\cdot))$ be three dimensional unimodular Lie group with left invariant Riemannian metric of the positive Ricci curvature. Then the Lie algebra g of Lie group G is an isomorphic to $su(2)$, and the set of left invariant Riemannian metrics with $r_{(\cdot,\cdot)}(\pi) > 0$ is defined by the system of inequalities*

$$\begin{cases} 0 < r_1, \\ 0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \lambda_1 + \lambda_2, \end{cases}$$

Besides this $d = \left(\min_{G_1} r_{(\cdot,\cdot)}(\pi)\right)\left(\max_{G_1} r_{(\cdot,\cdot)}(\pi)\right)^{-1} \in (0,1]$, and $d = 1$ if and only if $\lambda_1 = \lambda_2 = \lambda_3$.

Let us consider nonunimodular case. Then we have

Theorem 2.4. *Let $K_{(\cdot,\cdot)}(\pi) : G_2 \to R$ be the sectional curvature of the Riemannian manifold $(G, (\cdot,\cdot))$. Then there are following estimations*

$(a)\quad -1 - \xi^2 - \eta^2\xi^2 - 2\xi - 2\eta^2\xi \leq K_{(\cdot,\cdot)}(\pi) \leq -1 + 2\xi + 2\eta^2\xi - \eta^2\xi^2 - \xi^2,$
where $0 < \xi < 1$,

$(b)\quad -1 - \xi^2 - \eta^2\xi^2 - 2\xi - 2\eta^2\xi \leq K_{(\cdot,\cdot)}(\pi) \leq -1 + \eta^2\xi^2 + \xi^2,$
where $1 < \xi$.

Proof. As in unimodular case we have

$$\sigma_{12} - \sigma_{23} = -2\xi(1+\eta^2)(1+\xi)$$
$$\sigma_{23} - \sigma_{31} = 2\xi(1+\eta^2)(\xi - 1)$$
$$\sigma_{31} - \sigma_{12} = 4\xi(1+\eta^2)$$

The proof follows from the above formulas. □

Remark 3. The statements of the theorems for the estimations of one-dimensional curvature and Ricci curvature in nonunimodular case are analogous to unimodular case.

Remark 4. We note also that O.Kowalski and S.Nikcevic have determined the exact ranges of all triplets $(r_1, r_2, r_3) \in \mathbf{R^3}$ which can occur as Ricci eigenvalues of locally homogeneous (M^3, ρ) [8].

3. Schouten-Weil tensor

In this part we investigate the behavior of the Schouten-Weil tensor. In unimodular case using the notations of [1] (pp. 310-311) and formula (1.10) one can write all non-zero Christoffel's symbols

$$\begin{aligned}
&\Gamma^3_{12} = -\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 \quad &&\Gamma^2_{13} = -\frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_3 \\
&\Gamma^3_{21} = -\frac{1}{2}\lambda_3 + \frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_1 \quad &&\Gamma^1_{23} = \frac{1}{2}\lambda_1 - \frac{1}{2}\lambda_2 + \frac{1}{2}\lambda_3 \\
&\Gamma^2_{31} = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2 - \frac{1}{2}\lambda_3 \quad &&\Gamma^1_{32} = -\frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_3 - \frac{1}{2}\lambda_2
\end{aligned} \tag{3.1}$$

Correspondingly all nontrivial components of the Schouten-Weil tensor are equal

$$
\begin{aligned}
&S_{123} = -{\lambda_1}^3 + \frac{1}{2}{\lambda_2}^3 + \frac{1}{2}{\lambda_3}^3 + \frac{1}{2}{\lambda_1}^2\lambda_2 + \frac{1}{2}{\lambda_1}^2\lambda_3 - \frac{1}{2}{\lambda_2}^2\lambda_3 - \frac{1}{2}{\lambda_3}^2\lambda_2,\\
&S_{231} = \frac{1}{2}{\lambda_1}^3 - {\lambda_2}^3 + \frac{1}{2}{\lambda_3}^3 - \frac{1}{2}{\lambda_1}^2\lambda_3 - \frac{1}{2}\lambda_1{\lambda_3}^2 + \frac{1}{2}{\lambda_2}^2\lambda_1 + \frac{1}{2}{\lambda_2}^2\lambda_3,\\
&S_{312} = \frac{1}{2}{\lambda_1}^3 + \frac{1}{2}{\lambda_2}^3 - {\lambda_3}^3 - \frac{1}{2}{\lambda_1}^2\lambda_2 + \frac{1}{2}\lambda_1{\lambda_3}^2 - \frac{1}{2}{\lambda_2}^2\lambda_1 + \frac{1}{2}{\lambda_3}^2\lambda_2,\\
&S_{132} = -S_{123}, \quad S_{213} = -S_{231}, \quad S_{321} = -S_{312}.
\end{aligned} \tag{3.2}
$$

From this we have

Theorem 3.1. *The components of the Schouten-Weil tensor are equal to zero if and only if one from the following conditions occurs either*

$$\{\lambda_1 = \lambda_2 = \lambda_3\}, \ \textit{or}$$
$$\{\lambda_i = 0,\, \lambda_j = \lambda_k\}, \ \textit{if } i \neq j,\, j \neq k,\, k \neq \imath.$$

Let us consider nonunimodular case. Then all nonzero Christoffel's symbols have the form

$$
\begin{aligned}
&\Gamma^3_{12} = \eta, \quad \Gamma^2_{13} = -\eta, \quad \Gamma^2_{21} = -1-\xi, \quad \Gamma^3_{21} = -\eta\,\xi, \quad \Gamma^1_{22} = 1+\xi,\\
&\Gamma^1_{23} = \eta\,\xi, \quad \Gamma^2_{31} = -\eta\,\xi, \quad \Gamma^3_{31} = -1+\xi, \quad \Gamma^1_{32} = \eta\,\xi, \quad \Gamma^1_{33} = 1-\xi.
\end{aligned} \tag{3.3}
$$

Correspondingly all nontrivial components of the Schouten-Weil tensor have the form

$$
\begin{aligned}
&S_{123} = 4\,\eta\,\xi^2 + 4\,\eta^3\,\xi^2,\\
&S_{212} = 2\,\xi + 2\,\eta^2\,\xi - 2\,\eta^2\,\xi^3 - 2\,\xi^3,\\
&S_{231} = 4\,\eta\,\xi + 4\,\eta^3\,\xi - 2\,\eta^3\,\xi^2 - 2\,\eta\,\xi^2 + 2\,\eta^3\,\xi^3 + 2\,\eta\,\xi^3,\\
&S_{312} = -4\,\eta\,\xi - 2\,\eta\,\xi^2 - 2\,\eta^3\,\xi^2 - 2\,\eta^3\,\xi^3 - 2\,\eta\,\xi^3 - 4\,\eta^3\,\xi,\\
&S_{331} = 2\,\xi + 2\,\eta^2\,\xi - 2\,\eta^2\,\xi^3 - 2\,\xi^3,\\
&S_{132} = -S_{123},\ S_{221} = -S_{212},\ S_{213} = -S_{231},\ S_{321} = -S_{312},\ S_{313} = -S_{331}.
\end{aligned} \tag{3.4}
$$

From this we have

Theorem 3.2. *The components of the Schouten-Weil tensor are equal to zero if and only if one from the following conditions occurs either*

$$\{\eta = 0,\, \xi = 1\}, \ \textit{or} \ \{\xi = 0,\, \eta \geq 0\}.$$

Remark 5. We can notice that D.V. Alekseevskii and B.N. Kimelfeld obtained a classification of all conformall flat homogeneous Riemannian manifolds [5]. That is why theorems 3.1, 3.2 one can consider as a special cases of the Alekseevski-Kimelfeld theorem.

4. One-parameter conformal flat subgroups

In this section all conformal flat one-parameter subgroups will be found. Let $\tau = x^1 e_1 + x^2 e_2 + x^3 e_3$, $(x^1)^2 + (x^2)^2 + (x^3)^2 = 1$ be a unit tangent vector of g generating an one-parameter subgroup of G.

Using left invariance of considered metrics and the formula (1.11) we have $\frac{D\tau}{dt} = \Gamma^i_{jk}x^k x^j e_i$, $\frac{D^2\tau}{(dt)^2} = \Gamma^s_{ip}\Gamma^i_{jk}x^k x^j x^p e_s$, and equations of conformal flat curves have the form of the algebraic system

$$\Gamma^s_{ip}\Gamma^i_{jk}x^k x^j x^p + A_{sp}x^p = cx^s, \quad s = 1, 2, 3.$$

After substitution and simplification we obtain three equations

$$(-\frac{1}{2}\,{x_3}^2 - \frac{1}{2}\,{x_2}^2 + \frac{5}{8})\,{\lambda_1}^2 + ({x_2}^2 - \frac{1}{4} - \frac{1}{2}\,{x_3}^2)\,\lambda_1\,\lambda_2 + ({x_3}^2 - \frac{1}{2}\,{x_2}^2 - \frac{1}{4})\,\lambda_1\,\lambda_3 +$$
$$+(-\frac{3}{8} - \frac{1}{2}\,{x_2}^2)\,{\lambda_2}^2 + (\frac{1}{2}\,{x_3}^2 + \frac{1}{2}\,{x_2}^2 + \frac{3}{4})\,\lambda_2\,\lambda_3 + (-\frac{1}{2}\,{x_3}^2 - \frac{3}{8})\,{\lambda_3}^2 = C,$$

$$(-\frac{1}{2}\,{x_1}^2 - \frac{3}{8})\,{\lambda_1}^2 + ({x_1}^2 - \frac{1}{4} - \frac{1}{2}\,{x_3}^2)\,\lambda_1\,\lambda_2 + (\frac{1}{2}\,{x_3}^2 + \frac{1}{2}\,{x_1}^2 + \frac{3}{4})\,\lambda_1\,\lambda_3 +$$
$$+(-\frac{1}{2}\,{x_1}^2 - \frac{1}{2}\,{x_3}^2 + \frac{5}{8})\,{\lambda_2}^2 + (-\frac{1}{4} + {x_3}^2 - \frac{1}{2}\,{x_1}^2)\,\lambda_2\,\lambda_3 + (-\frac{1}{2}\,{x_3}^2 - \frac{3}{8})\,{\lambda_3}^2 = C,$$

$$(-\frac{1}{2}\,{x_1}^2 - \frac{3}{8})\,{\lambda_1}^2 + (\frac{3}{4} + \frac{1}{2}\,{x_2}^2 + \frac{1}{2}\,{x_1}^2)\,\lambda_1\,\lambda_2 + (-\frac{1}{2}\,{x_2}^2 + {x_1}^2 - \frac{1}{4})\,\lambda_1\,\lambda_3 +$$
$$+(-\frac{3}{8} - \frac{1}{2}\,{x_2}^2)\,{\lambda_2}^2 + ({x_2}^2 - \frac{1}{2}\,{x_1}^2 - \frac{1}{4})\,\lambda_2\,\lambda_3 + (-\frac{1}{2}\,{x_1}^2 - \frac{1}{2}\,{x_2}^2 + \frac{5}{8})\,{\lambda_3}^2 = C,$$

From this we have

Theorem 4.1. *Real solutions of this system exist if and only if the structure constans are either*

(1) $\lambda_1 = \lambda_2 = \lambda_3$ *and an arbitrary direction generates one-parameter conformal flat subgroup of G, or*
(2) $\lambda_1 = \lambda_2 < \lambda_3$, $2\lambda_1 < \lambda_3$ *and directions generating one-parameter conformal flat subgroups of G form the cone*
$$(x^1)^2 + (x^2)^2 = 1 - \frac{\lambda_3}{2(\lambda_3 - \lambda_1)}, \quad (x^3)^2 = \frac{\lambda_3}{2(\lambda_3 - \lambda_1)},$$
or
(3) $\lambda_1 < \lambda_2 = \lambda_3$, $3\lambda_1 < 2\lambda_3$, *and directions generating one-parameter conformal flat subgroups of G form the cone*
$$(x^2)^2 + (x^3)^2 = 1 - \frac{\lambda_1}{2(\lambda_2 - \lambda_1)}, \quad (x^1)^2 = \frac{\lambda_1}{2(\lambda_2 - \lambda_1)}.$$

For nonunimodular case we have analogous system

$$(-{x_2}^2\,x_1 - {x_3}^2\,x_1 - \frac{3}{2}\,x_1)\,\eta^2\,\xi^2 + (-{x_2}^2\,x_1 - {x_3}^2\,x_1 - \frac{3}{2}\,x_1)\,\xi^2 +$$
$$(-2\,{x_2}^2\,x_1 + 2\,{x_3}^2\,x_1)\,\xi + (-{x_3}^2\,x_1 + {x_2}^2\,x_1)\,\eta^2\,\xi$$
$$-6\,x_3\,x_1\,x_2\,\eta\,\xi - {x_2}^2\,x_1 - {x_3}^2\,x_1 - \frac{1}{2}\,x_1 = Cx_1,$$

$$(-2\,{x_3}^2\,x_2+\frac{1}{2}\,x_2)\,\eta^2\,\xi^2+({x_3}^3-3\,x_3\,{x_2}^2)\,\eta\,\xi^2+(-{x_2}^3+\frac{1}{2}\,x_2+$$
$${x_3}^2\,x_2)\,\xi^2+(x_2\,{x_1}^2-2\,x_2)\,\eta^2\,\xi+(-3\,x_3\,{x_2}^2-x_3\,{x_1}^2-{x_3}^3)\,\eta\,\xi+$$
$$(-2\,{x_2}^3-2\,x_2)\,\xi-{x_2}^3+\eta\,x_3\,{x_1}^2-\eta^2\,x_2\,{x_1}^2-\frac{1}{2}\,x_2-{x_3}^2\,x_2=Cx_2,$$

$$(-2\,x_3\,{x_2}^2+\frac{1}{2}\,x_3)\,\eta^2\,\xi^2+(-{x_2}^3+3\,{x_3}^2\,x_2)\,\eta\,\xi^2+$$
$$(\frac{1}{2}\,x_3+x_3\,{x_2}^2-{x_3}^3)\,\xi^2+(-x_3\,{x_1}^2+2\,x_3)\,\eta^2\,\xi$$
$$+(-x_2\,{x_1}^2-3\,{x_3}^2\,x_2-{x_2}^3)\,\eta\,\xi+(2\,{x_3}^3+2\,x_3)\,\xi-$$
$$\eta\,x_2\,{x_1}^2-x_3\,{x_2}^2-\eta^2\,x_3\,{x_1}^2-\frac{1}{2}\,x_3-{x_3}^3=Cx_3.$$

From this we have

Theorem 4.2. *The solution of the above system has the form:*

$$x_1=\pm\sqrt{\frac{3\xi\sqrt{1+\eta^2}}{\xi\sqrt{1+\eta^2}+\eta}},\quad x_2=\sqrt{-\frac{(\sqrt{1+\eta^2}-\eta)(\xi\sqrt{1+\eta^2}-\eta/2)}{\xi(1+\eta^2)+\eta\sqrt{1+\eta^2}}},$$
$$x_3=\sqrt{-\frac{\xi\sqrt{1+\eta^2}-\eta/2}{(\xi(1+\eta^2)+\eta\sqrt{1+\eta^2})(\sqrt{1+\eta^2}-\eta)}}.$$

Corollary. *If $\xi=1$, $\eta=0$ then there is no real directions generating one-parameter conformal flat subgroups of G*

If $\xi=0$, then directions generating one-parameter conformal flat subgroups of G are parallel to the plane $x^1=0$.

bf Remark 6. It can be note that for some cases G is conformal flat but all one-parameters subgroups of G are not conformal flat (see Theorems 3.2., 4.2.).

Remark 7. We note that all the calculations concerning sections three and four have been made with the help of the Maple system for analytical calculations.

References

[1] Milnor, J., *Curvature of left invariant metric on Lie groups*, Advances in mathematics 21, 1976, 293-329.

[2] Wallach, N., *Compact homogeneous Riemannian manifolds with stricly positive curvature*, Ann. Math. 96(1972), 277-295.

[3] Valiev, F.M., *Sectional curvature formula of left invariant Riemannian metrics on Spin*(3), Preprint of IM SB RAS, 1977, 9 pp.

[4] Rashevskii, P.K., *Riemannian geometry and analysis*, M., 1964, 663 p.

[5] Alekseevskii, D.V., Kimelfeld, B.N., *Classification of conformal flat homogeneous Riemannian spaces*, Mat. Zametki, 1978, V.24, No 1, 103-110.

[6] Slavskii, V.V., *On a theorem of Fermi*, Commentat. Math. Univ. Carol. 37, No.4, 867-872 (1996).
[7] Horn, R., Jonson, Ch., *Matrix analysis*, M. 1989, 655 pp.
[8] Kowalski, O., Nikcevic, S., *On Ricci eigenvalues of locally homogeneous Riemann 3-manifolds*, Geom. Dedicata 62, No.1 (1996), 65-72.

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