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LOCALLY SYMMETRIC SPACE STRUCTURES ON THE TANGENT BUNDLE

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ABSTRACT. One studies the conditions under which the tangent bundle TM of a Riemannian manifold (M, g), or a tube around the zero section in TM, endowed with an appropriate metric G, is a locally symmetric space. Locally, the matrix of G is obtained as a linear combination of the matrix associated to g and a matrix of rank one, the coefficients depending on the energy density only. The base manifold (M, g) must be a space form and (TM, G) must have a structure of Kähler Einstein manifold.

Introduction

The tangent bundle $\tau : TM \longrightarrow M$ of a Riemannian manifold (M, g) can be organized as an almost Kaehlerian manifold (see [3], [20]) by using the Sasaki metric and an almost complex structure defined by the splitting of the tangent bundle to TM into the vertical and horizontal distributions VTM, HTM (the last one being determined by the Levi Civita connection on M) (see also [17], [19], [8]).

A slight generalization of the Sasaki metric is a special natural lift G of g to TM considered by the first author in [14] (for the definition and the expression of the natural lifts of g to TM see [9], [10], [5], [6], [7]). He has also considered an almost complex structure J on TM, related to the metric G and has studied the conditions under which (TM, G, J) is a Kähler Einstein manifold.

The purpose of the present paper is to study the conditions under which the Riemannian manifold (TM, G) or a tube around the zero section in the Riemannian manifold (TM, G) is a locally symetric manifold, without using the almost complex structure J. Roughly speaking, if (TM, G) or a tube around the zero section in TM is a locally symmetric space, then (M, g) must have constant sectional curvature c and TM must carry a Kähler-Einstein structure. More precisely, in Theorem 5

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we study the general case when (TM, G) or a tube around the zero section in TM is a locally symmetric space. In this case (M, g) has constant sectional curvature and TM carries an integrable almost complex structure J such that (TM, G, J) is a Kähler manifold of constant holomorphic sectional curvature. In Theorem 6 the expression of G on TM is simpler (the functions u, v are constant). However, if in this case (TM, G) (or a tube around the zero section in TM) is locally symmetric then (M, g) has constant sectional curvature and we obtain only a Kähler Einstein structure on TM (on a tube around the zero section in TM). In a special case we obtain in Theorem 7, for an (M, g) of positive constant sectional curvature c, a Kähler Einstein structure in a tube $T_0^A M$ consisting of nonzero tangent vectors y to M such that $0 < g(y, y) < \frac{c}{A^2}$.

The manifolds, tensor fields and geometric objects we consider in this paper, are assumed to be differentiable of class C^∞ (i.e. smooth). We use the computations in local coordinates in a fixed local chart, but many results from this paper may be expressed in an invariant form. The well known summation convention is used throughout this paper, the range for the indices i, j, k, l, h, s, r being always $\{1, ..., n\}$. We shall denote by $\Gamma(TM)$ the module of smooth vector fields on TM.

1. A Riemannian structure on the tangent bundle

In this section we present some results obtained in [13], [14], [15], [16]. Let (M, g) be a smooth $n-$dimensional Riemannian manifold and denote its tangent bundle by $\tau : TM \longrightarrow M$. Recall that TM has a structure of $2n-$dimensional smooth manifold induced from the smooth manifold structure of M. A local chart $(U, \varphi) = (U, x^1, ..., x^n)$ on M induces a local chart $(\tau^{-1}(U), \Phi) = (\tau^{-1}(U), x^1, \ldots, x^n, y^1, ..., y^n)$ on TM where the local coordinates x^i, y^i, $i = 1, ..., n$ are defined as follows. The first n local coordinates $x^i = x^i \circ \tau$, $i = 1, ..., n$ on TM are the local coordinates in the local chart (U, φ) of the base point of a tangent vector from $\tau^{-1}(U)$. The last n local coordinates y^i, $i = 1, ..., n$ are the vector space coordinates of the same tangent vector, with respect to the natural local basis in the corresponding tangent space, defined by the local chart (U, φ).

This special structure of TM allows us to introduce the notion of $M-$ tensor field on it (see [12] for detailed explanations). An $M-$tensor field of type (p, q) on TM is defined by sets of n^{p+q} components (functions of x and y) with p upper indices and q lower indices assigned to induced local charts $(\tau^{-1}(U), \Phi)$ on TM, such that the change rule is that of the local coordinate components of a usual tensor field of type (p, q) on the base manifold, when a change of local charts on the base manifold is performed. E.g., the components y^i, assigned to $(\tau^{-1}(U), \Phi)$, define an $M-$tensor field of type $(1, 0)$ on TM. The local coordinate components of an ordinary tensor field T on the base manifold M defines an $M-$tensor field on TM, having the same type as T. E.g., the local coordinate components $g_{ij}(x)$ of g define an $M-$tensor field of type $(0, 2)$ on TM. The local coordinate components δ^i_j of the identity tensor field I on M define an $M-$tensor field of type $(1, 1)$ on TM.

The tangent bundle TM of a Riemannian manifold (M, g) can be organized as a Riemannian or a pseudo-Riemannian manifold in many ways. The most known such structures are given by the Sasaki metric on TM defined by g (see [17], [3]) and the complete lift type pseudo-Riemannian metric defined by g (see [19], [20]). Recall that the Levi Civita connection $\dot{\nabla}$ of g defines a direct sum decomposition

$$TTM = HTM \oplus VTM \tag{1}$$

of the tangent bundle to TM into the horizontal distribution HTM and the vertical distribution $VTM = Ker\ \tau_*$. The set of vector fields $(\frac{\partial}{\partial y^1},\ ...,\frac{\partial}{\partial y^n})$ defines a local frame field for VTM and for HTM we have the local frame field $(\frac{\delta}{\delta x^1},...,\ \frac{\delta}{\delta x^n})$, where

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - \Gamma^h_{i0}\ \frac{\partial}{\partial y^h},\ \ \Gamma^h_{i0} = \Gamma^h_{ik} y^k$$

and $\Gamma^h_{ik}(x)$ are the Christoffel symbols defined by the Riemannian metric g. Remark that

$$(\frac{\partial}{\partial x^i})^V = \frac{\partial}{\partial y^i},\ (\frac{\partial}{\partial x^i})^H = \frac{\delta}{\delta x^i},$$

where X^V,X^H denote the vertical and horizontal lifts of the vector field X on M.

The distributions VTM and HTM are isomorphic to each other. Consider the energy density (kinetic energy)

$$t = \frac{1}{2}\|y\|^2 = \frac{1}{2}g_{\tau(y)}(y,y) = \frac{1}{2}g_{ik}(x)y^i y^k;\ \ y \in \tau^{-1}(U) \tag{2}$$

of the tangent vector y, defined by the Riemannian metric g of M, where g_{ik} are the components of g in the local chart (U,φ). Let $u, v : [0,\infty) \longrightarrow R$ be two real smooth functions. We shall assume that the functions u, v satisfy the conditions $u(t) > 0, u(t) + 2tv(t) > 0$ for all $t \in [0,\infty)$. Then we may consider the following symmetric $M-$tensor field of type (0,2) on TM, defined by the components (see [14], [15], [13])

$$G_{ij} = u(t)g_{ij} + v(t)g_{0i}\ g_{0j}, \tag{3}$$

where $g_{0i} = g_{hi}y^h$. The matrix (G_{ij}) is symmetric and the above conditions concerning the functions $u(t), v(t)$ assure that the matrix (G_{ij}) is positive (see [14]). The inverse of the matrix (G_{ij}) has the entries

$$H^{kl} = \frac{1}{u}g^{kl} + wy^k y^l, \tag{4}$$

where g^{kl} are the components of the inverse of the matrix (g_{ij}) and

$$w = w(t) = -\frac{v}{u(u+2tv)}. \tag{5}$$

The components $H^{kl}(x,y)$ define a symmetric $M-$tensor field of type $(2,0)$ on TM. We shall also use the components $H_{ij}(x,y)$ of a symmetric $M-$tensor field of type $(0,2)$ obtained from the components H^{kl} by "lowering" the indices

$$H_{ij} = g_{ik}H^{kl}g_{lj} = \frac{1}{u}g_{ij} + w\ g_{0i}g_{0j}, \tag{6}$$

as well as the following $M-$tensor fields on TM

$$G^{kl} = g^{ki}G_{ij}g^{jl} = ug^{kl} + vy^k y^l \quad , \quad G^i_k = G^{ih}g_{hk} = G_{kh}g^{hi} = \tag{7}$$

$$= u\delta^i_k + v\ y^i g_{0k} \quad , \quad H^i_k = H^{ih}g_{hk} = H_{kh}g^{hi} = \frac{1}{u}\delta^i_k + w\ y^i g_{0k}.$$

Remark that the matrix (H^i_k) is the inverse of the matrix (G^i_k).

The following Riemannian metric may be considered on TM

$$G = G_{ij}dx^i dx^j + H_{ij}\dot{\nabla} y^i \dot{\nabla} y^j, \tag{8}$$

where $\dot{\nabla} y^i = dy^i + \Gamma^i_{j0}dx^j$ is the absolute differential of y^i with respect to the Levi Civita connection $\dot{\nabla}$ of g. Equivalently, we have

$$G(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G_{ij},\ G(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = H_{ij},\ G(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = \ G(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}) = 0.$$

Remark that HTM, VTM are orthogonal to each other with respect to G but are not isometric vector bundles on TM, so the considered metric G on TM is no longer a metric of Sasaki type. Remark also that the system of 1-forms $(dx^1, ..., dx^n, \dot{\nabla} y^1, ..., \dot{\nabla} y^n)$ defines a local frame of T^*TM, dual to the local frame $(\frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n}, \ \frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n})$ adapted to the direct sum decomposition (1).

The Levi Civita connection of the metric G has been obtained in [15], (see also [14]).

Theorem 1. *The Levi Civita connection ∇ of G has the following expression in the local adapted frame $(\frac{\partial}{\partial y^1}, ..., \frac{\partial}{\partial y^n}, \ \frac{\delta}{\delta x^1}, ..., \frac{\delta}{\delta x^n})$*

$$\begin{cases} \nabla_{\frac{\partial}{\partial y^i}} \frac{\partial}{\partial y^j} = Q^h_{ij}\frac{\partial}{\partial y^h} \ , \ \nabla_{\frac{\delta}{\delta x^i}} \frac{\partial}{\partial y^j} = \Gamma^h_{ij}\frac{\partial}{\partial y^h} + P^h_{ji}\frac{\delta}{\delta x^h} \\ \nabla_{\frac{\partial}{\partial y^i}} \frac{\delta}{\delta x^j} = P^h_{ij}\frac{\delta}{\delta x^h} \ , \ \nabla_{\frac{\delta}{\delta x^i}} \frac{\delta}{\delta x^j} = \ \Gamma^h_{ij}\frac{\delta}{\delta x^h} + S^h_{ij}\frac{\partial}{\partial y^h}, \end{cases} \tag{9}$$

where the $M-$tensor fields P^h_{ij}, Q^h_{ij}, S^h_{ij} are given by

$$\begin{cases} P^h_{ij} = \frac{1}{2}(\frac{\partial G_{jk}}{\partial y^i} + H_{il}R^l_{0jk})H^{kh} \ , \ S^h_{ij} = \frac{1}{2}(-R^h_{0ij} - \ \frac{\partial G_{ij}}{\partial y^k}G^{kh}) \\ Q^h_{ij} = \frac{1}{2}G^{hk}(\frac{\partial H_{jk}}{\partial y^i} + \ \frac{\partial H_{ik}}{\partial y^j} - \frac{\partial H_{ij}}{\partial y^k}), \end{cases} \tag{10}$$

R^h_{ijk} are the components of the curvature tensor field of $\dot{\nabla}$ and $R^h_{0jk} = y^i R^h_{ijk}$.

The curvature tensor field K of the connection ∇ is defined by the well known formula

$$K(X,Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,Y]}Z \ , \ \ X, Y, Z \in \Gamma(TM).$$

Its formal expression in the local frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, $i = 1, \dots, n$, is given by (see [14])

$$
(11) \quad \begin{cases}
K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = XXX^h_{kij}\frac{\delta}{\delta x^h} + \frac{1}{2}y^r\dot{\nabla}_k R^h_{rij}\frac{\partial}{\partial y^h}, \\
K(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = XXY^h_{kij}\frac{\partial}{\partial y^h} + \frac{1}{2u}y^r\dot{\nabla}_l R_{rkij}H^{lh}\frac{\delta}{\delta x^h}, \\
K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\delta}{\delta x^k} = YYX^h_{kij}\frac{\delta}{\delta x^h}\ , \ K(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = YYY^h_{kij}\frac{\partial}{\partial y^h}, \\
K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = YXX^h_{kij}\frac{\partial}{\partial y^h} + \frac{1}{2u}y^r\dot{\nabla}_j R_{rikl}H^{lh}\frac{\delta}{\delta x^h}, \\
K(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j})\frac{\partial}{\partial y^k} = YXY^h_{kij}\frac{\delta}{\delta x^h},
\end{cases}
$$

where the components $XXX^h_{kij}, \dots$ define $M-$tensor fields on TM and are expressed with the help of the curvature tensor field R of $\dot{\nabla}$ and the $M-$tensor fields defined by P^h_{ij}, Q^h_{ij}, S^h_{ij}.

2. The locally symmetric space structure of (TM, G)

The condition for a Riemannian manifold (M, g) to be locally symmetric can be expressed by the property $\dot{\nabla}R = 0$ where $\dot{\nabla}$ is the Levi Civita connection of g and R is the curvature tensor field of $\dot{\nabla}$ (see e.g. [4],[18]). We shall study the conditions under which the Riemannian manifold (TM, G) is a locally symmetric space i.e.

$$
(12) \qquad \nabla K = 0.
$$

Using the expressions of the components of K given by (11) we obtain a quite complicate system consisting of 24 equations. However some of these equations become trivial if we obtain some properties of the base manifold, from some specific equations involved in the condition $\nabla K = 0$.

Theorem 2. *If the Riemannian manifold (TM, G) is a locally symmetric space then the base Riemannian manifold (M, g) is also a locally symmetric space.*

Proof. Consider the equation $(\nabla_{\frac{\partial}{\partial y^l}} K)(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j})\frac{\delta}{\delta x^k} = 0$ and use the expressions (11) and (9) in order to obtain (from the coefficient of $\frac{\partial}{\partial y^h}$) the relation

$$
\frac{1}{2}\dot{\nabla}_k R^h_{lij} + y^r B^h_{rklij} = 0,
$$

where B^r_{rklij} are the components of a certain $M-$tensor field of type (1,5) on TM defined by ∇ and K. Taking $y^r = 0$ in this relation, it follows that (M, g) is locally symmetric. If we assume that the base Riemannian manifold (M, g) is an irreducible locally symmetric space, then it is an Einstein manifold, hence we have

$$
R^k_{ikj} = \kappa g_{ij},
$$

where κ is a constant for dim $M > 2$.

From now on we shall assume that (M,g) is an Einstein manifold.

Remark. From the above result we obtain simpler expressions for the components from (11) of K in the local frame $(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i})$, $i=1,...,n$.

Introduce the real valued smooth function $z : R_+ \longrightarrow R$ given by

$$z = z(t) = 2u(u+2tv)u'' - u(u')^2 + 4u'(u-tu')v - 2u(u-tu')v'.$$

Theorem 3. *Assume that (TM,G) is a locally symmetric space. Then the functions u, v must satisfy the relation*

$$(u - tu')z = 0. \tag{13}$$

Proof. We begin with the equation $(\nabla_{\frac{\partial}{\partial y^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$. Since in the explicite expression of the component YYY^h_{kij} from (11) there are not involved the components of the curvature tensor R of ∇ and since the coefficients Q^h_{ij} are, again, independent of R, we obtain, after a straightforward but long computation

$$(\nabla_{\frac{\partial}{\partial y^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = \{\frac{(u-tu')z}{4u^4}(\delta^h_l g_{jk}g_{0i} - \delta^h_j g_{kl}g_{0i}+$$

$$+\delta^h_i g_{kl}g_{0j} - \delta^h_l g_{ik}g_{0j} + \delta^h_i g_{jl}g_{0k} - \delta^h_j g_{il}g_{0k} + \delta^h_i g_{jk}g_{0l} - \delta^h_j g_{ik}g_{0l})+$$

$$+\frac{u^2z' + z(4tu'v - 4uv - uu')}{4u^4(u+2tv)}(\delta^h_i g_{0j} - \delta^h_j g_{0i})g_{0k}g_{0l}+$$

$$+\frac{z(u-tu')(u+2tv)}{4u^5}(g_{il}g_{jk} - g_{ik}g_{jl})y^h + \frac{zv(u-tu')}{4u^5}(g_{jl}g_{0i} - g_{il}g_{0j})g_{0k}y^h+$$

$$+\frac{u^2z' + z(tu'v - uv - uu')}{4u^5}(g_{jk}g_{0i} - g_{ik}g_{0j})g_{0l}y^h\}\frac{\partial}{\partial y^h}.$$

Thus the condition $(\nabla_{\frac{\partial}{\partial y^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$ implies the relation (13) fulfilled by the functions u, v and their derivatives.

Remark. The expression of $(\nabla_{\frac{\partial}{\partial y^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k}$ has also been obtained by using the RICCI program for doing tensor calculations with Mathematica, elaborated by John D. Lee (see [11]).

Remark. If we consider in the relation $z = 0$, v as an unknown function, we can obtain its expression by solving a linear first order differential equation. Thus, assuming that $u - tu' \neq 0$, we obtain

$$v = \frac{2Au^2 + 2u^2u' - tu(u')^2}{2(u-tu')^2}, \tag{14}$$

where A is an arbitrary real constant (However, from the condition $u + 2tv > 0$, it follows that $u + 2At > 0$).

We shall now study the equation $(\nabla_{\frac{\delta}{\delta x^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$. After some quite complicate computations made by using RICCI we have obtained an expression of the type

$$(\nabla_{\frac{\delta}{\delta x^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = DXYYY^h_{lkij}\frac{\delta}{\delta x^h},$$

where $DXYYY^h_{lkij}$ are the components of a certain $M-$tensor field of type (1,4), whose (quite complicate) expression shall be used in order to obtain some conditions fulfilled by the base manifold in the case where (TM, G) is a locally symmetric space.

Theorem 4. *If $u(0) \neq 0$, and*

$$[(n-3)Au(0) - \kappa][(3n+1)Au(0) - \kappa] \neq 0, \tag{15}$$

then the Riemannian manifold (TM, G) is a locally symmetric space only if (M, g) has constant sectional curvature.

Proof. We shall denote, for convenience, $u(0) = u_0,\ u'(0) = u'_0,\ v(0) = v_0$. From the condition $(\frac{\partial}{\partial y^l} DXYYY^h_{khij})_{|y^r=0} = 0$ we obtain

$$\frac{(n-4)u_0u'_0v_0 - (n-1)u_0v_0^2 + 3u_0(u'_0)^2 - (u'_0 - v_0)\kappa}{4u_0^3}(g_{il}g_{jk} - g_{ik}g_{jl}) + \tag{16}$$

$$+\frac{(n-3)u_0v_0 + 5u_0u'_0 - \kappa}{4u_0^4}R_{klij} - \frac{1}{4u_0^4}R_{hrij}R^{hr}_{kl} = 0.$$

On the other hand, from the condition $(\frac{\partial}{\partial y^l} DXYYY^h_{hkij})_{|y^r=0} = 0$ we obtain

$$[3nu_0^2(u'_0)^2 - (2n-8)u_0^2u'_0v_0 - 6u_0^2v_0^2](g_{il}g_{jk} - g_{ik}g_{jl}) - \tag{17}$$

$$-2R_{hrkl}R^{hr}_{ij} + R_{hrlj}R^{hr}_{ik} - R_{hrli}R^{hr}_{jk} = 0.$$

Eliminating the products of type $R_{hrij}R^{hr}_{kl}$ from (16) and (17) we obtain

$$3(\kappa - 5u_0u'_0 + 3u_0v_0 - nu_0v_0)R_{ijkl} + (9u_0^2v_0^2 - 3nu_0^2v_0^2 + 3\kappa u_0v_0 - \tag{18}$$

$$-3\kappa u_0u'_0 + 9u_0^2(u'_0)^2 - 3nu_0^2(u'_0)^2 - 20u_0^2u'_0v_0 + 5nu_0^2u'_0v_0)(g_{ik}g_{jl} - g_{il}g_{jk}) = 0.$$

Taking into account the expression of v_0, obtained from (14), we get

$$3(\kappa + 3Au_0 - nAu_0 - 2u_0u'_0 - nu_0u'_0)R_{ijkl} + (3A\kappa u_0 + 9A^2u_0^2 - \tag{19}$$

$$-3nA^2u_0^2 - 2Au_0^2u'_0 - nAu_0^2u'_0 - 2u_0^2(u'_0)^2 - nu_0^2(u'_0)^2)(g_{ik}g_{jl} - g_{il}g_{jk}) = 0.$$

If the coefficient of R_{ijkl} is not zero then the manifold (M, g) has constant sectional curvature. If the coefficient of R_{ijkl} in (19) is zero then the relation (19) cannot be fullfilled, due to the condition (15).

3. The general case

In this section we shall assume that the the Riemannian manifold (M,g) has constant sectional curvature c and that the following conditions are fulfilled:

$$u(0)\neq 0 \quad , \quad u'\neq 0 \quad , \quad u-tu'\neq 0 \quad , \quad u-2tu'\neq 0. \tag{20}$$

From the explicite expression of the condition $(\nabla_{\frac{\delta}{\delta x^l}}K)(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k}=0$ we obtain (among some other equations) the following equations

$$v=\frac{uu'(3u-2tu')}{2(u-2tu')(u-tu')}, \tag{21}$$

$$c^2u-2c^2tu'-cu^2u'-4ctuu'v+u^3u'v-u^3v^2+2tu^2u'v^2- \tag{22}$$

$$-2ctu^2v'-2tu^3vv'=0.$$

The relation (21) is obtained from the vanishing of the coefficient of $\delta_l^h g_{jk}g_{0i}$, while the relation (22) is obtained from the vanishing ot the coefficient of $\delta_k^h g_{jl}g_{0i}$. Eliminating v from (14) and (21), we obtain that the function u must fulfill the equation

$$uu'+4Atu'-2Au=0. \tag{23}$$

The general solution can be expressed in the form $u=B\pm\sqrt{B^2+Ct}$, where B,C are real constants related to A by $C=4AB$. In the general case we can assume $BC\neq 0$. Asking for the function u to verify the equation (22), we obtain $C=-2c$.

Remark. The solution $u=B-\sqrt{B^2-2ct}$ has the property $u(0)=0$, thus it should be excluded. Hence we shall work with the function

$$u=B+\sqrt{B^2-2ct} \quad , \quad B>0. \tag{24}$$

The expression of the function v is obtained easily from (21)

$$v=-\frac{c}{B}\frac{B+2\sqrt{B^2-2ct}}{B+\sqrt{B^2-2ct}}=\frac{1}{2Bt}(B^2-4ct-B\sqrt{B^2-2ct}). \tag{25}$$

Remark that $v(0)$ is well defined. Now we can check easily that the found functions u,v are related by

$$v=\frac{c-uu'}{2tu'-u}. \tag{26}$$

This relation has been obtained in [14] in the study of the integrability problem for a certain almost complex structure related to the Riemannian metric G on TM. Since the matrix (H_j^i) is the inverse of the matrix (G_j^i), we may define an almost complex structure J on TM by

$$J\frac{\delta}{\delta x^i}=G_i^k\frac{\partial}{\partial y^k} \quad , \quad J\frac{\partial}{\partial y^i}=-H_i^k\frac{\delta}{\delta x^k}. \tag{27}$$

In fact, if we take into account the conditions under which $u > 0$, $u + 2tv > 0$, it follows $B > 0$ (see [14]), (TM, G, J) is an almost Kählerian manifold and the almost complex stucture J is integrable, if and only if (M, g) has constant sectional curvature c and the function v is given by (26). Moreover, in this case, the Kaehlerian manifold (TM, G, J) has constant holomorphic sectional curvature, hence (TM, G) is a locally symmetric space. Hence we may state

Theorem 5. *Assume that the the conditions* (20) *are fulfilled. Then we have*

(i) *The whole* (TM, G) *is a locally symmetric space if and only if the base manifold* (M, g) *has negative constant sectional curvature* c*, the function* u *is given by* (24) *and the function* v *is given by* (26)*. In this case* (TM, G, J) *is a Kähler manifold of constant holomorphic sectional curvature.*

(ii) *The result from (i) is valid for the tube* $(T^B M, G)$ *around the zero section, defined by the condition* $0 \le t < \frac{B^2}{2c}$*, in the case where* (M, g) *has positive constant sectional curvature* c*, the function* u *is given by* (24) *and* v *is given by* (26)*. In this case* $(T^B M, G, J)$ *is a Kähler manifold of constant holomorphic sectional curvature.*

Remark. If we consider the solution $u = B - \sqrt{B^2 - 2ct}$, the corresponding v determined from (26), and assume that (M, g) has constant sectional curvature c, we must have $c > 0$ and we obtain a structure of locally symmetric space on the tube $(T_0^B M, G)$ of nonzero tangent vectors from TM, defined by $0 < t < \frac{B^2}{2c}$.

4. The case $u' = 0$

The case $u' = 0$ has been excluded by the conditions (20). In this case we have $u = A$ where A is a positive real constant and (M, g) must have constant sectional curvature c. From the condition $(\nabla_{\frac{\partial}{\partial y^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$ we obtain that v must be a constant too and from the condition $(\nabla_{\frac{\delta}{\delta x^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$ we obtain the value $v = -\frac{c}{A}$. It follows that the relation (26) is fulfilled and, in this case, we have that the almost compex structure J defined by (27) is integrable. In this case it has been shown in [13] (in the case where $A = 1$) that if $A^2 - 2ct > 0$ then (TM, G) is a locally symmetric space. Remark that, for $c < 0$ the condition $A^2 - 2ct > 0$ is satisfied for all $t \ge 0$ and for $c > 0$ the above condition is satisfied only for $0 \le t < \frac{A^2}{2c}$. For $c > 0$ we denote by $T^A M$ the tube around the zero section in TM defined by $0 \le t < \frac{A^2}{2c}$. Hence we may state

Theorem 6. *Assume that* $u = A$*, where* A *is a real positive constant. Then we have:*

(i) *The whole Riemannian manifold* (TM, G) *is a locally symmetric space if and only if the base manifold* (M, g) *has negative constant sectional curvature* c *and the function* v *is given by* $v = -\frac{c}{A}$*. In this case* (TM, G, J) *is a Kähler Einstein manifold.*

(ii) *The result from (i) is valid for the tube* $(T^A M, G)$*, defined by* $0 \le t < \frac{A^2}{2c}$*, in the case where the base manifold* (M, g) *has positive constant sectional curvature* c *and* $v = -\frac{c}{A}$*. In this case* $(T^A M, G, J)$ *is a Kähler Einstein manifold.*

5. Singular cases

Consider, first, the case $u - tu' = 0$, excluded by the conditions (20). In this case we have $u = At$, where A is a real positive constant. We see that $u(0) = 0$, hence we cannot conclude that (M, g) has constant sectional curvature. However, we shall assume that (M, g) is a real space form with the sectional curvature c and we shall study whether the bundle of the nonzero tangent vectors to M has a structure of locally symmetric space.

From the condition $(\nabla_{\frac{\delta}{\delta x^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$ we obtain $v = \frac{c}{At} - A$, hence $c > 0$. Then from the condition $u + 2tv > 0$ we get $0 < t < \frac{2c}{A^2}$, that is, we must restrict ourselves to the tube $T_0^A M$ around the zero section in TM consisting of nonzero tangent vectors, for which $0 < t < \frac{2c}{A^2}$. Even in this case, the function v can be obtained by using the formula (26), hence $(T_0^A M, G, J)$ is a Kähler manifold. Moreover, the present authors have obtained in [16] that $(T_0^A M, G)$ is a locally symmetric space. Hence we may state

Theorem 7. *Assume that $u = At$; $A > 0$ and that (M, g) has positive constant sectional curvature c. Then $(T_0^A M, G)$ is a locally symmetric space if and only if the function v is given by (26), i.e. $(T_0^A M, G, J)$ is a Kähler Einstein manifold.*

In the following case we consider $u - 2tu' = 0$, see (20). In this case we have $u = A\sqrt{t}$, where A is a real, positive constant. We shall restrict ourselves to the bundle $T_0 M$ of the nonzero tangent vectors to M and we shall assume that (M, g) has constant sectional curvature. From the condition $(\nabla_{\frac{\delta}{\delta x^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} = 0$, we obtain $A = \sqrt{2c}$ (hence $c > 0$) and $v = B - \sqrt{\frac{c}{2t}}$, where $B > 0$. However, we have still $(\nabla_{\frac{\delta}{\delta x^l}} K)(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j})\frac{\partial}{\partial y^k} \neq 0$. Therefore, in this case, $(T_0 M, G)$ cannot be a locally symmetric space.

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