DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 1–11

SOME GEOMETRIC ASPECTS OF COMPATIBLE COMPLEX STRUCTURES ON A QUATERNION KÄHLER MANIFOLD

D.V. ALEKSEEVSKY, S. MARCHIAFAVA, M. PONTECORVO

ABSTRACT. This is a report on joint work concerning almost complex structures on a quaternion Kähler manifold (M^{4n}, g, Q) which are compatible with the quaternionic structure Q, see [AM1,2], [P1,2] and [AMP1,2,3]. We point out the interest for conformally balanced complex structures, which are related to solutions of a "twistor equation" and on a compact positive quaternion Kähler manifold may be seen as Hamiltonian forms of Killing vector fields on M^{4n} or as Hamiltonian functions on the twistor space Z and also as eigenforms of the Laplacian acting on self-dual 2-forms.

1. Introduction

A main subject of our research during the past years was the study of complex and hypercomplex structures on a quaternionic manifold and, in particular, on a quaternion Kähler manifold. Here we focus on quaternion Kähler manifolds by reporting on several results that were obtained recently, see [AM1,2], [P1,2] and [AMP1,2,3].

A *quaternion Kähler manifold*[1] is a Riemannian manifold (M^{4n}, g), $n \geq 2$, whose holonomy group $Hol(g)$ is isomorphic to a subgroup of $Sp(n) \cdot Sp(1)$. We recall that the enlarged quaternionic unitary group $Sp(n) \cdot Sp(1)$ may be presented as the group of $\mathbb{R}-$linear transformations $T : \mathbb{H}^n \longrightarrow \mathbb{H}^n$ of the numerical n-dimensional (right) quaternionic space $\mathbb{H}^n$ which are of the form

$$T : \xi \mapsto \xi' = A\xi q \qquad \xi, \xi' \in \mathbb{H}^n$$

where $A \in Sp(n)$ is a quaternionic unitary transformation (with respect to the canonical Hermitian product $\eta \cdot \xi = \sum_\alpha \overline{\eta^\alpha} \xi^\alpha$) and q is a unitary quaternion which

1991 *Mathematics Subject Classification*. 53C10, 53C15, 32C10, 58G25.
Key words and phrases. Quaternion Kähler - Compatible complex structure - Conformally balanced complex structure - Hamiltonian form - Self-dual form.
Work done under the program of G.N.S.A.G.A. of C.N.R.; partially supported by M.U.R.S.T (Italy) and E.S.I. (Vienna).
This paper is in final form and no version of it will be submitted for publication elsewhere.

[1]For an updated introduction on this subject, see the lecture of S. Salamon in these proceedings, [S4].

multiplies on the right. Note that for any quaternion $p \in \mathbb{H}$ one has $T(\xi p) = (T\xi)p'$ with $p' = \bar{q}pq$; moreover the Euclidean scalar product is preserved, $T\eta \times T\xi = \mathbb{R}e(T\eta \cdot T\xi) = \mathbb{R}e(\bar{q}(\eta \cdot \xi)q) = \eta \times \xi$, see [Ma]. There is an identification

$$Sp(n) \cdot Sp(1) \equiv Sp(n) \times Sp(1)/\mathbb{Z}_2 , \qquad \mathbb{Z}_2 = \{(Id, 1), (-Id, -1)\} .$$

Classical examples of compact quaternion Kähler manifolds are the *Wolf spaces*: homogeneous symmetric Riemannian manifolds of the form $G/N(K)$ where G is a compact simple Lie group and $N(K)$ is the normalizer in G of the regular 3-dimensional subgroup K of G associated with the maximal root. The Wolf spaces of the classical groups are the following:

$$\mathbb{H}P_n = \frac{Sp(n+1)}{Sp(n) \times Sp(1)},$$

$$G_2(\mathbb{C}^{n+2}) = \frac{SU(n+2)}{S(U(n) \times U(2))} , \qquad G_4^+(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{S(O(n) \times O(4))} ,$$

see for ex. [Be, p. 408].

A quaternion Kähler manifold is always an *Einstein manifold* , whose scalar curvature we denote by K. It carries a *quaternionic structure* Q which can be identified with a rank-3 subbundle $Q \xrightarrow{\pi} M^{4n}$ of skew-symmetric endomorphisms of the tangent bundle TM, with (local) *admissible frames* $H = (J_1, J_2, J_3 = J_1 J_2)$ where J_1, J_2, J_3 are anti-commuting almost complex structures. Notice that because of the right factor $Sp(1)$ of the holonomy group, a global admissible frame does not exist in general.

The bundle Q is parallel with respect to the Levi-Civita connection of g, denoted by $\nabla = \nabla^g$, that is for any admissible frame $H = (J_\alpha)$ one has

$$\nabla J_\alpha = \omega_\gamma \otimes J_\beta - \omega_\beta \otimes J_\gamma \tag{1.1.1}$$

for any circular permutation (α, β, γ) of $(1, 2, 3)$, where the ω_α, $\alpha = 1, 2, 3$, are 1-forms.

We shall denote by (M^{4n}, g, Q) a quaternion Kähler manifold.

Duality with respect to the metric g defines a rank-3 subbundle $\Lambda_+ \to M$ of the bundle of 2-forms Λ_2 which we call the bundle of *self-dual* forms.

$$\begin{array}{ccc} Q \longrightarrow End^{sk}(TM) & & \Lambda_+ \longrightarrow \Lambda_2(TM) \\ \pi \big\downarrow \qquad\qquad & \overset{g}{\Longrightarrow} & \pi' \big\downarrow \qquad\qquad \\ M \qquad\qquad & & M \qquad\qquad \end{array}$$

Λ_+ is equipped with (local) admissible frames $(F_\alpha = g \circ J_\alpha)$, $\alpha = 1, 2, 3$, where $H = (J_\alpha)$ is an admissible frame of Q and, of course,

$$\nabla F_\alpha = \omega_\gamma \otimes F_\beta - \omega_\beta \otimes F_\gamma \tag{1.1.2}$$

The following *integrability conditions* hold:

$$d\omega_\alpha + \omega_\beta \wedge \omega_\gamma = -\nu F_\alpha \qquad (\alpha = 1, 2, 3) \tag{1.2}$$

where $\nu = K/4n(n+2)$ is the *reduced scalar curvature*

We recall that $\nu = 0$ if and only if the bundle $Q \to M^{4n}$ (resp., $\Lambda_+ \to M^{4n}$) is locally flat and there exist local admissible frames $H = (J_\alpha)$ (resp., $(F_\alpha = g \circ J_\alpha)$) which are parallel. In turn, this conditions are equivalent to the restricted holonomy group $Hol_0(g)$ of g being isomorphic to a subgroup of $Sp(n)$, that is (M^{4n}, g) is *locally hyperKähler* .

2. Compatible almost complex structures on (M^{4n}, g, Q)

Let (M^{4n}, g, Q) be a quaternion Kähler manifold. A *compatible almost complex structure* , shortly a *compatible a.c.s.* , is a section $J \in \Gamma(Q)$ of $Q \to M^{4n}$ such that $J^2 = -Id$, i.e. for any $H = (J_\alpha)$,

$$J = \sum_{\alpha=1}^{3} c_\alpha J_\alpha$$

for some smooth functions $c_\alpha, \alpha = 1, 2, 3,\quad \sum_{\alpha=1}^{3} c_\alpha^2 = 1$.

When J is integrable one says that it is a *compatible complex structure.*

From the integrability conditions (1.2) it follows immediately that *if there exists a compatible complex structure J which is parallel then the manifold must be locally hyperKähler* . In fact, for any admissible frame $H = (J_1 = J, J_2, J_3)$ we get $\nabla J = \omega_3 \otimes J_2 - \omega_2 \otimes J_3 = 0$. Hence $\omega_2 = \omega_3 = 0$ and

$$0 = d\omega_2 + \omega_3 \wedge \omega_1 = -\nu F_2$$

which implies $\nu = 0$.

Global hypothesis on M^{4n} imply the following stronger result.

Proposition ([AMP2]). *Let J be a parallel complex structure (not necessarily compatible). Assume M^{4n} simply connected, complete, irreducible. Then*

(1) *J is compatible if and only if $\nu = 0$ and there exists a (global) parallel admissible frame $H = (J_\alpha)$ such that*

$$J = \sum_{\alpha=1}^{3} c_\alpha J_\alpha$$

where the c_α are constants.

(2) *Otherwise, (M^{4n}, g) is homothetic to the complex Grassmannian $G_2(\mathbb{C}^{n+2})$ or its non compact dual and J is the unique parallel complex structure.*

From now on J will denote a compatible a.c.s. and $F = g \circ J$ its *Kähler form.*

The following result shows that local conditions on J apparently weaker are still equivalent to the condition that J is parallel.

Proposition ([AMP2]).

(1) J *is* locally conformally symplectic, i.e. $dF = \alpha \wedge F$, if and only if $\nabla J = 0$.
(2) J is cosymplectic, i.e. $\delta F = 0$, only if $\nu \leq 0$ and $\nu = 0$ only when $\nabla J = 0$.
(3) A self-dual form $\omega \in \Gamma(\Lambda_+)$ is closed, $d\omega = 0$, only if it is parallel, $\nabla \omega = 0$.

Let us also mention another useful local result.

Proposition ([AMP2]). *If J is integrable then the differential $d\theta$ of its Lee form $\theta := -\delta F \circ J$ is a Q-Hermitian harmonic 2-form.*

On M^{4n} the *fundamental 4-form* Ω,

$$\Omega = \sum_{\alpha=}^{3} F_\alpha \wedge F_\alpha$$

is globally defined, non degenerate and parallel. Let assume that the scalar curvature is non-zero, that is $\nu \neq 0$, then

$$\Omega = \frac{1}{\nu^2} d\Big(\sum_\alpha \omega_\alpha \wedge d\omega_\alpha + 2\omega_1 \wedge \omega_2 \wedge \omega_3 \Big)$$

with respect to any admissible frame $H = (J_\alpha)$; this was shown in [AM1] and used to prove that *on a compact quaternion Kähler manifold (M^{4n}, g, Q) with $\nu \neq 0$ no global admissible frame $H = (J_\alpha)$ exists.*

On the other hand, if $\nu < 0$ a complete M^{4n} may admit a global admissible frame, and even a hypercomplex frame $H = (J_\alpha)$, that is the J_α are integrable, $\alpha = 1, 2, 3$: e.g. the hyperbolic quaternionic space $\widetilde{\mathbb{H}P}_n$.

Proposition ([AMP2]). *Let (M^{4n}, g, Q) be compact and $\nu > 0$. Then no compatible a. c. s. J exists on M^{4n}.*

S. Salamon proved that locally there exist many compatible almost complex structures which are integrable [S2]. On the other hand M. Pontecorvo proved that *on a compact M^{4n} a (global) compatible complex structure may exist only if it is parallel and hence the manifold is locally hyperKähler*, see [P1].

There are general interesting questions about existence of compatible almost complex structures from the local point of view which as far as we know are still open.

Problem 1. [AM2] *Beside $\mathbb{H}P^n$ and its non compact dual $\widetilde{\mathbb{H}P}_n$, which other Wolf space admits local hypercomplex* frames $H = (J_\alpha)$?

The solution would be a first step to solve the following more general problem.

Problem 2. *Which quaternion Kähler manifolds are locally hypercomplex?*

Other natural questions concerning global or local compatible complex structures will arise from the following discussion.

From now on we will assume $\nu \neq 0$.

3. Killing vector fields and conformally balanced compatible complex structures J

Associated to any self-dual form $\eta \in \Gamma(\Lambda_+)$ there is a compatible almost complex structure $J = \frac{g^{-1} \circ \eta}{|\eta|}$ defined on the open set $U = \{x \in M, \eta_x \neq 0\}$ and in what follows we will point out a special class of compatible *complex* structures J which are associated to Hamiltonian 2-forms defined by Killing fields.

Let X be a Killing vector field on M and denote by the same symbol the 1-form $g \circ X$ associated to it by the metric g. Then the 2-form $\nabla X = \frac{1}{2} dX$ decomposes as

$$\nabla X = \nabla X^{\mathfrak{sp}_1} + \nabla X^{\mathfrak{sp}_n} \tag{3.1}$$

where

$$\nabla X^{\mathfrak{sp}_1} = -\frac{1}{4n} \sum_{\alpha=1}^{3} Trace(J_\alpha \nabla X) F_\alpha$$

is the self-dual part of ∇X and $\nabla X^{\mathfrak{sp}_n} = \nabla X - \nabla X^{\mathfrak{sp}_1}$ is a field of endomorphisms which commute with Q, $[\nabla X^{\mathfrak{sp}_n}, Q] = 0$.

The *Hamiltonian 2-form* of X is

$$\eta_X = \frac{4}{\nu} \nabla X^{\mathfrak{sp}_1}$$

and satisfies

$$d\eta_X = i_X \Omega \tag{3.2}$$

As a self-dual 2-form the Hamiltonian η_X is uniquely defined by (3.2) and the correspondence

$$\mathfrak{isom}(g) \ni X \mapsto \eta_X \in \Gamma(\Lambda_+)$$

is injective.

The following result was first proved in [S1], [GL]; see also [AMP2].

Proposition. *On a compact quaternion Kähler manifold (M^{4n}, g, Q) with $\nu \neq 0$, the Hamiltonian 2-forms η_X of Killing vector fields $X \in \mathfrak{isom}(g)$ are exactly the solutions of the following* twistor equation

$$\bar{D}\eta := \nabla \eta + \frac{1}{3} \sum_{\alpha} \delta\eta \circ J_\alpha \otimes J_\alpha = 0$$

We proved the same result by looking at the solutions of the twistor equation from the following other point of view.

We recall that an (integrable) *complex* structure J on M is called *conformally balanced* if its Lee form $\theta = -\delta(g \circ J) \circ J$ is exact: $\theta = dh$.

Proposition ([AMP2]). *Let (M^{4n}, g, Q) be a quaternion Kähler manifold with $\nu \neq 0$. If $\eta = \eta_X$ is the Hamiltonian 2-form of a Killing vector field, then the associated almost complex structure J is a conformally balanced complex structure with Lee form $\theta = d\log|\eta|^2$.*

Conversely, if the almost complex structure J associated with a self-dual 2-form η is a conformally balanced complex structure then η is the Hamiltonian of a Killing vector field X given by $X = (df \circ J)^{\#}$ where $\theta = dlogf^2$ and $f = |\eta|$.

Our next result shows that compatible conformally balanced complex structures can also be seen as eigenforms of the Laplacian.

Proposition ([AMP3]). *On a quaternion Kähler manifold (M^{4n}, g, Q) with $\nu \neq 0$ the Hamiltonian 2-form η_X of a Killing vector field X is an eigenform of the Laplacian with eigenvalue $\lambda = 2(n+2)\nu$, i.e. $\lambda = \frac{K}{2n}$.*

The statement can be proved by recalling a classical result which states that for a Killing vector field X on an m-dimensional Einstein manifold the 2-form dX is an eigenform which is associated to the eigenvalue $\lambda = 2K/m$ and, moreover, by observing that the decomposition (3.1) is holonomy invariant and hence the orthogonal projection onto the space of self-dual forms commutes with the Laplacian.

A more direct proof that the conformally balanced compatible complex structures correspond to eigenforms of the Laplacian is presented in the last section of this paper, see also [M].

In the next section we will discuss the following natural question: *is it true that any eigenform $\eta \in \Lambda^2_+$ of the Laplacian with eigenvalue $\lambda = \frac{K}{2n}$ is the Hamiltonian form of a Killing vector field $X = (g)^{-1} \circ (df \circ J) = (\delta\eta)^{\#}$ and hence satisfies also the twistor equation?* .

4. Spectral properties of the twistor fibration

Let (M^{4n}, g, Q) be a *positive* quaternion Kähler manifold, that is $\nu > 0$.

The unit bundle $Z := S_1(Q) = \{J \in Q | J^2 = -Id\}$ has a natural complex structure J' and also a Kähler-Einstein metric g' such that the *twistor fibration* $t : Z \to M$ is a Riemannian submersion with totally geodesic fiber isometric to $\mathbb{C}P_1$:

$$\begin{array}{ccc} \mathbb{C}P_1 & \longrightarrow & Z \\ & & \big\downarrow t \\ & & M \end{array} \qquad \left(\text{prototype:} \quad \begin{array}{ccc} \mathbb{C}P_1 & \longrightarrow & \mathbb{C}P_{2n+1} \\ & & \big\downarrow t \\ & & \mathbb{H}P_n \end{array}\right)$$

The scalar curvatures of g and g' are related by $K' = \left[\frac{(n+1)(2n+1)}{2n(n+2)}\right] K$, which can also be written as

$$\frac{K'}{2n+1} = 2(n+1)\nu$$

where $m = 2n+1$ is the complex dimension of Z.

The value $\frac{K'}{m}$ is related to spectral properties of Z. Indeed, let us recall the Lichnerowicz-Matsushima theorem, see [Be p. 330].

Theorem. *Let (Z, g', J') be a compact Kähler-Einstein manifold with $K' > 0$, $dim_{\mathbb{C}} Z = m$. Then:*

(Lichnerowicz) the first non-zero eigenvalue of $\triangle$ on functions is greater or equal to $\frac{K'}{m}$;

(Matsushima) there is an isomorphism

$$\Psi : \mathfrak{isom}(g') \to E_{\frac{K'}{m}}(C^\infty(Z'))$$

where $E_{\frac{K'}{m}}(C^\infty(Z'))$ is the eigenspace for the Laplacian with eigenvalue $\lambda = \frac{K'}{m}$.

Precisely:

$$\Psi : \mathfrak{isom}(g') \ni X \mapsto f = -\frac{m}{K'} trace(J'\nabla X) \in E_{\frac{K'}{m}}(C^\infty(Z'))$$

where $f = H_X$ is a Hamiltonian of X. Viceversa:

$$\Psi^{-1} : E_{\frac{K'}{m}}(C^\infty(Z')) \ni f \mapsto X = (J' df)^\# \in \mathfrak{isom}(g')$$

Since by a classical result of B. Watson, see [B-B], for any function $f : M \to \mathbb{R}$ one has $(\triangle^M f) \circ t = \triangle^Z (f \circ t)$, we are able to deduce immediately that $2(n+1)\nu$ is the minimal non-zero eigenvalue of the Laplacian acting on functions on M^{4n}, because otherwise $\frac{K'}{2n+1}$ would not be minimal for functions on Z; this result was already proved in [Marg] and [AM3] using different methods.

A quaternionic version of the Lichnerowicz-Matsushima Theorem is obtained by the following correspondence between Hamiltonians of Killing fields on the positive quaternion Kähler manifold (M^{4n}, g, Q) and on its twistor space. In fact, for any Killing vector field $X \in \mathfrak{isom}(g)$ there is a natural extension X^Z on Z, which is also Killing:

$$\begin{array}{ccc} \mathfrak{isom}(g) \ni X & \longrightarrow & X^Z \\ \Big\downarrow & & \Big\downarrow \\ \eta_X & \longrightarrow & H_{X^Z} \end{array} \tag{4.1}$$

The lower arrow in the diagram above is induced by a natural *injective* correspondence

$$\Phi : \Gamma(\Lambda_+) \ni \eta \longrightarrow f_\eta \in C^\infty(Z)$$

given by

$$f_\eta(z) = c\langle (g^{-1} \circ \eta), J_z \rangle \qquad \forall z \in Z$$

where $\langle \cdot, \cdot \rangle$ is the natural scalar product in $Q_{\pi(z)} \subset \mathrm{End}(T_{\pi(z)} M)$, J_z is the complex structure of the tangent space $T_{\pi(z)} M^{4n}$ represented by z and $c = -1/8(n+1)$ is an appropriate constant. Indeed, by restricting Φ to Hamiltonian forms of Killing vector fields X and using the notation

$$f_X := \Phi(\eta_X)$$

the following proposition holds.

Proposition ([AMP3]). *For any Killing vector field $X \in \mathfrak{isom}(g)$ one has*

$$f_X = H_X z$$

The following theorem says that Φ *relates the spectra of the Laplacian acting on self-dual 2-forms of M with the spectra of functions on Z* . (We denote by $E_\lambda(\mathcal{A})$ the eigenspace of a Laplacian acting on the space $\mathcal{A}$).

Theorem ([AMP3]). *For all real numbers λ the restriction of Φ to $E_\lambda(\Lambda_+(M))$ is a linear injection*

$$\Phi : E_\lambda(\Lambda_+(M)) \longrightarrow E_{\lambda-2\nu}(C^\infty(Z))$$

In particular we have the following result which can be thought of as a quaternionic version of Lichnerowicz-Matsushima Theorem.

Proposition ([AMP3]). *Let (M^{4n}, g, Q) be a compact, positive quaternion Kähler manifold. Then the first non-zero eigenvalue $\bar\lambda$ of the Laplacian on self-dual forms is bounded below by $2(n+2)\nu$,*

$$\bar\lambda \geq 2(n+2)\nu = \frac{K}{2n}$$

Equality holds if and only if there are non trivial Killing vector fields on M and on Z.

Remark. The equality always holds if $n \leq 4$, see [S1]. By the above results, the following theorem relates compatible conformally balanced complex structures on M^{4n} to eigenfunctions of the Laplacian on the twistor space Z.

Theorem ([AMP3]). *Let (M^{4n}, g, Q) be a compact, positive quaternion Kähler manifold. The map*

$$\Phi : E_{\frac{K}{2n}}(\Lambda_+(M)) \longrightarrow E_{\frac{K'}{2n+1}}(C^\infty(Z))$$

is a linear isomorphism unless M^{4n} is the quaternionic projective space.

We conclude the paper by giving another approach to the relationship between conformally balanced complex structures and eigenforms of the Laplacian.

5. Self-dual 2-forms and eigenforms of the Laplacian

Let $\sigma \in \Gamma(\Lambda_+)$ be a self-dual form and J the associated compatible complex structure which is defined on $U = \{x \in M^{4n}, \sigma_x \neq 0\}$ by $J = g^{-1} \circ F$ where $F = |\sigma|^{-1}\sigma$. In the formulas below we will refer to an admissible frame $H = (J_1 = J, J_2, J_3)$.

We recall the following formula

$$\begin{aligned} \triangle(fF) &= [f(4\nu + \|\omega_2\|^2 + \|\omega_3\|^2) + \triangle f]F \\ &+ [f(\delta\omega_3 - \langle\omega_1, \omega_2\rangle) - 2\langle df, \omega_3\rangle]F_2 \\ &+ [-f(\delta\omega_2 + \langle\omega_1, \omega_3\rangle) + 2\langle df, \omega_2\rangle]F_3 \end{aligned} \tag{5.1}$$

see [AMP2]. We recall also the following result which holds for a complex J.

Lemma ([AMP2]). *Let $J = J_1$ be a compatible* complex *structure on M^{4n}. Then*

$$\delta\omega_2 + \langle\omega_1, \omega_3\rangle = 0 \quad , \quad \delta\omega_3 - \langle\omega_1, \omega_2\rangle = 0 \tag{5.2}$$

Proof. By expressing that for $\alpha = 3$ the 2-form at left member of integrability condition (1.2) is Q-Hermitian one has the identities

$$\begin{aligned}&(\nabla_X\omega_3)(J_3Y) - (\nabla_{J_3Y}\omega_3)(X) + \omega_1(X)\omega_2(J_3Y) - \omega_1(J_3Y)\omega_2(X)\\ &\quad = (\nabla_{J_\rho X}\omega_3)(J_3J_\rho Y) - (\nabla_{J_3J_\rho Y}\omega_3)(J_\rho X) \qquad\qquad (\rho = 1,2,3)\\ &\quad + \omega_1(J_\rho X)\omega_2(J_3J_\rho Y) - \omega_1(J_3J_\rho Y)\omega_2(J_\rho X)\end{aligned}$$

We take also into account the identity

$$\nabla_X\omega_3 = -(\nabla_X\omega_2)\circ J_1 + \omega_2(X)(\omega_2\circ J_3) + \omega_2(J_1X)(\omega_2\circ J_2) \tag{5.3}$$

which is obtained by differentiating the identity $\omega_2\circ J_2 = \omega_3\circ J_3$ (equivalent to the integrability of J). Then for $\rho = 1$, by substituting J_2Y to Y and then contracting with respect to g, one obtains $\delta\omega_2 + \langle\omega_1, \omega_3\rangle = 0$. By interchanging J_3 and J_2 one gets also $\delta\omega_3 - \langle\omega_1, \omega_2\rangle = 0$ □

Proposition. *On a quaternion Kähler manifold (M^{4n}, g, Q) with $\nu \neq 0$, a self-dual 2-form σ whose associated a. c. s. J is a conformally balanced complex structure with $\theta = d\log|\sigma|^2$ is an eigenform of the Laplacian with eigenvalue $\lambda = 2(n+2)\nu$, i.e. $\lambda = \frac{K}{2n}$.*

Proof. We prove that a self-dual 2-form $\sigma = fF$ where F is the Kähler form of a conformally balanced complex structure J with Lee form $\theta = dlogf^2$, is an eigenform for the Laplacian with eigenvalue $\lambda = 2(n+2)\nu$.

Let $H = (J_1 = J, J_2, J_3)$ be an admissible basis of Q. We have $df = \frac{1}{2}f\theta$ and $\omega_2 = -\frac{1}{2}\theta\circ J_2$, $\omega_3 = -\frac{1}{2}\theta\circ J_3$ since $\theta = \omega_2\circ J_2 + \omega_3\circ J_3$ and $\omega_2\circ J_2 = \omega_3\circ J_3$ because of the integrability, see [AMP1]. Then, by taking account also of (5.2), formula (5.1) reduces to

$$\triangle(fF) = [f(4\nu + \frac{1}{2}||\theta||^2) + \triangle f]F \tag{5.4}$$

Moreover $\triangle f = \delta df = \frac{1}{2}\delta(f\theta) = \frac{1}{2}(-\langle df, \theta\rangle + f\delta\theta) = \frac{1}{2}(-\frac{1}{2}f||\theta||^2 + f\delta\theta) = \frac{1}{4}f(2\delta\theta - ||\theta||^2)$ and by using the identity[2]

$$||\theta||^2 + 2\delta\theta = 8n\nu \tag{5.5}$$

see [AMP2], it follows

$$\triangle f = f(2n\nu - \frac{1}{2}||\theta||^2)$$

Hence by substitution in previous formula (5.4) we get $\triangle\sigma = 2(n+2)\nu\sigma$ as wanted. □

[2] This identity holds for any compatible complex structure and is also useful to prove the above mentioned result of Pontecorvo, see [AMP2].

References

[AM1] Alekseevsky, D. V., Marchiafava, S., *Quaternionic structures on a manifold and subordinated structures*, Annali di Mat. Pura e Appl. **171** (1996), 205–273.

[AM2] ______, *Hypercomplex structures on Quaternionic Manifolds*, New Developments in Differential GeometryProceed. of the Colloquium on Diff. Geom., Debrecen, Hungary, July 26–30, 1994 , Kluwer Academic Publishers, 1996, pp. 1–19.

[AM3] ______, *Transformations of a quaternion-Kähler manifold*, C.R. Acad. Paris **320** (1996), 703–708.

[AMP1] Alekseevsky, D. V., Marchiafava, S., Pontecorvo, M., *Compatible complex structures on almost quaternionic manifolds*, Preprint ESI n. 404 http://www.esi.ac.at (1996)(To appear on Trans. AMS) .

[AMP2] Alekseevsky, D. V., Marchiafava, S., Pontecorvo, M., *Compatible almost complex structures on quaternion-Kähler manifolds*, Preprint ESI n. 419 http://www.esi.ac.at (1997) Ann. of Global. Anal. and Geom., vol. 16 (1998), 419–444 .

[AMP3] Alekseevsky, D. V., Marchiafava, S., Pontecorvo, M., *Spectral properties of the twistor fibration of a quaternion Kähler manifolds*, Preprint ESI (Vienna) (to appear) (1998).

[B-B] Berard-Bergery, L., Bourguignon, J-P., *Laplacians and Riemannian submersions with totally geodesic fibers*, Illinois J. of Math. **26** (1982), 181–200.

[Be] Besse, A., *Einstein manifolds*, Ergebnisse der Math., vol. 3, Springer-Verlag, 1987.

[BGMR] Burstall, F. E., Grantcharov, G., Muškarov, O., Rawnsley, J. H., *Hermitian structures on Hermitian symmetric spaces*, J. Geom. Phys. **10** (1993), 245–249.

[G] Galicki, K., *A generalization of the momentum mapping construction for quaternionic Kähler manifolds*, Comm. Math. Phys. **108** (1987), 117–138.

[GL] Galicki, K., Lawson, H. B., *Quaternionic reduction and quaternionic orbifolds*, Math. Ann. **282** (1988), 1–21.

[G1] Gauduchon, P., *Complex structures on compact conformal manifolds of negative type*, Complex analysis and geometry (Trento, 1993), Lecture Notes in Pure and Appl. Math., vol. 173, Dekker, New York, 1996, pp. 201–212.

[LS] LeBrun, C., Salamon, S., *Strong rigidity of positive quaternion Kähler manifolds*, Invent. Math. **118** (1994), 109–132.

[M] Marchiafava, S., *The twistor equation, moment forms and compatible complex structures on a quaternion Kähler manifold*, Proceedings of the "Third Int. Workshop on Diff. Geom. Appl., Sibiu, Romania, September 1997", to appear.

[Marg] Margerin, C., *A sharp lower bound for the first eigenvalue of a quaternion Kähler manifold*, Preprint n. 1094.1193, Centre de mathematiques, Ecole Polytechnique, Palaiseau.

[Ma] Martinelli, E., *Varietà a struttura quaternionale generalizzata*, Rend. Accad. Lincei **26** (1959), 353–362.

[OP] Ornea, L., Piccinni, P., *Una classe di varietà quaternionali che ammettono una struttura complessa compatibile*, Rend. Mat. Accad. Lincei **8** (1997), 293–298.

[P1] Pontecorvo, M., *Complex structures on quaternionic manifolds*, Diff. Geometry and its Applications **4** (1992), 163–177.

[P2] ______, *Complex structures on Riemannian 4-manifolds*, Math. Ann. **309** (1997), 159–177.

[S1] Salamon, S., *Quaternionic Kähler manifolds*, Invent. Math. **67** (1982), 143–171.

[S2] ______, *Riemannian Geometry and Holonomy Groups*, Pitman Research Notes in Mathematics, vol. 201, Longman Scientific & Technical, UK, 1989.

[S3] ______, *Orthogonal complex structures*, Proceedings of the Conference Differential Geometry and Applications (Brno, 1995) (1996), 103–117.

[S4] ______, *A survey of quaternionic manifolds*, Proceedings of the Conference Differential Geometry and Applications, Brno, August 1998 (to appear).

[Sw] Swann, A., *Hyperkähler and quaternionic Kähler geometry*, Math. Ann. **289**, 421–450.

Gen. Antonova 2 - 99, 117279 Moskow, RUSSIA
E-mail: daleksee@esi.ac.at ; fort@rsuh.ru

Dipartimento di Matematica, Università di Roma "La Sapienza"
P.le A. Moro 2, 00185 Roma, ITALY
E-mail: marchiafava@axrma.uniroma1.it

Dipartimento di Matematica, Università di Roma Tre
Largo S.L.Murialdo 1, 00146 Roma, ITALY
E-mail: max@mat.uniroma3.it