DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 89–97

PLURIHARMONIC MORPHISMS BETWEEN COMPLEX MANIFOLDS

E. LOUBEAU

ABSTRACT. We study maps between complex manifolds which preserve, by post-composition, local pluriharmonic functions. We characterise such maps, called *pluriharmonic morphisms*, as $\pm$holomorphic maps.

All objects considered will be of class C^∞. The Einstein convention on the summation of repeated indices will be implicitly used.

1. Introduction

Harmonic maps are one of the main subjects of Differential Geometry. Their rich history has branched into many sub-fields, two of these we combine in this work.

The first, *pluriharmonic maps*, stems from one of the most exploited feature of harmonic maps from a surface: conformal invariance of the energy functional. A detailed description of this property forms the early part of Section 3.

A particularly geometrical subclass of harmonic maps, is the *harmonic morphisms*. Though defined in more analytical than geometrical terms, as pulling local harmonic functions back onto harmonic functions (this could be as well looked at in the Potential Theory language of harmonic sheaves), harmonic morphisms turn out to be exactly harmonic maps satisfying a geometrical condition called *horizontal weak conformality* (see § 4 for details, properties and some examples).

The question raised in Section 5 of the article, is of the characterisation of maps which play, for the pluriharmonic operator, the role harmonic morphisms perform for the Laplacian, i.e. pulling back local solutions onto local solutions.

2. Harmonic maps

In 1958, J. Eells published an article [9] which upholds the theory of harmonic maps

Theorem 1. [9] *Let (M,g) and (N,h) be (smooth) Riemannian manifolds, then $C^\infty(M,N)$, the set of smooth maps from M to N, is an infinite dimensional C^∞ manifold.*

1991 *Mathematics Subject Classification.* Primary 32F05; Secondary 31C10, 53C55, 58E20.
Key words and phrases. Pluriharmonic maps, harmonic maps, pluriharmonic morphisms.
This paper is in final form and no version of it will be submitted for publication elsewhere.

A few years later, Eells and Sampson published their seminal 1964 paper [10] which laid down the first stone to the edifice of an infinite dimensional Morse theory on the space $C^{\infty}(M,N)$.

The functional used to study $C^{\infty}(M,N)$ is the Energy Functional

$$E(\phi) = \frac{1}{2}\int_M \mid d\phi_x \mid^2 v_g$$

where v_g is the volume element of (M,g) and $\mid d\phi_x \mid^2$ is the Hilbert-Schmidt norm of $d\phi_x \in \bigwedge^1 \left(T_xM \otimes T_{\phi(x)}N\right)$. When M is not compact, the energy is defined on each compact subset of M.

Critical points of this functional are called *harmonic maps.*

Once we have remarked that the homotopy classes of maps from M to N are precisely the connected components of $C^{\infty}(M,N)$, the next theorem, due to Eells and Sampson and proved using the heat flow, takes its place in the aforementioned infinite dimensional Morse theory:

Theorem 2. [10] *Let (M,g) and (N,h) be compact Riemannian manifolds with $Riem(N) \leq 0$. Then for any map $h : M \to N$ there exists a harmonic map $\tilde{h} : M \to N$ homotopic to h.*

The Euler-Lagrange equation of the functional E, called *the tension field*, is

$$\tau(\phi) = \operatorname{trace} \nabla d\phi = \sum_{i=1}^{\dim M} \left({}^{\phi}\nabla_{e_i} d\phi(e_i) - d\phi\left(\nabla_{e_i} e_i\right)\right) \tag{1}$$

for an orthonormal frame $\{e_i\}_{i=1,\ldots,\dim M}$ and where ∇ is the Levi-Civita connection on (M,g) and ${}^{\phi}\nabla$ the connection on the pull-back bundle $\phi^{-1}TN$.

If we equip the manifold M^m with local coordinates $(x^i)_{i=1,\ldots,m}$ at the point $x \in M$, and the manifold N^n with local coordinates $(y^{\alpha})_{\alpha=1,\ldots,n}$ around the point $\phi(x)$, then the local expression for the tension field is

$$\tau^{\alpha}(\phi) = g^{ij}\left(\frac{\partial^2\phi^{\alpha}}{\partial x^i \partial x^j} - {}^{M}\Gamma^k_{ij}\frac{\partial \phi^{\alpha}}{\partial x^k} + {}^{N}\Gamma^{\alpha}_{\beta\gamma}\frac{\partial\phi^{\beta}}{\partial x^i}\frac{\partial\phi^{\gamma}}{\partial x^j}\right), \tag{2}$$

where ${}^{M}\Gamma^k_{ij}$ and ${}^{N}\Gamma^{\alpha}_{\beta\gamma}$ are the Christoffel symbols of (M,g) and (N,h).

A mere glance at Equation 2, is enough to see that harmonic maps generalise harmonic functions ($N = \mathbb{R}$) and (closed) geodesics ($M = \mathbb{S}^1$).

A major application of harmonic maps has been the construction of new minimal surfaces via:

Theorem 3. [10] *If $\phi : (M,g) \to (N,h)$ is an isometric immersion, then ϕ is harmonic if and only if ϕ is minimal.*

3. Pluriharmonic maps

Let $\phi : (M^2, g) \to (N,h)$ be a map from a two-dimensional Riemannian manifold, i.e. a surface.

Consider a metric $\tilde{g}$ on M^2, in the conformal class of g, i.e. such that there exists a function f on M for which $\tilde{g} = f^2 g$.

It is easy to see that for any map $\phi : M^2 \to N$ the (Hilbert-Schmidt) norm $\mid d\phi_x \mid^2_g$ of $d\phi_x$, $x \in M^2$, with respect to the metric g on M^2 and the norm $\mid d\phi_x \mid^2_{\tilde{g}}$ w.r.t. to $\tilde{g}$ are related by:

$$\mid d\phi_x \mid^2_g = f^2 \mid d\phi_x \mid^2_{\tilde{g}},$$

while the volume forms satisfy

$$v_{\tilde{g}} = f^{\dim M} v_g = f^2 v_g \,.$$

Therefore

$$\begin{aligned} E_{\tilde{g}}(\phi) &= \frac{1}{2}\int_M \mid d\phi_x \mid_{\tilde{g}}^2 \, v_{\tilde{g}} \\ &= \frac{1}{2}\int_M f^{-2} \mid d\phi_x \mid_g^2 \, f^2 v_g \\ &= E_g(\phi)\,. \end{aligned}$$

A direct application of this property is that it now makes sense to speak of a harmonic map from a Riemann surface, since the choice of a complex structure on an orientable two-dimensional manifold is equivalent to the adoption of a conformal class of metrics. Besides any Hermitian metric on a Riemann surface is in fact a Kähler metric.

In summary, harmonic maps from Riemann surfaces are harmonic w.r.t. any Kähler metric on the domain space.

When the real dimension of the domain manifold is strictly greater than two, the energy functional is no longer conformally invariant and harmonic maps from Hermitian manifolds of complex dimension strictly greater than one, are not necessarily harmonic w.r.t. any Kähler metric. Maps which satisfy this condition are called *pluriharmonic maps*, and can be defined from a complex manifold to a Riemannian manifold as follows:

Let (M, J) be a complex manifold of complex dimension m and (N, h) a Riemannian manifold of (real) dimension n. The tangent bundle $TM \to M$ can be complexified into the complex bundle $T^{\mathbb{C}}M$ and the eigenspaces corresponding to the eigenvalues $+i$ and $-i$ of the complex structure J, are called the holomorphic bundle $T^{(1,0)}M$ and antiholomorphic bundle $T^{(0,1)}M$. Over each of these two bundles, the exterior differential (extended to $T^{\mathbb{C}}M$ by complex linearity) $d^{\mathbb{C}}$ splits into its holomorphic, ∂, and antiholomorphic, $\overline{\partial}$, parts. Therefore if $\{e_i\}$ is a holomorphic frame on $T^{\mathbb{C}}M$ and $Z = Z^i e_i$ and $W = W^i e_i$ are two holomorphic vector fields then $\overline{W} = \overline{W^j} e_{\overline{j}} \in T^{(0,1)}M$ is antiholomorphic and we can define on the bundle $T^{(0,1)}M \to M$ the operator $\overline{\partial}_Z$ by:

$$\overline{\partial}_Z \overline{W} = Z^i \overline{\partial}_{e_i}\left(\overline{W^j} e_{\overline{j}}\right) = Z^i \frac{\partial \overline{W^j}}{\partial z^i} e_{\overline{j}}\,.$$

A map $\phi : M \to N$ is said to be *pluriharmonic* if

$$\nabla^{(1,0)} d''\phi = 0 \tag{3}$$

where

$$\left(\nabla^{(1,0)} d''\phi\right)(Z, \overline{W}) = {}^{\phi}\nabla_Z d''\phi(\overline{W}) - d''\phi\left(\overline{\partial}_Z \overline{W}\right) \quad \forall Z, W \in T^{(1,0)}M\,,$$

and ${}^{\phi}\nabla$ is the pull-back connection on $\phi^{-1}TN$, induced by the Levi-Civita connection on the Riemannian manifold (N, h).

Note that the choice of a Hermitian metric on M would infer a (uniquely defined) holomorphic connection on $T^{\mathbb{C}}M$ which is compatible with the Hermitian metric and whose antiholomorphic (or $(0,1)$) part coincides with the operator $\overline{\partial}$. Since a manifold is Kähler if and only if the holomorphic and Levi-Civita connections coincide, in this situation, we clearly see that the trace of the operator $\nabla^{(1,0)}d''$ is indeed the tension field τ of (1). If we equip the complex manifold (M, J) with the complex local coordinates $(z^i)_{i=1,\dots,m}$ and the Riemannian manifold (N,h) with the (real) coordinates $(y^\alpha)_{\alpha=1,\dots,n}$, then the local expression of (3) is

$$\frac{\partial^2\phi^\alpha}{\partial z^i\partial z^{\bar{\jmath}}} + {}^N\Gamma^\alpha_{\beta\gamma}\frac{\partial\phi^\beta}{\partial z^i}\frac{\partial\phi^\gamma}{\partial z^{\bar{\jmath}}} = 0 \quad \forall\alpha = 1,\dots,n, \quad \forall i,j = 1,\dots,m\,.$$

In particular a function $f : M \to \mathbb{R}$ is pluriharmonic if in local coordinates

$$\frac{\partial^2 f}{\partial z^i\partial z^{\bar{\jmath}}} = 0 \quad \forall i,j = 1,\dots,m\,.$$

Remark 1.

1. A direct implication of the last equation is that $\pm$holomorphic functions are pluriharmonic, a fact we will rely on in Section 5.
2. From a complex to a Hermitian manifold, all $\pm$holomorphic maps will be pluriharmonic if and only if the target space is Kähler (cf. [15]).
3. Finally, pluriharmonic maps are always harmonic if and only if the domain space is a cosymplectic manifold [20].

4. Harmonic morphisms

Harmonic morphisms are maps which preserve the harmonic sheaves between Brelot (or harmonic) spaces ([8]). In the context of Differential Geometry, we restrict ourselves to Riemannian manifolds, and use the following definition:

Definition 1. *A map $\phi : (M,g) \to (N,h)$ between Riemannian manifolds is a* harmonic morphism *if for any $f : U \subset N \to \mathbb{R}$ such that ${}^N\Delta f = 0$, then $f\circ\phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ satisfies ${}^M\Delta(f\circ\phi) = 0$.*

Remark 2.

1. This definition is purely local.
2. The composition of harmonic morphisms is again a harmonic morphism, something certainly not true for harmonic maps.
3. Even when the target manifold is compact, it admits many, in fact infinitely many as Lemma 1 on page 93 shows, local harmonic functions.

Definition 1 is rather awkward to exploit and the theory of harmonic morphisms relies heavily on their characterisation obtained independently in 1978-1979 by Fuglede [11] and Ishihara [13].

Theorem 4. [11, 13] *A map between Riemannian manifolds is a harmonic morphism if and only if it is a horizontally weakly conformal harmonic map.*

Aside from providing a handle on harmonic morphisms, Theorem 4 expresses in analytical and geometrical terms an essentially analytical object.

We now give an account of the condition of Horizontal Weak Conformality:
Let $\phi : (M,g) \to (N,h)$ be a map between two Riemannian manifolds. Let $x \in M$ be a non-degenerate point, i.e. such that the linear map $d\phi_x : T_xM \to T_{\phi(x)}N$ does not identically vanish.

We decompose the space T_xM into its *vertical space* $V_x = \ker d\phi_x$ and its *horizontal space* $H_x = \ker d\phi_x{}^\perp$, so that $T_xM = V_x \oplus H_x$. The map ϕ is then said to be *horizontally weakly conformal* if for any regular point $x \in M$, $d\phi_x$ is surjective (that is of maximal rank) and

$$d\phi_x\big|_{H_x} : H_x \to T_{\phi(x)}N$$

is conformal.

The conformal factor, denoted by $\lambda(x)$, is called the *dilation*.

Horizontal weak conformality is best summarised by the formula:

$$h\left(d\phi(X), d\phi(Y)\right) = \lambda^2 g\left(X, Y\right), \quad \forall X, Y \in H_x .$$

The definition implies that for a horizontally weakly conformal map, the differential, at a given point, is either of maximal rank (regular points) or is identically zero (critical points, which form a set of measure zero, see [11]).

When $\dim M < \dim N$, the only horizontally weakly conformal maps are the constant mappings.

It is easy to see that for a horizontally weakly conformal map the energy density $e(\phi) = \frac{1}{2} \mid d\phi_x \mid^2 = \frac{\dim N}{2}\lambda^2$. This implies that λ^2 is a smooth function on M (if extended by the value 0 over critical points), while λ is only continuous.

If $\lambda^2 \equiv 1$, the map is called a *Riemannian submersion*, and if $\operatorname{grad}\lambda^2 \in V$ (or equivalently $d\phi(\operatorname{grad}\lambda^2) = 0$), ϕ is called *horizontally homothetic*.

The chain rule for the Laplacian, for a horizontally weakly conformal map $\phi : M \to N$ and a function $f : U \subset N \to \mathbb{R}$, is

$$^{M}\Delta(f \circ \phi) = \lambda^{2N}\Delta(f) \circ \phi + df\left(\tau(\phi)\right) ,$$

from which the "if" statement of Theorem 4 easily follows. The "only if" part requires the combination of the general case of the chain rule for the Laplacian and the next lemma due to Ishihara:

Lemma 1. [13] *Let (N^n, h) be a Riemannian manifold and $q \in N$. Then for any set of constants $\{C_\alpha, C_{\alpha\beta}\}_{\alpha,\beta=1,\dots,n}$ such that $C_{\alpha\beta} = C_{\beta\alpha}$ and $\sum_\alpha C_{\alpha\alpha} = 0$, there exist a neighbourhood U of q and a harmonic function $f : U \subset N \to \mathbb{R}$ such that:*

$$\frac{\partial^2 f}{\partial y^\alpha \partial y^\beta}(q) = C_{\alpha\beta} \quad \frac{\partial f}{\partial y^\alpha}(q) = C_\alpha ,$$

for any choice of local coordinates $\{y^\alpha\}_{\alpha=1,\dots,n}$ around the point q.

Remark 3.

1. This result on the existence of local harmonic functions with prescribed 2-jets was extended by Hörmander in the appendix of [12] where Fuglede generalises Theorem 4 to semi-Riemannian geometry (after adjusting the definition of horizontally weakly conformal to non positive-definite metrics).
2. In fact this local existence result remains true for a large class of operator, in particular quasi-linear elliptic operators (cf. [1]). Following the original method of Ishihara [13], a characterisation similar to Theorem 4 was obtained for p-harmonic morphisms ([16]) and exponential harmonic morphisms ([17]).

The characterisation of harmonic morphisms leads to

Proposition 1. *Let $\phi : (M, g) \to (N, h)$ be a harmonic morphism between Riemannian manifolds. Then ϕ pulls back (local) harmonic maps onto harmonic maps, i.e. for any (local) harmonic map $\psi : U \subset (N, h) \to (P, h')$, where (P, h') is a Riemannian manifold, the map $\psi \circ \phi : \phi^{-1}(U) \subset (M, g) \to (P, h')$ is again harmonic.*

Remark 4. Between surfaces, harmonic morphisms are exactly (weakly) conformal maps. Therefore when the target is two-dimensional, a harmonic morphism will remain a harmonic morphism for any conformal change of metric and thus the concept of harmonic morphism into Riemann surfaces is well-defined.

So far the main application of harmonic morphisms has been the construction of new minimal submanifolds via the following result:

Theorem 5. [4] *Let $\phi : (M, g) \to N^2$ be a submersive harmonic morphism into a Riemann surface, then the fibres $\{\phi^{-1}(y); y \in N^2\}$ are minimal submanifolds of M.*

If we assume the domain manifold to be compact and three-dimensional, we have a statement of a more topological nature.

Theorem 6. [5] *Let $\phi : (M^3, g) \to N^2$ be a submersive harmonic morphism from a compact three-dimensional Riemannian manifold to a Riemann surface, then the set of fibres $\{\phi^{-1}(y); y \in N^2\}$ is a Seifert foliation, i.e. a decomposition into circles each having, on a neighbourhood, a fibre-preserving homeomorphism to a fibred (twisted) solid torus.*

Example 1.
1. *The most celebrated example of harmonic morphisms, is the Hopf map from $\mathbb{S}^3$ to $\mathbb{S}^2$. It actually turns out to be a Riemannian submersion ($\lambda^2 \equiv 1$) with totally geodesic fibres (diffeomorphic to $\mathbb{S}^1$). It can be constructed from the normed product in $\mathbb{C}$ and as such leads, via the normed products in $\mathbb{H}$ and Ca, to Hopf maps $\mathbb{S}^7 \to \mathbb{S}^4$ and $\mathbb{S}^{15} \to \mathbb{S}^8$ which again are harmonic morphisms (cf.* [3]*).*
2. *Since $\pm$holomorphic maps between Kähler manifolds are automatically harmonic (*[14]*) and maps to a surface are always horizontally weakly conformal, any $\pm$holomorphic map from a Kähler manifold to a Riemann surface is a harmonic morphism.*

Finally, I would like to mention several other avenues of investigation, notably the interaction between harmonic morphisms and (almost) complex structures ([23, 6]), harmonic morphisms and stability and minimality ([7, 19, 18]) and quite recently the development of a discrete version of harmonic morphisms by Urakawa ([22, 2].

An extensive list of articles on harmonic morphisms has been compiled by Gudmundsson and can be found on the internet at the address: `http://www.maths.lth.se/matematiklu/personal/sigma/harmonic/publications.html`

5. Pluriharmonic morphisms

Since pluriharmonic maps generalise, in a certain sense, harmonic maps, it is legitimate to introduce the notion of pluriharmonic morphisms.

Definition 2. *A map $\phi : M \to N$ between complex manifolds is a* pluriharmonic morphism *if it pulls back (local) pluriharmonic functions on N back onto (local) pluriharmonic functions on M. In other words, for any $f : U \subset N \to \mathbb{R}$ such that in local complex coordinates $\{y^\alpha\}$ on N*

$$\frac{\partial^2 f}{\partial y^\alpha \partial y^{\overline{\beta}}} = 0 \quad \forall \alpha, \beta = 1, \ldots, \dim N \,,$$

the function $f \circ \phi : \phi^{-1}(U) \subset M \to \mathbb{R}$ is pluriharmonic, i.e.

$$\frac{\partial^2 f}{\partial z^i \partial z^{\overline{\jmath}}} = 0 \quad \forall i, j = 1, \ldots, \dim M \,,$$

for a system of local complex coordinates $\{z^i\}$ on M.

Remark 5. We would obtain the same definition of pluriharmonic morphisms had we considered complex-valued functions. This is due to the fact that a complex-valued function is pluriharmonic if and only if its real and complex parts are (real-valued) pluriharmonic functions.

In contrast to harmonic morphisms between Riemannian manifolds, the characterisation we obtain for pluriharmonic morphisms between complex manifolds does not require the choice of metrics or connections.

Theorem 7. *Let (M, J) and $(N, \widetilde{J})$ be two complex manifolds. Then a map $\phi : M \to N$ is a pluriharmonic morphism if and only if ϕ is a $\pm$holomorphic map.*

Idea of the proof. Let $\phi : (M, J) \to (N, \widetilde{J})$ be a map between complex manifolds and $f : U \subset N \to \mathbb{C}$ a (locally defined) function. Equip the manifold M with the local complex coordinates $\{z^i\}$ at a point $q \in M$ and the manifold N with local complex coordinates $\{y^\alpha\}$ around $\phi(q) \in N$.

The method consists of two steps:

First, using the chain rule, we obtain that:

$$\frac{\partial^2 (f \circ \phi)}{\partial z^i \partial z^{\bar{j}}} = \frac{\partial f}{\partial y^C} \frac{\partial^2 \phi^C}{\partial z^i \partial z^{\bar{j}}} + \frac{\partial \phi^A}{\partial z^i} \frac{\partial \phi^B}{\partial z^{\bar{j}}} \frac{\partial^2 f}{\partial y^A \partial y^B} . \tag{4}$$

From this identity, we clearly see that if f is pluriharmonic (i.e. $\frac{\partial^2 f}{\partial y^\alpha \partial y^{\bar{\beta}}} = 0$) and ϕ is $\pm$holomorphic, then

$$\frac{\partial^2 (f \circ \phi)}{\partial z^i \partial z^{\bar{j}}} = 0, \quad \forall \alpha, \beta$$

so that ϕ does pull local pluriharmonic functions on N back onto (local) pluriharmonic functions on M and thus is a pluriharmonic morphism.

The reverse implication requires a Ishihara type of lemma on the local existence of pluriharmonic functions with prescribed 2-jets on a complex manifold. This is easily obtained since $\pm$holomorphic functions are pluriharmonic and the sum of two pluriharmonic functions is again pluriharmonic. The integrability of the complex structure enables us to use local complex coordinates with which we form complex polynomials with the needed first and second derivatives at the chosen point.

Plugging these special pluriharmonic functions into Equation (4), yields

$$\frac{\partial^2 \phi^\alpha}{\partial z^i \partial z^{\bar{j}}} = 0, \tag{5}$$

$$\frac{\partial \phi^\beta}{\partial z^{\bar{j}}} \frac{\partial \phi^\alpha}{\partial z^i} = 0 \quad \forall \alpha, \beta, i, j . \tag{6}$$

Either (i) for all $\alpha = 1, \dots, n$, and all i:

$$\frac{\partial \phi^\alpha}{\partial z^i} = 0$$

or (ii) there exists α_0 and i_0 such that:

$$\frac{\partial \phi^{\alpha_0}}{\partial z^{i_0}} \neq 0 .$$

Then Equation (6) shows that:

$$\frac{\partial \phi^\alpha}{\partial z^{\bar{i}}} = 0 ,$$

for all $\alpha = 1, \dots, n$ and all i.

Therefore at each point $p \in M$, ϕ is either antiholomorphic or holomorphic, but an argument of Siu [21] combined with Equation (5) shows that ϕ is either holomorphic or antiholomorphic on the whole of M (cf. [15] for details). □

However when dealing with maps, a connection on the target manifold is needed to make sense of pluriharmonicity.

Proposition 2. *A pluriharmonic morphism from a complex manifold to a Kähler manifold pulls back (local) pluriharmonic* maps *to (local) pluriharmonic* maps*.*

Proof. Let $\phi : M \to N$ be a pluriharmonic morphism (i.e. a $\pm$holomorphic map) and $\psi : N \to P$ a pluriharmonic map. Since N is Kähler, ϕ is a pluriharmonic map ([15]).

The map $\psi \circ \phi$ is pluriharmonic if

$$\nabla^{(0,1)} d'(\psi \circ \phi)(\overline{Z}, W) = 0 \qquad \forall Z, W \in T^{(1,0)}M\,.$$

But

$$\begin{aligned} &\nabla^{(0,1)} d'(\psi \circ \phi)(\overline{Z}, W) = \\ &d\psi \circ \nabla^{(0,1)} d'\phi(\overline{Z}, W) + \nabla d\psi(d\phi(\overline{Z}), d\phi(W)) = \qquad \text{as shown in [15]} \\ &\nabla d\psi(d\phi(\overline{Z}), d\phi(W)) = \\ &\nabla_{d\phi(\overline{Z})} d\psi(d\phi(W)) - d\psi\left(\nabla_{d\phi(\overline{Z})} d\phi(W)\right)\,. \end{aligned} \tag{7}$$

Suppose that ϕ is holomorphic then $d\phi(W) \in T^{(1,0)}N$, therefore

$$d\psi(d\phi(W)) = d'\psi(d\phi(W))$$

and, as the bundle $T^{(1,0)}N \to N$ is holomorphic and N is Kähler,

$$\nabla_{d\phi(\overline{Z})} d\phi(W) = \overline{\partial}_{d\phi(\overline{Z})} d\phi(W).$$

Therefore the right-hand side of (7) can be written

$$\nabla^{(0,1)} d'\psi(d\phi(\overline{Z}), d\phi(W))\,,$$

which vanishes since ψ is pluriharmonic. □

References

[1] Alinhac, S., Gerard, P., *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs Actuels. InterEditions \ Editions du C.N.R.S., 1991.

[2] Anand, C.K., Harmonic morphisms of metric graphs, In: C.K. Anand, P. Baird, E. Loubeau and J.C. Wood, editors, *Harmonic morphisms, harmonic maps and related topics. Brest 1997*, *Pitman Res. Notes Math. Ser.,* Pitman, Boston, London, Melbourne, to appear.

[3] Baird, P., *Harmonic maps with symmetry, harmonic morphisms, and deformation of metrics*, volume 87 of *Pitman Res. Notes Math. Ser.* Pitman, Boston, London, Melbourne, 1983.

[4] Baird, P., Eells, J., A conservation law for harmonic maps. In E. Looijenga, D. Siersma, and F. Takens, editors, *Geometry Symposium. Utrecht 1980*, volume 894 of *Lecture Notes in Math.*, pages 1–25. Springer, Berlin, Heidelberg, New York, 1981.

[5] Baird, P., Wood, J.C., *Harmonic morphisms and conformal foliations by geodesics of three-dimensional space forms*, J. Austral. Math. Soc. Ser. A, 51: 118–153, 1991.

[6] Baird, P., Wood, J.C., *Hermitian structures and harmonic morphisms on higher dimensional Euclidean spaces*, Internat. J. Math., 6: 161–192, 1995.

[7] Chen, J.Y., *Stable harmonic maps into S^2*, In: T. Kotake, S. Nishikawa, and R. Schoen, editors, Geometry and Global Analysis, 431–436. Tôhoku Univ., Sendai, 1993.

[8] Constantinescu, C., Cornea, A., *Potential Theory on Harmonic Spaces*, volume 158 of Die Grundlehren der mathematischen Wissenschaften, Springer-Verlag, 1972.

[9] Eells, J., *On the geometry of function spaces*, In: International Symposium in Algebraic Topology, 303–308. Universidad Nacional Aotonoma de Mexico and UNESCO, 1958.

[10] Eells, J., Sampson, J. H., Harmonic mappings of Riemannian manifolds. *Amer. J. Math.*, 86:109–160, 1964.

[11] Fuglede, B., *Harmonic morphisms between Riemannian manifolds*, Ann. Inst. Fourier (Grenoble), 28: 107–144, 1978.

[12] Fuglede, B., *Harmonic morphisms between semi-Riemannian manifolds*, Ann. Acad. Sci. Fenn. Ser. A I Math., 21: 31–50, 1996.

[13] Ishihara, T., *A mapping of Riemannian manifolds which preserves harmonic functions*, J. Math. Kyoto Univ., 19: 215–229, 1979.

[14] Lichnerowicz, A., *Applications harmoniques et variétés kählériennes*, Sympos. Math., 341–402, 1970.

[15] Loubeau, E., *Pluriharmonic morphisms*, to appear in Mathematica Scandinavica.

[16] Loubeau, E., *On p-harmonic morphisms*, preprint.

[17] Loubeau, E., Montaldo, S., *A note on exponentially harmonic morphisms*, to appear in Glasgow Mathematical Journal.

[18] Montaldo, S., *Stability of harmonic morphisms to a surface*, *Internat. J. Math.*, 9: 865–875, 1998.

[19] Montaldo, S., *Stability of harmonic maps and morphisms*, Leeds PhD thesis, 1996.

[20] Ohnita, Y., Valli, G., *Pluriharmonic maps into compact Lie groups and factorization into unitons*, Proc. London Math. Soc., 61: 546–570, 1990.

[21] Siu, Y. T., *The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds*, Ann. of Math., 112: 73–111, 1980.

[22] Urakawa, H., *A discrete analogue of the harmonic morphism and Green kernel comparison theorems*, preprint, 1997.

[23] Wood, J. C., *Harmonic morphisms and Hermitian structures on Einstein 4-manifolds*, Internat. J. Math., 3: 415–439, 1992.

Université de Bretagne Occidentale, UFR Sciences et Techniques
Departement de Mathematiques, 6, avenue Victor Le Gorgeu
BP 809, 29285 Brest Cedex, FRANCE
E-mail: loubeau@univ-brest.fr