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CURVATURE BOUNDS FOR THE SPECTRUM OF A COMPACT EINSTEIN-LIKE MANIFOLD

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Abstract. Let (M,g) be a connected compact n-dimensional, Riemannian manifold which is of class $\mathcal{A}$ according to Gray [3, p.261]. If the sectional curvatures of M are bounded below by a constant k_0 and the square of the length of curvature tensor field $\|R_X\|^2$ in the direction of a vector field X satisfies $\|R_X\|^2 \leq 2(n-1)k_0^2 L\|X\|^2$, where L is a constant (for the unit n-sphere S^n it satisfies $\|R_X\|^2 = 2(n-1)\|X\|^2$), then it is shown that either M is isometric to a sphere or else each nonzero eigenvalue λ of the Laplacian satisfies one of the following (i) $\lambda > \frac{2(3n-2)}{3}k_0$, (ii) $\lambda < nk_0L$, (iii) $\frac{2(3n-2)}{3}k_0 < \lambda < nk_0L$.

In addition to this, an estimate on $\|R_X\|^2$ is used to characterize spheres among the Riemannian manifolds in the class $\mathcal{A}$.

1. Introduction

One of the interesting questions in the Geometry of a Riemannian manifold (M,g) is to obtain lower bounds for the eigenvalues of the Laplacian operator Δ acting on the smooth functions on M. One of the classical results in this direction states that if the Ricci curvature of (M,g) satisfies $Ric \geq (n-1)k, k > 0$, $n = \dim M$ and M is compact, then the first nonzero eigenvalue λ_1 of Δ satisfies $\lambda_1 \geq nk$ (cf. [1]). Simon [5] considered compact Einstein manifolds (M,g), $\dim M \geq 2$ of constant scalar curvature (which is automatic for $\dim M \geq 3$), and proved that either they are isometric to a sphere or that each eigenvalue λ of the Laplacian satisfies $\lambda > 2nk_0$, where k_0 is the lower bound of the sectional curvatures. A similar result is obtained by Tanno [6]. Motivated by the work of Simon [5], we consider in this paper a compact Riemannian manifold (M,g) whose Ricci tensor satisfies $(\nabla_X Ric)(X,X) = 0, X \in \chi(M)$, where $\chi(M)$ is the Lie algebra of the smooth vector fields on M and ∇ is the Riemannian connection, and obtain the lower bounds for the eigenvalues of the Laplacian under some curvature

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assumptions. These Riemannian manifolds have been studied by Gray [3], where the class of such manifolds is denoted by $\mathcal{A}$ and that of Einstein manifolds by $\mathcal{E}$ and proved the strict inclusion $\mathcal{E} \subset \mathcal{A}$ (cf. §4 in [3]). In our work we use the square of the length $\|R_X\|^2$ of the curvature tensor field in the direction of X defined by

$$\|R_X\|^2 = \sum_{ij} \|R(e_i, e_j)X)\|^2, \quad X \in \chi(M)$$

for a local orthonormal frame $\{e_1, \dots, e_n\}$. For a sphere S^n of constant curvature k_0 this turns out to be $\|R_X\|^2 = 2k_0^2(n-1)\|X\|^2$. Our main result is the following

Theorem. *Let (M, g) be an n-dimensional compact and connected Riemannian manifold whose Ricci curvature satisfies*

$$(\nabla_X Ric)(X, X) = 0, \quad X \in \chi(M) .$$

If all the sectional curvatures of M are bounded below by a constant k_0 and the square of the length $\|R_X\|^2$ of the curvature tensor field in the direction of $X \in \chi(M)$ satisfies $\|R_X\|^2 \leq 2k_0^2(n-1)\|X\|^2 L$, where L is a constant, then either M is isometric to a sphere or else each eigenvalue λ of the Laplacian satisfies one of the following

$$\text{(i) } \lambda > \frac{2(3n-2)}{3}k_0, \quad \text{(ii) } \lambda < nk_0L, \quad \text{(iii) } \frac{2(3n-2)}{3}k_0 < \lambda < nk_0L .$$

We have also shown that an appropriate upper bound of $\|R_X\|^2$ in terms of the first nonzero eigenvalue of the Laplacian gives a characterization of spheres among the Riemannian manifolds in the class $\mathcal{A}$ (see theorems 4.1 and 4.2).

2. Preliminaries

Throughout this section (M, g) will be an n-dimensional Riemannian manifold without boundary and with Riemannian connection ∇. For a smooth function $f : M \to R$, the Laplacian of f is defined by $\Delta f = -div(grad\ f)$.

For a smooth function $f : M \to R$, we define an operator $A : \chi(M) \to \chi(M)$ by $A(X) = \nabla_X grad\ f,\ X \in \chi(M)$. This operator has the following properties:

$$tr.\ A = -\Delta f, \quad g(AX, Y) = g(X, AY), \tag{2.1}$$

$$(\nabla A)(X, Y) = (\nabla A)(Y, X) + R(X, Y)\ grad\ f , \tag{2.2}$$

where $(\nabla A)(X, Y) = \nabla_X A(Y) - A(\nabla_X Y)$ and R is the curvature tensor field given by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z .$$

$$g((\nabla A)(X, Y), Z) = g(Y, (\nabla A)(X, Z)), \quad X, Y, Z \in \chi(M). \tag{2.3}$$

This follows from the fact that A is symmetric (see (2.1)).

We denote by Q the Ricci operator defined by

$$Ric(X, Y) = g(Q(X), Y), \quad X, Y \in \chi(M) . \tag{2.4}$$

The second covariant derivative $(\nabla^2 A)(X,Y,Z)$, is defined by

$$(2.5)\ (\nabla^2 A)(X,Y,Z) = \nabla_X(\nabla A)(Y,Z) - (\nabla A)(\nabla_X Y, Z) - (\nabla A)(Y, \nabla_X Z)\,,$$

and we have the following Ricci identity

$$(2.6)\quad \begin{aligned} &(\nabla^2 A)(X,Y,Z) - (\nabla^2 A)(Y,X,Z) = R(X,Y)AZ - A(R(X,Y)Z),\\ &\qquad\qquad X,Y,Z \in \chi(M)\,. \end{aligned}$$

Lemma 1.

$$\begin{aligned} &(\nabla^2 A)(X,X,Z) =\\ &\quad (\nabla^2 A)(Z,X,X) + 2R(X,Z)AX - A(R(X,Z)X) + (\nabla_X R)(X,Z)\ grad\ f\,, \end{aligned}$$

where $A(X) = \nabla_X\ grad\ f,\ X \in \chi(M)$.

Proof. This follows immediately from (2.2), (2.5), (2.6) and the definition of the operator A.

Lemma 2. *For a local orthonormal frame* $\{e_1, \dots, e_n\}$ *of* M

$$\sum_i (\nabla A)(e_i, e_i) = -grad \Delta f + Q(grad\ f)\,,$$

where $A(X) = \nabla_X\ grad\ f,\ X \in \chi(M)$.

Proof. Using (2.1), we have $-\Delta f = \sum_i g(Ae_i, e_i)$, and thus for $X \in \chi(M)$, we have

$$\begin{aligned} -X(\Delta f) &= \sum_i g((\nabla A)(X, e_i), e_i) = \sum_i g((\nabla A)(e_i, X) + R(X, e_i) grad\ f, e_i)\\ &= \sum_i g(X, (\nabla A)(e_i, e_i)) + \sum_i g(R(X, e_i) grad\ f, e_i)\\ &= g(\sum_i (\nabla A)(e_i, e_i), X) - Ric(X, grad\ f), \end{aligned}$$

where we have used (2.2) and (2.3). The last equation proves the Lemma.

Lemma 3. *Let* (M, g) *be a compact Riemannian manifold and* $f : M \to R$ *be a smooth function. Then*

$$\int_M \{(\Delta f)^2 - g(grad\ f, grad \Delta f)\} dv = 0\,.$$

Proof. Using $\Delta f = -div(grad\ f)$, we obtain

$$\Delta(f \Delta f) = f \Delta^2 f + (\Delta f)^2 - 2g(grad\ f, grad \Delta f)\,.$$

Integrating this equation and using the fact that Δ is self-adjoint together with Stokes theorem we get the desired result.

Lemma 4. *Let* (M, g) *be a compact Riemannian manifold and* $f : M \to R$ *be a smooth function. Then*

$$\int_M \{\|A\|^2 - (\Delta f)^2 + Ric(grad\ f, grad\ f)\} dv = 0$$

where $A(X) = \nabla_X \, grad \, f, \; X \in \chi(M)$.

Proof. We use definition of A and equation (2.3) to compute

$$\begin{aligned} div(A(grad\, f)) &= \sum_i g(\nabla_{e_i} A(grad\, f), e_i) \\ &= \sum_i g((\nabla A)(e_i, grad\, f) + A(\nabla e_i grad\, f), e_i) \\ &= \sum_i g(grad\, f, (\nabla A)(e_i, e_i)) + \sum_i g(Ae_i, Ae_i) \\ &= g(grad\, f, \sum_i (\nabla A)(e_i, e_i)) + \|A\|^2 . \end{aligned}$$

Integrating this equation and using Lemma 2, we get the result.

Remark. If λ is an eigenvalue of Δ, i.e., $\Delta f = \lambda f$, then Lemma 3 gives $\int_M \{(\Delta f)^2 - \lambda \|grad\, f\|\} dv = 0$. Using this equation and the Schwarz inequality $\|A\|^2 \geq \frac{1}{n}(tr.A)^2 = \frac{1}{n}(\Delta f)^2$ in Lemma 4, we arrive at

$$\int_M \{-\frac{(n-1)}{n}\lambda \|grad\, f\|^2 + Ric(grad\, f, grad\, f)\} dv \leq 0 \,. \tag{2.7}$$

As an application of this we have the following corollary which is a well known result [1].

Corollary. *Let (M, g) be a compact and connected n- dimensional Riemannian manifold with all sectional curvatures bounded below by a constant k_0. Then the first nonzero eigenvalue λ_1 of the Laplacian satisfies $\lambda_1 \geq nk_0$ and the equality holds if and only if M is isometric to a sphere.*

Proof. Since all the sectional curvatures are bounded below by k_0, we have for a unit vector field $X \in \chi(M)$ extended to a local orthonormal frame $\{e_1, \ldots, e_n\}$ with $e_1 = X$, that

$$Ric(X, X) = \sum_i g(R(e_i, X)X, e_i) \geq (n-1)k_0 \,.$$

Thus (2.7) gives $\left(k_0 - \frac{\lambda}{n}\right) \int_M \|grad\, f\|^2 dv \leq 0$, which proves $\lambda \geq nk_0$, and the equality holds if and only if the equality $\|A\|^2 = \frac{1}{n}(tr.A)^2$ holds, which is equivalent to $A = \mu I$ for the function $\mu = -\frac{\lambda f}{n}$. Thus $H_f(X, Y) = -\frac{-\lambda f}{n} g(X, Y)$, $X, Y \in \chi(M)$, where H_f is the Hessian of the function f. By Obata's theorem M is isometric to a sphere [4].

Lemma 5. *Let (M,g) be a compact Riemannian manifold and $\{e_1,\dots,e_n\}$ be a local orthonormal frame. Then*

$$\int_M \{\sum_{i\alpha} g(\nabla_{e_\alpha} R)(e_\alpha, e_i) grad\ f, Ae_i)\} dv$$
$$= \int_M \{\sum_{i\alpha} R(e_i, e_\alpha; Ae_\alpha, Ae_i) - \frac{1}{2}\sum_{i\alpha} \|R(e_i,e_\alpha) grad\ f\|^2\} dv\,,$$

where $R(X,Y;Z,W) = g(R(X,Y)Z,W)$, $X,Y,Z,W \in \chi(M)$ and $AX = \nabla_X grad\ f$.

Proof. For a local orthonormal frame $\{e_1,\dots,e_n\}$ consider

$$X = \sum_i R(grad\ f, Ae_i)e_i$$

and compute

$$\begin{aligned} divX &= \sum_\alpha g(\nabla_{e_\alpha} X, e_\alpha) = \sum_{\alpha i} g(\nabla_{e_\alpha} R(grad\ f, Ae_i)e_i, e_\alpha) \\ &= \sum_{i\alpha} g((\nabla_{e_\alpha} R)(grad\ f, Ae_i)e_i, e_\alpha) + \sum_{i\alpha} g(R(Ae_\alpha, Ae_i)e_i, e_\alpha) \\ &\quad + \sum_{i\alpha} g(R(grad\ f, (\nabla A)(e_\alpha, e_i))e_i, e_\alpha) \end{aligned}$$

$$\begin{aligned} &= \sum_{i\alpha} (\nabla_{e_\alpha} R)(grad\ f, Ae_i; e_i, e_\alpha) + \sum_{i\alpha} R(Ae_\alpha, Ae_i; e_i, e_\alpha) \\ &\quad + \sum_{i\alpha} R(grad\ f, (\nabla A)(e_\alpha, e_i); e_i, e_\alpha)\,, \end{aligned}$$

where $R(X,Y;Z,W) = g(R(X,Y)Z,W)$, $X,Y,Z,W \in \chi(M)$. This gives

$$\begin{aligned} divX &= -\sum_{i\alpha} (\nabla e_\alpha R)(e_\alpha, e_i; grad\ f, Ae_i) + \sum_{i\alpha} R(e_i, e_\alpha; Ae_\alpha, Ae_i) \\ &\quad + \sum_{i\alpha} R(e_i, e_\alpha; grad\ f, (\nabla A)(e_\alpha, e_i)) \\ divX &= -\sum_{i\alpha} g((\nabla e_\alpha R)(e_\alpha, e_i; grad\ f, Ae_i) + \sum_{i\alpha} R(e_i, e_\alpha; Ae_\alpha, Ae_i) \\ &\quad + \sum_{i\alpha} g((\nabla A)(e_i, e_\alpha) - (\nabla A)(e_\alpha, e_i), (\nabla A)(e_\alpha, e_i))\,. \end{aligned}$$

Thus

$$\begin{aligned} divX &= -\sum_{i\alpha} g((\nabla e_\alpha R)(e_\alpha, e_i) grad\ f, Ae_i) + \sum_{i\alpha} R(e_i, e_\alpha; Ae_\alpha, Ae_i) \\ &\quad + \sum_{i\alpha} g((\nabla A)(e_i, e_\alpha), (\nabla A)(e_\alpha, e_i)) - \|\nabla A\|^2\,. \end{aligned} \tag{2.8}$$

Next, we have

$$\sum_{i\alpha} \|R(e_i, e_\alpha) grad\ f\|^2$$
$$= \sum_{i\alpha} g((\nabla A)(e_i, e_\alpha) - (\nabla A)(e_\alpha, e_i), (\nabla A)(e_i, e_\alpha) - (\nabla A)(e_\alpha, e_i))$$
$$= \sum_{i\alpha} \|(\nabla A)(e_i, e_\alpha)\|^2 + \sum_{i\alpha} \|(\nabla A)(e_\alpha, e_i)\|^2 - 2\sum_{i\alpha} g((\nabla A)(e_i, e_\alpha), (\nabla A)(e_\alpha, e_i))$$
$$= 2\|(\nabla A)\|^2 - 2\sum_{i\alpha} g((\nabla A)(e_i, e_\alpha), (\nabla A)(e_\alpha, e_i)),$$

which gives

$$-\frac{1}{2}\sum_{i\alpha} \|R(e_i, e_\alpha) grad\ f\|^2 = \sum_{i\alpha} g((\nabla A)(e_i, e_\alpha), (\nabla A)(e_\alpha, e_i)) - \|\nabla A\|^2$$

Using above equation in (2.8) and integrating over M we get the result.

Lemma 6. *Let (M, g) be a compact Riemannian manifold whose Ricci curvature satisfies $(\nabla_X Ric)(X, X) = 0$, $X \in \chi(M)$. Then for a local orthonormal frame $\{e_1, \ldots, e_n\}$,*

$$\int_M \{\sum_i g(\nabla e_i Q(grad\ f), Ae_i)\}$$
$$= \frac{1}{3}\int_M \{\sum_{i\alpha} R(e_i, e_\alpha; Ae_\alpha, Ae_i) - \frac{1}{2}\sum_{i\alpha} \|R(e_i, e_\alpha) grad\ f\|^2\} dv$$
$$+ \int_M \{\sum_i Ric(Ae_i, Ae_i)\} dv.$$

Proof. The condition $(\nabla_X Ric)(X, X) = 0, \quad X \in \chi(M)$ is equivalent to

(2.9)
$$(\nabla_X Ric)(Y, Z) + (\nabla_Y Ric)(Z, X) + (\nabla_Z Ric)(X, Y) = 0, \quad X, Y, Z \in \chi(M).$$

Using curvature properties and the second Bianchi identity we obtain

$$\sum_{i\alpha} g((\nabla_{e_\alpha} R)(e_\alpha, e_i) grad\ f, Ae_i)$$
$$= \sum_{i\alpha} (\nabla e_\alpha R)(e_\alpha, e_i; grad\ f, Ae_i) = -\sum_{i\alpha} (\nabla e_\alpha R)(grad\ f, Ae_i; e_i, e_\alpha)$$
$$= \sum_{i\alpha} (\nabla_{grad\ f} R)(Ae_i, e_\alpha; e_i, e_\alpha) + \sum_{i\alpha} (\nabla_{Ae_i} R)(e_\alpha, grad\ f; e_i, e_\alpha)$$
$$= -\sum_i (\nabla_{grad\ f} Ric)(Ae_i, e_i) + \sum_i (\nabla_{Ae_i} Ric)(grad\ f, e_i).$$

Now use (2.9) in the first term of the above equation to arrive at

$$(2.10)\quad \sum_{i\alpha} g((\nabla_{e_\alpha} R)(e_\alpha, e_i)grad\ f, Ae_i)$$
$$= 2\sum_i (\nabla_{Ae_i} Ric)(grad\ f, e_i) + \sum_i (\nabla_{e_i} Ric)(grad\ f, Ae_i)\,.$$

Since $(\nabla_X Ric)(Y,Z) = g((\nabla_X Q)(Y), Z)$ and $Ae_i = \sum_k g(Ae_i, e_k)e_k$, we get

$$\sum_i (\nabla_{Ae_i} Ric)(grad\ f, e_i)$$
$$= \sum_{ik} g(Ae_i, e_k)(\nabla_{e_k} Ric)(grad\ f, e_i) = \sum_{ik} g(Ae_i, e_k) g((\nabla_{e_k} Q)(grad\ f), e_i)$$
$$= \sum_k g((\nabla_{e_k} Q)(grad\ f), \sum_i g(Ae_k, e_i)e_i) = \sum_k g((\nabla_{e_k} Q)(grad\ f), Ae_k)$$
$$= \sum_k (\nabla_{e_k} Ric)(grad\ f, Ae_k)\,.$$

Thus (2.10) becomes

$$\frac{1}{3}\sum_{i\alpha} g((\nabla_{e_\alpha} R)(e_\alpha, e_i)grad\ f, Ae_i)$$
$$= \sum_i g((\nabla_{e_i} Q)(grad\ f), Ae_i) = \sum_i g(\nabla_{e_i} Q(grad\ f), Ae_i)$$
$$- \sum_i g(Q(Ae_i), Ae_i) = \sum_i g(\nabla_{e_i} Q(grad\ f), Ae_i) - \sum_i Ric(Ae_i, Ae_i)\,.$$

Integrating this last equation and using Lemma 5, we get the result.

Lemma 7. *Let (M, g) be an n-dimensional connected Riemannian manifold. Then*

$$\|\nabla A\|^2 \geq \frac{1}{n}\|grad\Delta f\|^2$$

and for a positively curved M, the equality holds if and only if $A(X) = \left(-\frac{\Delta f}{n}\right) X$, $X \in \chi(M)$, where $AX = \nabla_X grad\ f$, $X \in \chi(M)$.

Proof. Define $B(X,Y) = g(AX,Y) + \frac{\Delta f}{n} g(X,Y)$, $X, Y \in \chi(M)$. Then we obtain $(\nabla B)(X,Y,Z) = g((\nabla A)(X,Y),Z) + \frac{X(\Delta f)}{n} g(Y,Z)$, $X,Y,Z \in \chi(M)$.
Thus

$$\|\nabla B\|^2 = \sum_{ijk} [(\nabla B)(e_i, e_j, e_k)]^2$$
$$= \|\nabla A\|^2 + \frac{1}{n}\|grad\Delta f\|^2 + \frac{2}{n}\sum_{ij} g((\nabla A)(e_i, e_j), e_j) g(grad\Delta f, e_i)\,.$$

Now use equation (2.2) to obtain

$$\begin{aligned}
||\nabla B||^2 &= ||\nabla A||^2 + \frac{1}{n}||grad\Delta f||^2 \\
&\quad + \frac{2}{n}\sum_{ij} g((\nabla A)(e_j,e_i) + R(e_i,e_j)grad\ f, e_j)g(grad\Delta f, e_i) \\
&= ||\nabla A||^2 + \frac{1}{n}||grad\Delta f||^2 + \frac{2}{n}\sum_{ij}[g(e_i,(\nabla A)(e_j,e_j))g(grad\Delta f, e_i) \\
&\qquad - Ric(e_i, grad\ f)g(grad\Delta f, e_i)]\,.
\end{aligned}$$

Using Lemma 2, we find

$$||\nabla B||^2 = ||\nabla A||^2 + \frac{1}{n}||grad\Delta f||^2 - \frac{2}{n}||grad\Delta f||^2 = ||\nabla A||^2 - \frac{1}{n}||grad\Delta f||^2\,,$$

which proves $||\nabla A||^2 \geq \frac{1}{n}||grad\Delta f||^2$. If the equality holds, we should have $\nabla B = 0$, and as M is irreducible (M being of positive curvature), we get $B(X,Y) = ag(X,Y)$ for some constant $a \in R$. This together with the definition of B gives $A(X) = \left(a - \frac{\Delta f}{n}\right)(X)$. Since $tr.A = -\Delta f$, we get $-\Delta f = \left(a - \frac{\Delta f}{n}\right)n$, which implies $a = 0$; that is, $A(X) = \left(-\frac{\Delta f}{n}\right)X, \quad X \in \chi(M)$.

Lemma 8. *Let (M,g) be a compact Riemannian manifold and $\{e_1,\dots,e_n\}$ be a local orthonormal frame which diagonalizes A with $Ae_i = \lambda_i e_i$. Then*

$$\int_M \{\sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2\}dv = (n-1)\int_M (\Delta f)^2 dv - n\int_M Ric(grad\ f, grad\ f)dv\,,$$

where $A(X) = \nabla_X grad\ f,\ X \in \chi(M)$.

Proof. We have $\sum_{i\alpha}(\lambda_i - \lambda_\alpha)^2 = \sum_{i\alpha}\lambda_i^2 + \sum_{i\alpha}\lambda_\alpha^2 - 2\sum_{i\alpha}\lambda_i\lambda_\alpha = n||A||^2 + n||A||^2 - 2\sum_\alpha(-\Delta f)\lambda_\alpha = 2n||A||^2 - 2(\Delta f)^2$ and $\sum_{i\alpha}(\lambda_i - \lambda_\alpha^2) = 2\sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2$. Integrating the resulting equation and using Lemma 4, we get the result.

3. Proof of the Theorem

Define $F : M \to R$ by $F = \frac{1}{2}||A||^2$. Then it is straight-forward to see that the Hessian H_F of F is given by

$$\begin{aligned}
H_F(X,Y) &= \sum_{ij} g((\nabla A)(X,e_i),e_j)g((\nabla A)(Y,e_i),e_j) \\
&\qquad + \sum_{ij} g(Ae_i,e_j)g((\nabla^2 A)(X,Y,e_i),e_j)\,,
\end{aligned}$$

which gives

$$H_F(X,X) = \sum_{ij} g((\nabla A)(X,e_i),e_j)^2 + \sum_i g((\nabla^2 A)(X,X,e_i),Ae_i)\,.$$

Now, using equation (2.2) and Lemma 1 in the second term of the above equation, we arrive after some simple calculations, at

$$\begin{aligned} H_F(X,X) = &\sum_{ij} g((\nabla A)(X,e_i),e_j)^2 \\ &+ \sum_i [g((\nabla^2 A)(e_i,X,X),Ae_i) + 2R(X,e_i;AX,Ae_i) \\ &\qquad - R(X,e_i;X,A^2e_i) + g((\nabla_X R)(X,e_i)grad\ f, Ae_i)]\,. \end{aligned}$$

Using $-\Delta F = \sum_\alpha H_F(e_\alpha,e_\alpha)$ in this equation we get

$$\begin{aligned} -\Delta F = \|\nabla A\|^2 + &\sum_{i\alpha}[g((\nabla^2 A)(e_i,e_\alpha,e_\alpha),Ae_i) + 2R(e_\alpha,e_i;Ae_\alpha,Ae_i) \\ &- R(e_\alpha,e_i;e_\alpha,A^2e_i) + g((\nabla_{e_\alpha}R)(e_\alpha,e_i)grad\ f,Ae_i)] \end{aligned} \tag{3.1}$$

From Lemma 2, we find

$$\sum_\alpha (\nabla^2 A)(X,e_\alpha,e_\alpha) = -\nabla_X grad\Delta f + \nabla_X Q(grad\ f)\,,$$

which, together with (3.1), gives

$$\begin{aligned} -\Delta F = \|\nabla A\|^2 + &\sum_i [-g(\nabla_{e_i} grad\Delta f, Ae_i) + g(\nabla_{e_i} Q(grad\ f),Ae_i)] \\ &+ \sum_{i\alpha}[2R(e_\alpha,e_i;Ae_\alpha,Ae_i) - R(e_\alpha,e_i;e_\alpha,A^2e_i)] \\ &\qquad + \sum_{i\alpha} g((\nabla_{e_\alpha}R)(e_\alpha,e_i)grad\ f,Ae_i)\,. \end{aligned} \tag{3.2}$$

Next, we compute

$$\begin{aligned} div(A(grad\Delta f)) &= \sum_i g((\nabla A)(e_i,grad\Delta f) + A(\nabla_{e_i} grad\Delta f),e_i) \\ &= \sum_i g(grad\Delta f,(\nabla A)(e_i,e_i)) + \sum_i g(\nabla_{e_i} grad\Delta f,Ae_i) \\ &= -\|grad\Delta f\|^2 + Ric(grad\ f, grad\Delta f) + \sum_i g(\nabla_{e_i} grad\Delta f, Ae_i)\,. \end{aligned} \tag{3.3}$$

Substituting the value of $\sum_i g(\nabla_{e_i} grad\Delta f, Ae_i)$ as expressed by (3.3) into (3.2), integrating the resulting equation and using Lemmas 5 and 6, we get

$$\int_M \{\|\nabla A\|^2 - \|grad\Delta f\|^2 + Ric(grad\ f, grad\Delta f)$$
$$+ \frac{1}{3}\sum_{i\alpha}[4R(e_i, e_\alpha; Ae_\alpha, Ae_i) - 2\|R(e_i, e_\alpha) grad\ f\|^2]$$
$$+ \sum_i Ric(Ae_i, Ae_i) + \sum_{i\alpha}[2R(e_\alpha, e_i, Ae_\alpha, Ae_i)$$
$$- R(e_\alpha, e_i, e_\alpha, A^2 e_i)]\}dv = 0\,.$$

Taking $Ae_j = \lambda_j e_j$ in the above integral and noting that

$$\sum_i Ric(Ae_i, Ae_i) - \sum_{i\alpha} R(e_i, e_\alpha; Ae_\alpha, Ae_i)$$
$$= \sum_{i\alpha} \lambda_i^2 K_{i\alpha} - \sum_{i\alpha} \lambda_i\lambda_\alpha K_{i\alpha} = \frac{1}{2}[\sum_{i\alpha}\lambda_i^2 K_{i\alpha} + \sum_{i\alpha}\lambda_\alpha^2 K_{i\alpha} - 2\sum_{i\alpha}\lambda_i\lambda_\alpha K_{i\alpha}]$$
$$= \frac{1}{2}\sum_{i\alpha}(\lambda_i - \lambda_\alpha)^2 K_{i\alpha} = \sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2 K_{i\alpha}\,,$$

where $K_{i\alpha} = R(e_i, e_\alpha; e_\alpha, e_i)$ is the sectional curvature of the plane section spanned by $\{e_i, e_\alpha\}$, we get

$$(3.4)\quad \int_M \{\|\nabla A\|^2 - \|grad\Delta f\|^2 + Ric(grad\ f, grad\Delta f)$$
$$+ \frac{2}{3}\sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2 K_{i\alpha} + \frac{4}{3}\sum_i Ric(Ae_i, Ae_i) - \frac{2}{3}\sum_{i\alpha}\|R(e_i, e_\alpha) gradf\|^2\}dv = 0\,.$$

Since all the sectional curvatures are bounded below by k_0, we have

$$\begin{aligned}\int_M \{\sum_i Ric(Ae_i, Ae_i)\}dv &\geq k_0(n-1)\int_M \|A\|^2 dv\\ &= k_0(n-1)\int_M \{(\Delta f)^2 - Ric(grad\ f, grad\ f)\}dv\end{aligned}$$

where we used Lemma 4. Now, using Lemma 8, we obtain

$$\begin{aligned}\int_M \{\sum_i Ric(Ae_i, Ae_i)\}dv &\geq k_0(n-1)\int_M (\Delta f)^2 dv\\ &\quad -\frac{k_0(n-1)}{n}\int_M \{(n-1)(\Delta f)^2 - \sum_{i<\alpha}(\lambda_i - \lambda_k)^2\}dv\end{aligned}$$

that is,

$$(3.5)\quad \int_M \{\sum_i Ric(Ae_i, Ae_i)\}dv \geq k_0\frac{(n-1)}{n}\int_M (\Delta f)^2 dv + k_0\frac{(n-1)}{n}\int_M \{\sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2\}dv\,.$$

Now suppose that $\Delta f = \lambda f,\ \lambda > 0$, then we have

$$\int_M Ric(grad\ f, grad\Delta f)dv = \lambda\int_M Ric(grad\ f, grad\ f)dv = \frac{\lambda}{n}\int_M nRic(grad\ f, grad\ f)dv\,,$$

which in view of Lemma 8, gives

$$(3.6)\quad \int_M Ric(grad\ f, grad\Delta f)dv = \frac{\lambda(n-1)}{n}\int_M (\Delta f)^2 dv - \frac{\lambda}{n}\int_M \{\sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2\}dv\,.$$

Substituting from (3.5) and (3.6) into (3.4) and using the relations

$\sum_{i\alpha} \|R(e_i, e_\alpha)grad\ f\|^2 = \|R_{grad\ f}\|^2 \leq 2k_0^2(n-1)L\|grad\ f\|^2$, and

$\int_M (\Delta f)^2 dv = \lambda\int_M \|grad\ f\|^2 dv$ which is an outcome of Lemma 3, we obtain

$$(3.7)\quad \int_M (\|\nabla A\|^2 - \frac{1}{n}\|grad\nabla f\|^2)dv + \frac{2}{3n}\left((3n-2)k_0 - \frac{3\lambda}{2}\right)\int_M \{\sum_{i<\alpha}(\lambda_i - \lambda_\alpha)^2\}dv + \frac{4}{3}k_0(n-1)[\frac{\lambda}{n} - Lk_0]\int \|grad\ f\|^2 dv \leq 0\,.$$

If both $\frac{\lambda}{n} - Lk_0 \geq 0$ and $(3n-2)k_0 - \frac{3\lambda}{2} \geq 0$, then we must have by Lemma 7, $\|\nabla A\|^2 = \frac{1}{n}\|grad\Delta f\|^2$ and $\lambda = nLk_0 > 0$,(as f is a non-constant function $\Delta f = \lambda f,\ \lambda > 0$), which again by Lemma 7 implies that $A(X) = -\frac{\lambda f}{n}X$, that is, $H_f(X,Y) = \frac{-\lambda f}{n}g(X,Y)$. Hence by Obata's theorem M is isometric to a sphere. Otherwise we have one of the following (i) $(3n-2)k_0 - \frac{3\lambda}{2} < 0$, (ii) $\frac{\lambda}{n} - Lk_0 < 0$ (iii) $(3n-2)k_0 - \frac{3\lambda}{2} < 0$ and $\frac{\lambda}{n} - Lk_0 < 0$, and this proves the theorem.

4. Characterization of spheres

In this section we shall use the length $\|R_X\|$ of the curvature tensor field in the direction of a vector field X to obtain a characterization of the spheres. Note that for a sphere $S^n(k_0)$ we have $\|R_X\|^2 = 2(n-1)k_0^2 \|X\|^2$ and the first nonzero eigenvalue λ_1 is given by $\lambda_1 = nk_0$. Thus we have

$$\|R_X\|^2 = \frac{(n-1)}{2n}\lambda_1(4k_0)\|X\|^2 = \frac{n-1}{2n}\lambda_1\left[(3n+4)k_0 - 3\lambda_1\right]\|X\|^2 .$$

This raises a question, "Is a compact and connected positively curved n-dimensional Riemannian manifold satisfying

$$\|R_X\|^2 = \frac{(n-1)}{2n}\lambda_1\left[(3n+4)k_0 - 3\lambda_1\right]\|X\|^2 ,$$

where k_0 is the infimum of the sectional curvatures, isometric to S^n?"

We show that the answer to this question is in affirmative for the Riemannian manifolds of class $\mathcal{A}$. In fact we prove the following:

Theorem 4.1. *Let (M,g) be an n-dimensional compact and connected Riemannian manifold whose Ricci curvature satisfies*

$$(\nabla_X Ric)(X,X) = 0, \quad X \in \chi(M),$$

and all its sectional curvatures be bounded below by a constant $k_0 > 0$. If the first nonzero eigenvalue λ_1 of the Laplacian satisfies

$$\|R_X\|^2 \le \frac{(n-1)}{2n}\lambda_1\left[(3n+4)k_0 - 3\lambda_1\right]\|X\|^2, \quad X \in \chi(M),$$

where $\|R_X\|^2 = \sum_{ij}\|R(e_i,e_j)X\|^2$, then M is isometric to $S^n(k_0)$.

Proof. Let $f : M \to R$ be a smooth function such that $\Delta f = \lambda f,\ \lambda > 0$. Then equation (3.4) takes the form

$$\begin{aligned}
&\int_M \left\{\|\nabla A\|^2 - \frac{1}{n}\|grad\Delta f\|^2\right\} dv \\
&\quad + \int_M \left\{\left(-\frac{(n-1)}{n}\lambda^2 + (n-1)k_0\lambda\right)\|grad\ f\|^2 - \frac{2}{3}\|R_{grad\ f}\|^2\right\} dv \\
&\qquad + \frac{4}{3}\int_M \left\{\sum_i Ric(Ae_i, Ae_i)\right\} dv + \frac{2}{3}k_0\int_M \left\{\sum_{i<j}(\lambda_i - \lambda_j)^2\right\} dv \le 0
\end{aligned}$$

Using Lemma 3 in §2, in (3.5) we arrive at

$$\int_M \left\{ \sum_i Ric(Ae_i, Ae_i) \right\} dv$$
$$\geq k_0 \frac{(n-1)}{n} \lambda \int_M \|grad\ f\|^2\, dv + \frac{k_0(n-1)}{n} \int_M \left\{ \sum_{i<j} (\lambda_i - \lambda_j)^2 \right\} dv\,.$$

Thus (4.1) takes the form

$$\int_M \left\{ \|\nabla A\|^2 - \frac{1}{n} \|grad \Delta f\|^2 \right\} dv$$
$$+ \frac{2}{3} \int_M \left\{ \lambda \frac{(n-1)}{2n} [(3n+4)k_0 - 3\lambda] - \|R_{grad\ f}\|^2 \right\} dv$$
$$+ \frac{2}{3} \frac{(3n-2)}{n} k_0 \int_M \left\{ \sum_{i<j} (\lambda_i - \lambda_j)^2 \right\} dv \leq 0\,.$$

If the given condition holds, as the integrand in first integral is non-negative, we have from the last integral $A = \mu I$, which gives $\mu = -\frac{\lambda_1 f}{n}$, consequently $H_f(X,Y) = -\frac{\lambda_1 f}{n} g(X,Y)$ holds. Hence by Obata's theorem M is isometric to the sphere $S^n(c)$ whose $\|R_{grad\ f}\|$ according to (4.2) is given by

$$\|R_{grad\ f}\|^2 = \frac{\lambda_1(n-1)}{2n} [(3n+4)k_0 - 3\lambda_1] \|grad\ f\|^2$$
$$= \frac{(n-1)}{2} c\, [(3n+4)k_0 - 3nc] \|grad\ f\|^2\ ,$$

as $\lambda_1 = nc$. However $\|R_{grad\ f}\|$ for $S^n(c)$ is $\|R_{grad\ f}\|^2 = 2(n-1)c^2 \|grad\ f\|^2$ using this in (4.3), we get $c = k_0$ and consequently M is isometric to $S^n(k_0)$.

Similarly for Einstein manifolds we prove the following

Theorem 4.2. *Let (M, g) be an n-dimensional compact and connected non-negatively curved Einstein manifold of constant positive scalar curvature S. If the length $\|R_X\|$ and the first nonzero eigenvalue λ_1 of the Laplacian satisfy*

$$\|R_X\|^2 \leq \frac{(n-1)}{2n} \lambda_1 \left[\frac{(3n+4)S}{n(n-1)} - 3\lambda_1 \right] \|X\|^2\,, \quad X \in \chi(M)\,,$$

then M is isometric to $S^n\left(\frac{S}{n(n-1)}\right)$.

Proof. Since, $Ric = \frac{S}{n} g$, for a smooth $f : M \to R$ with $\Delta f = \lambda f$, $\lambda > 0$, the equation (3.4) gives

$$\int_M \left\{ \|\nabla A\|^2 - \frac{1}{n}\|grad \Delta f\|^2 \right\} dv$$

$$+\int_M \left\{ -\frac{(n-1)}{n}\lambda^2 \|grad\ f\|^2 + \frac{S}{n}\lambda \|grad\ f\|^2 + \frac{4}{3}\|A\|^2 - \frac{2}{3}\|R_{grad\ f}\|^2 \right\} dv$$

$$+\int_M \left\{ \sum_{i<j} (\lambda_i - \lambda_j)^2 K_{ij} \right\} dv = 0\,.$$

Now, using lemmas 4 and 8, we get

$$\int_M \left\{ \|\nabla A\|^2 - \frac{1}{n}\|grad \Delta f\|^2 \right\} dv$$

$$+\int_M \left\{ \frac{\lambda}{n}\left[\frac{S(3n+4)}{3n} - (n-1)\lambda \right] \|grad\ f\|^2 - \frac{2}{3}\|R_{grad\ f}\|^2 \right\} dv$$

$$+\int_M \left\{ \sum_{i<j} (\lambda_i - \lambda_j)^2 (K_{ij} + \frac{4}{3}\frac{S}{n^2}) \right\} dv = 0\,.$$

If the given condition holds, then as in the proof of last theorem, from above equation we conclude that M is isometric to $S^n(c)$ and it can be similarly shown that $c = \frac{S}{n(n-1)}$.

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