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ON THE PROPERTIES OF CERTAIN METRICS WITH CONSTANT SCALAR CURVATURE

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ABSTRACT. In a preceding paper [Ch1], we showed some curvature properties of the Delaunay type metrics. These metrics are locally conformally flat of constant scalar curvature on the Riemannian product $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$, a circle of length T crossed with the standard (n-1)-sphere. In particular we have determined the number of these metrics in a conformal class $[g_0]_T$. These metrics have a harmonic Riemannian curvature and a non parallel Ricci tensor, except for the cylindric one. We also remark a natural link between these metrics and the Derdzinski metrics, which are warped products: $dt^2 + f^2(t)d\xi^2$, and classify a family of Riemannian manifolds. Furthermore, we interest in the singularities of these Delaunay solutions for certain dimensions.

1. Introduction

Let us consider a smooth Riemannian manifold (M, g_0) of dimension $n \geq 3$, where g_0 is a Riemannian metric on M. dv_{g_0} denotes the volume form of g_0, and R_{g_0} is the associated scalar curvature.

$[g_0]$ denotes the conformal class of g_0 : $g \in [g_0]$, if and only if there exists a C^∞ positive function u such that $g = u^{\frac{4}{n-2}} g_0$ is a complete Riemannian metric on M. From analytical point of view, the problem is equivalent to put u as a positive solution of the differential equation

$$4\frac{n-1}{n-2}\Delta_{g_0} u + R_{g_0} u - R_g u^{\frac{n+2}{n-2}} = 0\,,$$

where $g = u^{\frac{4}{n-2}} g_0$ is complete on M.

R_{g_0} and R_g are the scalar curvatures associated respectively to g_0 and g, Δ_{g_0} is the Laplace-Beltrami operator on (M, g_0).

The Yamabe problem asserts that every conformal class $[g_0]$ contains at least one metric, say g, whose scalar curvature is constant.

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This problem has been completely resolved in the compact case of the manifold by H. Yamabe, T. Aubin and R. Schoen since 1984.

Notice that the Yamabe metric is unique up to homothety when

- The compact manifold (M, g_0) is such that its total curvature is non positive, i.e. $\int_M R_{g_0} dv \leq 0$.
- (M, g_0) is an Einstein manifold.

Multiplicity results may occur if the total curvature is positive. Nevertheless, the Yamabe problem remains open in the non compact case of the manifold (when (M, g_0) is complete).

There are some reductions of this problem, one of them is known as

Singular Yamabe problem: *Let* $\Lambda \subset (S^n, g_0)$ *a closed subset of the standard sphere of radius* r*. We seek a complete metric* g *on* $S^n - \Lambda$*, conformally equivalent to the standard metric, whose scalar curvature* R_g *is constant positive (here* $R_{g_0} = \frac{n(n-1)}{r^2}$*).*

In others words, the following differential equation

$$4\frac{n-1}{n-2}\Delta_{g_0} u + R_{g_0} u - R_g u^{\frac{n+2}{n-2}} = 0\,, \tag{1}$$

$g = u^{\frac{4}{n-2}} g_0$ *is complete on* $S^n - \Lambda$*,* $R_g = constant > 0$*, has a solution.*

Let $\mathcal{L}_g = \Delta_g + \frac{n-2}{4(n-1)} R_g$ denotes the conformal Laplacian of any Riemannian manifold (M, g). $\mathcal{L}_g$ posseses a special feature, remarked by W. Ding, [D]:
Invariance property of the conformal Laplacian: Let (M, g) *and* (N, h) *two Riemannian manifolds, such that there exists a conformal diffeomorphism* $f : M \to N$ *(that means if* f *is a diffeomorphism, there is a function* $v : M \to \mathbb{R}^+$*, such that* $f^* h = v^{\frac{4}{n-2}} g$*).*

Thus, for any function $\varphi : N \to \mathbb{R}$ *of class* C^2*, we have*

$$\mathcal{L}_h(\varphi) \circ f = v^{\frac{n+2}{n-2}} \mathcal{L}_g(v\varphi \circ f)$$

($\varphi = 1$ *and* $f = Id$ *gives the Yamabe equation.)*

Consequently, we may deduce from the above, if v is any positive solution of

$$\mathcal{L}_{g_0} v - \frac{n-2}{4(n-1)} K v^{\frac{n+2}{n-2}} = 0\,, \tag{2}$$

then K is precisely the scalar curvature of the metric $g = v^{\frac{4}{n-2}} g_0$.

Moreover, condition g complete requires that $lim_{x \to \Lambda} v(x) = \infty$.

2. Isolated singularities

Consider now the case where the singularities of a (singular) Yamabe solution are isolated. Here $\Lambda = \Lambda_k = (p_1, p_2, \ldots, p_k)$ is a finite point-set of the standard sphere.

Let $\Pi : S^n \setminus \{pt\} \to \mathbb{R}^n$ the stereographic projection. We know that the converse Π^{-1} defined by

$$\Pi^{-1}(x) = [\frac{2x}{|x|^2+2}, \frac{|x|^2-1}{|x|^2+1}]$$

is a conformal diffeomorphism, and we have $(\Pi^{-1})^\star(g_0) = \frac{4}{(|x|^2+1)^2}dx^2$, where dx^2 is the euclidean metric of $\mathbb{R}^n$.

Thus, the resolution of (2) is equivalent to the resolution of

$$\Delta u + \frac{n(n-2)}{4} u^{\frac{n+2}{n-2}} = 0 \tag{3}$$

on $\mathbb{R}^n - \tilde{\Lambda}_k$, where $\tilde{\Lambda}_k = \Pi(\Lambda_k)$. Moreover, u verify $lim_{x \to \tilde{\Lambda}_k} u(x) = \infty$.

2.1. Existence results for k = 2. Examine now the case $k = 2$. We shall see it will be possible to descrive all the Yamabe metrics. It is known that if u is a solution on $S^n - \{p_1, p_2\}$, then it is invariant with respect to any conformal transformation fixing these points. If these points are antipodal (as we may assume) then u is rotationally invariant. It is then convenient to project the sphere S^n to $\mathbb{R}^n$, from one of the singular point, p_1, so that p_2 sent to $0 \in \mathbb{R}^n$.

In this case, we have choosen $\tilde{\Lambda}_k = (0, \infty)$.

On the other hand, Cafarelli, Gidas and Spruck [C,G,S] proved that all solutions of (3) on $\mathbb{R}^n - \tilde{\Lambda}$ are radial. We then define the change

$$u(x) = |x|^{\frac{2-n}{2}} v(log\frac{1}{|x|}) .$$

In fact, this change corresponds to the conformal diffeomorphism

$$\beta : \mathbb{R}^n - \{0\} \to \mathbb{R} \times S^{n-1}$$

where $\beta(x) = (log\frac{1}{|x|}, \frac{x}{|x|})$.

Consequently, $v : \mathbb{R} \to \mathbb{R}^+$ is a solution of the autonomous ODE

$$\frac{d^2}{dt^2}v - \frac{(n-2)^2}{4}v + \frac{n(n-2)}{4}v^{\frac{n+2}{n-2}} = 0 \tag{4}$$

The analysis of (4) show us, there is only one center say $(\alpha, 0)$ corresponding to the constant solution $\alpha = (\frac{n-2}{n})^{\frac{n-2}{2}}$. All the periodic orbits denoted by$\gamma_c(t)$, are parametrized by $(v_c(t), \frac{d}{dt}v_c(t))$, c verifying $0 < c < c_0$. They are surrounded by the homoclinic orbit γ_{c_0}. Here $v_{c_0}(t) = (\cosh t)^{\frac{n-2}{2}}$. The period is also depending on $c : T = T(c)$.

The metrics

$$g = v_c^{\frac{4}{n-2}}(dt^2 + d\xi^2)$$

are called Delaunay type metrics. They are defined on the product manifold $(S^1 \times S^{n-1}, dt^2 + d\xi^2)$ a circle of length T crossed with the standard (n-1)-dimensional sphere. They belong to the conformal class of the cylindric metric $[dt^2 + d\xi^2]$.

Sum up the properties of the Delaunay metrics, see Schoen [Sc]:

- They are rotationally invariant, conformaly flat metrics
- They are finite and their number depends on T values ${g_c}^j = ({v_c}^j)^{\frac{4}{n-2}}(dt^2 + d\xi^2)$, where $j = 1, 2, \dots, k$ and $0 < c < c_0$.

In particular, if $T \leq 2\pi\sqrt{\frac{n-1}{R_{g_0}}}$ $\quad (= \frac{2\pi}{\sqrt{n-2}}$ if radius of S^{n-1} is1), we get only one metric. We have non unicity result if $T > 2\pi\sqrt{\frac{n-1}{R_{g_0}}}$.

2.2. Curvature properties. Consider a compact Riemannian manifold (M, g), $\mathcal{R}(g)$ is its Riemannian curvature, $r = Ric(g)$ is its Ricci tensor, and D is the Levi-Civita connexion. This curvature is harmonic if the formal divergence vanishes: $\delta\mathcal{R} = 0$.

It can be expressed in a system of local coordinates $D_k\mathcal{R}_{ij} = D_i\mathcal{R}_{kj}$. In this case, $Ric(g)$ is a Codazzi tensor. In particular, any Riemannian manifold with Ricci parallel tensor $Dr = 0$ (i.e. $D_i\mathcal{R}_{kj} = 0$) has an harmonic curvature. Notice that the Riemannian curvature must be harmonic for all Einstein manifolds and for all conformally flat manifolds with constant scalar curvature, see Besse [B], chap. 16.

Derdzinski had given examples of metrics with harmonic curvature but non parallel Ricci tensor $Dr \neq 0$, [De]. The corresponding manifolds are bundles with fibres N over the circle S^1 (parametrized by arc length t) with allowed warped metrics $dt^2 + h^{4/n}(t)g_0$ on the product $S^1 \times N$. Here, (N, g_0) is an Einstein manifold of dimension $n \geq 3$, and the function $h(t)$ on the prime factor is a periodic solution of the ODE, established by Derdzinski

$$h'' - \frac{nR}{4(n-1)}h^{1-4/n} = -\frac{n}{4}Ch \qquad \text{for some constant} \qquad C > 0\,. \tag{5}$$

This function must be non constant, otherwise the corresponding metric has a parallel Ricci tensor. In fact, as we shall see below the curvatures of the Delaunay metrics also share this property.

Our main result is the following

Theorem (2-1). *Consider the product manifold $(S^1(T) \times S^{n-1}, dt^2 + d\xi^2)$ a circle of length T crossed with the standard (n-1) sphere of radius 1.*

Suppose that $T > \frac{2\pi}{\sqrt{n-2}}$, the Ricci tensors of the Delaunay type metrics are non parallel and the Riemannian curvature are harmonic.

Moreover, any Delaunay type metric may be identified to a Derdzinski metric up to a conformal diffeomorphism.

Proof. Since the Delaunay metrics are conformally flat, then their Riemannian curvature are harmonic, (see Besse [B], Chap. 16).
Furthermore, if $T \leq \frac{2\pi}{\sqrt{n-2}}$, the only solution is the product metric, then its Ricci tensor is parallel.

If $T > \frac{2\pi}{\sqrt{n-2}}$, there exists at least one non trivial Delaunay, say $g_c = v_c^{\frac{4}{n-2}}(dt^2 + d\xi^2)$,[Sc], and [Ch]. We denote $g_0 = dt^2 + d\xi^2$.

Consider now a local chart in this product manifold, where the local coordonates are $(x^0, x^1, \dots, x^{n-1})$, and $x^0 = t$ is the coordinate corresponding to the circle factor S^1. Any metric tensor $\bar{g}$ in the conformal class $[g_0]$ can be written: $\bar{g} = v^{\frac{4}{n-2}} g_0$, where the C^∞ function v is defined on the circle. Then we obtain the expressions in a local coordonates system

$$\bar{g}_{00} = v^{\frac{4}{n-2}}\,, \quad \bar{g}_{0i} = 0\,, \quad \bar{g}_{ij} = v^{\frac{4}{n-2}}(g_0)_{ij}\,, \quad \text{where} \quad i,j = 1,2,\dots,n-1\,.$$

We get the Christoffel symbols $\bar{\Gamma}^i_{jk}$ associated to the metric $\bar{g}$. They are defined by the expression

$$\bar{\Gamma}^i_{jk} = \frac{1}{2}\bar{g}^{il}\left(\frac{\partial \bar{g}_{jl}}{\partial x^i} + \frac{\partial \bar{g}_{il}}{\partial x^j} - \frac{\partial \bar{g}_{jk}}{\partial x^l}\right).$$

Then, we obtain

$$\bar{\Gamma}^0_{jk} = -\frac{2}{n-2}\frac{v'}{v}\bar{g}_{jk}\,, \quad \bar{\Gamma}^i_{j0} = \frac{2}{n-2}\frac{v'}{v}\bar{\delta}^i_j\,, \quad \bar{\Gamma}^0_{00} = \frac{2}{n-2}\frac{v'}{v} \quad \text{and} \quad \bar{\Gamma}^i_{00} = 0\,.$$

The Ricci tensor components are

$$\bar{\mathcal{R}}_{ij} = \mathcal{R}_{ij} - \frac{2\nabla_{ij}v}{v} + \frac{2n}{n-2}\frac{\nabla_i v \nabla_j v}{v^2} - \frac{2}{n-2}\frac{\mid \nabla v \mid^2 + v\Delta v}{v^2} g_{ij}\,.$$

We have

$$\bar{\mathcal{R}}_{0i} = 0\,, \qquad \bar{\mathcal{R}}_{00} = 2\frac{n-1}{n-2}\left[\frac{v''}{v} + \frac{v'^2}{v^2}\right].$$

The associated scalar curvatures are

$$R(g_0) = (n-1)(n-2) \qquad \text{and} \qquad \bar{R}(v^{\frac{4}{n-2}} g_0) = n(n-1)\,.$$

So, the trace of the Ricci tensor gives the Yamabe equation

$$\bar{g}^{ij}\bar{\mathcal{R}}_{ij} = \bar{\mathcal{R}} = v^{\frac{-4}{n-2}}\left(R + \frac{4(n-1)}{(n-2)}\frac{\Delta v}{v}\right),$$

and the function v satisfies the above equation (4).

The covariant derivative of the Ricci tensor gives

$$D_k\bar{\mathcal{R}}_{ij} = \frac{\partial \bar{\mathcal{R}}_{ij}}{\partial x^k} - \bar{\Gamma}^l_{ik}\bar{\mathcal{R}}_{lj} - \bar{\Gamma}^l_{jk}\bar{\mathcal{R}}_{li}\,,$$

where D denotes the Riemannian connexion of the metric g.

In particular we get

$$D_0\bar{\mathcal{R}}_{00} = \frac{d\bar{\mathcal{R}}_{00}}{dt} - 2\bar{\Gamma}^k_{00}\bar{\mathcal{R}}_{k0}\,.$$

Thus, we obtain

$$D_0\bar{\mathcal{R}}_{00} = 2\frac{n-1}{n-2}\frac{v'''}{v} + 2\frac{n+2}{n-2}\frac{v'v''}{v^2} - 4\frac{v'^3}{v^3}\,.$$

By comparison with (4), v should be a periodical function. So, we deduce that, if the function u is non constant, then $D_0\bar{\mathcal{R}}_{00}\neq 0$. The Ricci curvature of the Delaunay metrics is then non parallel, except for the cylindric one.

Remark (2-2). More generally, Derdzinski established a classification of certain compact n-dimensional Riemannian manifolds $(M_n, g), n \geq 3$, with harmonic curvature. If the Ricci tensor $Ric(g)$ is not parallel and has less than three distinct eigenvalues at each point, then (M, g) is covered isometrically by a manifold

$$(S^1(T) \times N, dt^2 + h^{4/n}(t) g_0),$$

where the non constant positive periodic solutions verify the equation (5). Here $(N, g_0)$is a (n-1)-dimensional Einstein manifold with positive (constant) scalar curvature.

Return now to the Derdzinski metrics descrived above. We remark that, in the locally conformally flat case of this product, the factor N may be identified with the standard sphere S^{n-1}. Hence, these metrics must be conformally flat and will be conformal to the cylindric one.

Indeed, for the metric $dt^2 + f^2(t)d\xi^2$ we can write

$$dt^2 + f^2(t)d\xi^2 = f^2(t)[(\frac{dt}{f})^2 + d\xi^2].$$

After a change of variables and by using the conformal flatness of the product metric, we get

$$dt^2 + f^2(t)g_0 = \phi^2(\theta)[d\theta^2 + d\xi^2]$$

which is conformally flat. We then get

Lemma (2-3). *Any warped metric $dt^2 + f^2(t)g_0$ on the product Riemannian manifold $(S^1(T) \times S^{n-1}, dt^2 + g_0)$ must be conformally flat and conformal to a Riemannian metric product $d\theta^2 + d\xi^2$ where θ is a S^1-parametrisation of length $\int_{S^1} \frac{dt}{f(t)}$.*

Thus, any Derdzinski metric may be identified with a Delaunay type metric up to conformal diffeomorphism, let F. Then the metrics are relied

$$dt^2 + f^2(t)d\xi^2 = F^*({v^j}_c^{\frac{4}{n-2}}(dt^2 + d\xi^2)),$$

where ${v^j}_c$ are the Delaunay solutions belonging to the (same) conformal class.

3. Global estimates of the solutions

Sometimes, Equation (4) may be explicitly solved, as it is for certain dimensions. More precisely, for the dimensional cases $n = 4$ and 6, we are able [Ch1] to resolve the equation (4) by using the elliptic functions. For these dimensions, the functions ${v^j}_c$which have (necessairely) a real period T, turn out to be meromorphic. Moreover, T is precisely the period of $\gamma_c(t)$, the integral curve of Equation (4), which is an elliptic curve.

Recall that any correponding solution of Equation (3) (which is positive), must have two singular points in 0 and ∞, as has been shown by Cafarelli-Gidas-Spruck

[C-G-S]. Moreover, they have given an estimates near a singular point $p \in \Lambda$, that is for r near 0, u satisfies

$$C^{-1} r^{\frac{2-n}{2}} \leq u(r) \leq C r^{\frac{2-n}{2}}$$

for some constant C, which depends only on n. In others words the metrics ${g^j}_c = ({u^j}_c)^{\frac{4}{n-2}} g_0$ must be cylindrically bounded near $p \in \Lambda$. In fact, the estimate of the solution u does not need the hypothesis, r near 0, at least for the dimensions 4 and 6.

We obtain the following result, [Ch1].

Proposition (3-1). *Consider the Delaunay type metrics ${g^j}_c = ({u^j}_c)^{\frac{4}{n-2}} g_0$. For the dimensional values $n = 4$ and 6, we get the estimates*

$$r^{\frac{n-2}{2}} {u^j}_c(r) \leq {C^1}_n(T) \qquad \textit{and} \qquad r^{\frac{n}{2}} \frac{d{u^j}_c(r)}{dr} \leq C_n^2(T)\,,$$

where the constants ${C^1}_n$ and ${C^2}_n$ depend on the period T.

Proof of the proposition (3-1). Consider the orbits of the autonomous system associated to Equation (4)

$$v'^2 = (\frac{n-2}{2})^2 \left[v^2 - v^{\frac{2n}{n-2}} \right] - c\,. \tag{6}$$

These orbits are closed only if the energy constant satisfies the following condition $0 < c < c_0$ where

$$c_0 = \frac{2}{n} (\frac{n-2}{2})^2 (\frac{n-2}{n})^{\frac{n-2}{2}}\,.$$

As we have also used the change

$$u(x) = |x|^{\frac{2-n}{2}} v(log \frac{1}{|x|})\,,$$

it suffices to prove that the function $v(t)$ is bounded.
For $n = 4$, (6) becomes

$$v'^2 = v^2 - v^4 - c\,,$$

and all their solutions express by means elliptic Jacobi functions. For $n = 6$, (6) becomes

$$v'^2 = 4(v^2 - v^3)$$

all their solutions are the elliptic Weierstrass functions: $v(t) = \frac{1}{3} - \wp(t + \beta)$.
Moreover, it is known when t increases in the interval $]0, \frac{T}{2}[$, then the Weierstrass function $\wp(t + \beta)$ decreases monotonically in $[\wp(\frac{T}{2} + \beta), +\infty[$. So, we obtain explicitly the bounds for $n = 6$

$$r^2 u(r) \leq e^{-T} u(e^{\frac{-T}{2}})\,.$$

Also, for the dimension $n = 4$, we get an analog result

$$r u(r) \leq e^{\frac{-T}{2}} u(e^{\frac{-T}{2}})\,.$$

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