DIFFERENTIAL GEOMETRY AND APPLICATIONS
Satellite Conference of ICM in Berlin, Aug. 10 – 14, 1998, Brno
Masaryk University in Brno (Czech Republic) 1999, 13–21

NATURALLY REDUCTIVE ALMOST PRODUCT MANIFOLDS

VITALY V. BALASHCHENKO

ABSTRACT. It is proved that, in accordance with the Naveira classification, there are exactly three classes of invariant naturally reductive almost product structures. A remarkable collection of such structures is presented. These structures are canonical almost product structures on Riemannian k-symmetric spaces. Some particular cases including non-naturally reductive the 6-dimensional generalized Heisenberg group are considered in more detail.

Introduction

Riemannian almost product manifolds have been studied by many authors from various view-points. The survey of some investigations on the subject was included in [Mi]. The general classification of Riemannian almost product manifolds has been obtained by A. M. Naveira (see [Na]) following the method of A. Gray and L. M. Hervella for almost Hermitian structures (see [GH]). The examples of all the classes were constructed in [Mi], and the geometric properties for some of the classes were indicated in [GM], [GMN], [Mo] and others. It should be mentioned that, in contrast to almost Hermitian structures, homogeneous Riemannian almost product structures have not practically been considered.

The goals of this paper are

(i) to specify the Naveira classification for homogeneous naturally reductive almost product spaces and

---

1991 *Mathematics Subject Classification.* Primary 53C15, 53C30; Secondary 53C10, 53C35.
Key words and phrases. Almost product structure, homogeneous space, naturally reductive space, k-symmetric space.

This paper is in final form and no version of it will be submitted for publication elsewhere.

(ii) to present a large set of naturally reductive spaces (Riemannian k-symmetric spaces) with canonical almost product structures.

The paper is organized as follows.

Some preliminary notions and results are given in Section 1. In particular, the basic algebraic conditions for the distributions of an almost product structure are indicated.

In Section 2, we consider the Naveira classification in the case of invariant almost product structures on naturally reductive homogeneous spaces. It turns out that, in accordance with the classification, there are exactly three classes of such structures. They are $(\mathbf{TGF}, \mathbf{TGF}), (\mathbf{TGF}, \mathbf{AF}), (\mathbf{AF}, \mathbf{AF})$. The corresponding algebraic criteria for all the cases are obtained.

The main class of examples is illustrated in Section 3. It is well-known (see [Ko], [St]) that Riemannian k-symmetric spaces (homogeneous Φ-spaces of order k) with a suitable metric are naturally reductive. On the other hand, these spaces possess a large class of canonical classical structures, in particular, almost product structures (see [BS]). As a result, pseudo-Riemannian k-symmetric spaces with canonical structures provide a remarkable collection of naturally reductive almost product manifolds.

Finally, in Section 4, we examinate in more detail pseudo-Riemannian 4-symmetric spaces. In naturally reductive case they generally belong to the class $(\mathbf{TGF}, \mathbf{AF})$. As an example, the sphere S^5 is also of such a kind. The last example is the 6-dimensional generalized Heisenberg group that is not naturally reductive but admits the structure of a Riemannian homogeneous Φ-space of order 4.

1. Riemannian almost product manifolds

We recall some notions and results following generally to the notations of [Na], [GMN], [Mi].

Let (M, g) be a Riemannian manifold, P an almost product structure on M, i.e. P is a smooth tensor field of type $(1,1)$ such that $P^2 = id$. The pair (g, P) is called a Riemannian almost product structure if $g(P\mathbf{X}, P\mathbf{Y}) = g(\mathbf{X}, \mathbf{Y})$ for any vector fields $\mathbf{X}, \mathbf{Y}$ on M.

Any Riemannian almost product manifold (M, g, P) naturally admits two complementary mutually orthogonal distributions corresponding to the eigenvalues 1 and -1 of P. They are usually called vertical $\mathbf{V}$ and horizontal $\mathbf{H}$ respectively. We will denote by $A, B, C, ...$ vertical vector fields, X, Y, Z horizontal vector fields, and $S, T, U, ...$ arbitrary vector fields on M. Further, let p and q be the dimension of the distributions $\mathbf{V}$ and $\mathbf{H}$ respectively, $\{E_a\}_{a=1}^{p}$ a local orthonormal frame of $\mathbf{V}$, $\{E_u\}_{u=1}^{q}$ a local orthonormal frame of $\mathbf{H}$. As usual, denote by ∇ the Levi-Civita connection of the metric g.

The classification of Riemannian almost product structures was obtained by A. M. Naveira in [Na]. It turned out (see [GM]) that all the thirty six classes in the Naveira classification could be constructed by considering several conditions on each of the distributions $\mathbf{V}$ and $\mathbf{H}$. For simplicity we will formulate the conditions in terms of a vertical distribution.

AF (anti-foliation): $\nabla_A(P)A = 0$;
$\mathbf{D}_1$ (minimality): $\alpha(X) = \sum_{a=1}^{p} g(\nabla_{E_a}(P)E_a, X) = 0$;
$\mathbf{D}_2$ (umbilicity):

$$g(\nabla_A(P)B, X) + g(\nabla_B(P)A, X) = \frac{2}{p} g(A, B)\alpha(X);$$

$\mathbf{\Delta}$: no condition;
F (foliation): $\nabla_A(P)B = \nabla_B(P)A$;
TGF (totally geodesic foliation): $\nabla_A P = 0$;
$\mathbf{F}_1$ (minimal foliation): $\mathbf{D}_1$ and **F**;
$\mathbf{F}_2$ (umbilical foliation): $\mathbf{D}_2$ and **F**.

As a result, all the classes in the Naveira classification can be identified with the pairs $(\mathbf{D}, \mathbf{D}')$ (up to permutation) , where $\mathbf{D}$ and $\mathbf{D}'$ are the above conditions.

Finally, we recall the following two results

Theorem 1 [GM].
(i) $\{\mathbf{AF} \text{ and } \mathbf{F}\} \Longleftrightarrow \mathbf{TGF}$;
(ii) $\{\mathbf{D}_1 \text{ and } \mathbf{D}_2\} \Longleftrightarrow \mathbf{AF}$.

2. The Naveira classification for naturally reductive almost product manifolds

Naturally reductive homogeneous spaces were introduced as a natural generalization of Riemannian symmetric spaces. These spaces were shown to be of great importance in differential geometry and some related topics.

Let G be a connected Lie group, H its closed subgroup, g an invariant pseudo-Riemannian structure on the homogeneous space G/H. Denote by $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to G and H respectively. A homogeneous space $(M = G/H, g)$ is said to be naturally reductive [KN] if there exists a reductive decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ such that

$$\langle [S, T]_{\mathfrak{m}}, U\rangle = \langle S, [T, U]_{\mathfrak{m}}\rangle \qquad (1)$$

for any $S, T, U \in \mathfrak{m}$. Here $\langle\cdot,\cdot\rangle$ denotes the induced metric on $\mathfrak{m}$ and the subscript $\mathfrak{m}$ the projection onto $\mathfrak{m}$ with respect to the reductive decomposition.

Further, suppose P is an invariant pseudo-Riemannian almost product structure on $(G/H, g)$. The structure P defines the orthogonal decomposition $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$, where $\mathfrak{m}_1(\mathfrak{m}_2)$ is vertical (horizontal) subspace.

For simplicity let us denote by P both an invariant structure P and its value P_o at the point $o = H \in G/H$. It is concerned with all kinds of invariant structures (affinor structures, vector fields, Riemannian metrics etc.) throughout the paper.

Lemma 1. *Let $(G/H, \langle\cdot,\cdot\rangle, P)$ be a naturally reductive space equipped with an invariant metric $\langle\cdot,\cdot\rangle$ and an invariant almost product structure P, $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ the corresponding decomposition of $\mathfrak{m}$. Then the following relations are satisfied:*

$$\langle [A, B]_{\mathfrak{m}_1}, C\rangle = \langle A, [B, C]_{\mathfrak{m}_1}\rangle;$$

$$\langle [X,Y]_{\mathfrak{m}_2}, Z\rangle = \langle X, [Y,Z]_{\mathfrak{m}_2}\rangle;$$
$$\langle [A,B]_{\mathfrak{m}_2}, X\rangle = \langle A, [B,X]_{\mathfrak{m}_1}\rangle;$$
$$\langle [X,Y]_{\mathfrak{m}_1}, A\rangle = \langle X, [Y,A]_{\mathfrak{m}_2}\rangle.$$

Proof. For instance, we prove the first relation. Using (1), we get $\langle [A,B]_{\mathfrak{m}}, C\rangle = \langle A, [B,C]_{\mathfrak{m}}\rangle$. Now taking into account the fact that $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonal, we obtain$\langle [A,B]_{\mathfrak{m}_1}, C\rangle = \langle A, [B,C]_{\mathfrak{m}_1}\rangle$. The other relations can be proved in exactly the same way. □

Now we are able to algebraically characterize all the types of distributions on naturally reductive almost product manifolds.

Lemma 2. *Let $(G/H, \langle\cdot,\cdot\rangle, P)$ be a naturally reductive space with invariant structures $\langle\cdot,\cdot\rangle$ and P, $\mathfrak{m}_1$ and $\mathfrak{m}_2$ vertical and horizontal subspaces respectively. Then*

(i) $\mathfrak{m}_1$ *is of type* **TGF** *if and only if both of the following hold:*

$$[\mathfrak{m}_1,\mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}, \quad [\mathfrak{m}_1,\mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}. \tag{2}$$

Respectively $\mathfrak{m}_2$ *is of type* **TGF** *if and only if*

$$[\mathfrak{m}_2,\mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}, \quad [\mathfrak{m}_2,\mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}. \tag{3}$$

(ii) $\mathfrak{m}_1$ *is of type* **F** *iff (2). Respectively* $\mathfrak{m}_2$ *is of type* **F** *iff (3);*

(iii) $\mathfrak{m}_1$ *and* $\mathfrak{m}_2$ *are always of type* **AF**.

Proof. Without loss of generality it is sufficient to consider the vertical distribution only.

(i) The condition $\nabla_A P = 0$ means that $\nabla_A PS - P\nabla_A S = 0$. The Levi-Civita connection ∇ is invariant, and denote by α the Nomizu function of ∇ (see [No]). Now using special vector fields in a neighborhood of the point $o = H \in G/H$ (see [No]) it can be shown that the last condition is equivalent to the condition $\alpha(A, PS) = P\alpha(A,S)$. Since the metric g is naturally reductive, we have $\alpha(S,T) = \frac{1}{2}[S,T]_{\mathfrak{m}}$ (see [KN]). Hence we obtain

$$[A, PS]_{\mathfrak{m}} = P([A,S]_{\mathfrak{m}}) \tag{4}$$

for any $A \in \mathfrak{m}_1, S \in \mathfrak{m}$. It is easy to show that (4) is equivalent to the following pair of relations:

$$[A,B]_{\mathfrak{m}} \in \mathfrak{m}_1, \quad [A,X]_{\mathfrak{m}} \in \mathfrak{m}_2.$$

Evidently, the last system is equivalent to (2).

(ii) On analogy with the previous item one can prove that the condition $\nabla_A(P)B = \nabla_B(P)A$ is equivalent to that of $[\mathfrak{m}_1,\mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h}$. Using the last one and Lemma 1, we get

$$0 = \langle [A,B]_{\mathfrak{m}_2}, X\rangle = \langle A, [B,X]_{\mathfrak{m}_1}\rangle.$$

Since $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are orthogonal, we obtain $\langle S, [B,X]_{\mathfrak{m}_1}\rangle = 0$. It immediately follows that $[B,X]_{\mathfrak{m}_1} = 0$. Hence we have the condition $[\mathfrak{m}_1,\mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h}$. This proves that the condition **F** for $\mathfrak{m}_1$ implies (2). The converse is obvious. This completes the proof of item (ii).

(iii) On analogy one can get that the condition $\nabla_A(P)A = 0$ is equivalent to (4) for $S = A$, i.e. $[A, PA]_{\mathfrak{m}} = P([A, A]_{\mathfrak{m}})$. It follows $0 = 0$. This concludes the proof. □

Corollary. *Let $(G/H, \langle\cdot,\cdot\rangle, P)$ be a naturally reductive almost product manifold. Any foliation on G/H defined by the structure P is totally geodesic.*

Indeed, it is sufficient to apply items (i) and (ii).

Remark 1. The last statement can be proved in a different way. Namely, it is possible to use Lemma 2(iii) and Theorem 1(i).

To sum up, we have

Theorem 2. *In accordance with the Naveira classification, there are exactly three classes of invariant naturally reductive almost product structures. They are* $(\mathbf{TGF}, \mathbf{TGF}), (\mathbf{TGF}, \mathbf{AF}), (\mathbf{AF}, \mathbf{AF})$. *More precisely,*

(i) *G/H is $(\mathbf{TGF}, \mathbf{TGF})$ if and only if the following relations are satisfied*

$$[\mathfrak{m}_i, \mathfrak{m}_i] \subset \mathfrak{m}_i \oplus \mathfrak{h},\ i = 1, 2,\quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{h};$$

(ii) *G/H is $(\mathbf{TGF}, \mathbf{AF})$ if and only if the following relations are satisfied*

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{m}_1 \oplus \mathfrak{h},\quad [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2 \oplus \mathfrak{h};$$

(iii) *Any G/H belongs to $(\mathbf{AF}, \mathbf{AF})$.*

Remark 2. It is clear that $(\mathbf{TGF}, \mathbf{TGF})$ means a (local) product structure. As to $(\mathbf{TGF}, \mathbf{AF})$, it is a foliation with bundle-like metric, also known as the Reinhart foliation (see [GMN], [R]).

3. Canonical almost product structures on k-symmetric spaces

Here we give a brief survey on canonical affinor structures on homogeneous regular Φ-spaces (generalized symmetric spaces). For simplicity we concentrate on homogeneous Φ-spaces of finite order k (k-symmetric spaces). As to details, we refer to [BS], [Ko], [St].

Let G be a connected Lie group, Φ its automorphism. Denote by G^Φ the subgroup of all fixed points of Φ and G^Φ_o the identity component of G^Φ. Suppose a closed subgroup H of G satisfies the condition $G^\Phi_0 \subset H \subset G^\Phi$. Then G/H is called a homogeneous Φ-space. In the case $\Phi^2 = id$ we have exactly a homogeneous symmetric space.

The class of homogeneous Φ-spaces is fairly large and contains even non-reductive homogeneous spaces. Nevertheless any homogeneous Φ-space of order k ($\Phi^k = id$) is reductive (see [St]). We describe the situation in more detail.

Let G/H be a homogeneous Φ-space of order k, $\mathfrak{g}$ and $\mathfrak{h}$ the Lie algebras corresponding to G and H respectively, $\varphi = d\Phi_e$ the automorphism of $\mathfrak{g}$. Consider the linear operator $A_\varphi = \varphi - id$ of $\mathfrak{g}$. It had been proved by N. A. Stepanov [St] that G/H is a reductive homogeneous space. The corresponding reductive decomposition of the Lie algebra $\mathfrak{g}$ is of the form

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}, \quad \mathfrak{m} = A_\varphi \mathfrak{g}. \tag{5}$$

(see [St]). The decomposition (5) is called a canonical reductive decomposition of G/H. The canonical reductive supplement $\mathfrak{m}$ is invariant with respect to φ. Denote by θ a restriction of φ to $\mathfrak{m}$. As usual, we can identify $\mathfrak{m}$ with the tangent space $T_o(G/H)$.

Recall that an affinor structure on a manifold M is a smooth tensor field T of type $(1,1)$. An invariant affinor structure T on a homogeneous Φ-space G/H of order k is called canonical if its value at the point $o = H$ is a polynomial in θ: $T = T(\theta)$ (see [BS]). The set $\mathcal{A}(\theta)$ of all canonical affinor structures on G/H is a commutative algebra and $dim\mathcal{A}(\theta) = deg\,\nu \leq dim G/H$, where ν is the minimal polynomial of the operator θ. It is interesting to note that the algebra $\mathcal{A}(\theta)$ is trivial in the case of any symmetric Φ-space ($\Phi^2 = id$).

Generally, the algebra $\mathcal{A}(\theta)$ contains affinor structures of classical type (almost product, almost complex, f-structures). All the structures mentioned have been recently described (see [BS]). Here we will concentrate on the canonical almost product structures only. Denote by s the number of factors irreducible over $\mathbf{R}$ in the minimal polynomial ν of the operator θ.

Theorem 3 [BS]. *Let G/H be a homogeneous Φ-space of order k.*

i) *There exist precisely 2^s canonical almost product structures on G/H.*

ii) *Put $u = n$ if $k = 2n+1$, and $u = n-1$ if $k = 2n$. All canonical almost product structures on G/H can be defined by the polynomials $P = \sum_{m=0}^{k-1} a_m \theta^m$ such that*

1) *if $k = 2n+1$, then*

$$a_m = a_{k-m} = \frac{2}{k}\sum_{j=1}^{u} \xi_j cos\frac{2\pi mj}{k};$$

2) *if $k = 2n$, then*

$$a_m = a_{k-m} = \frac{1}{k}\left(2\sum_{j=1}^{u} \xi_j cos\frac{2\pi mj}{k} + (-1)^m \xi_n\right).$$

Here $\xi_j \in \{-1,1\}$, $j = 1, ..., u$.

As an example, in each of the cases $k = 4$ or $k = 5$ there is one unique (up to sign) generally non-trivial almost product structure. They are $P = \theta^2$ and $P = \frac{1}{\sqrt{5}}(\theta - \theta^2 - \theta^3 + \theta^4)$ respectively.

Further, suppose that G is a semisimple Lie group. In this case an invariant pseudo-Riemannian metric g on a regular Φ-space G/H can be defined by means of the Killing form of the corresponding Lie algebra $\mathfrak{g}$. This is the so-called standard metric. It is well-known that the standard metric on any regular Φ-space G/H of a semisimple Lie group G is naturally reductive (see [St]). Hence this is valid for any homogeneous Φ-space of order k.

Now we are able to indicate a large set of invariant pseudo-Riemannian almost product structures.

Theorem 4 [B]. *Let G be a semisimple Lie group, $(G/H, g)$ a homogeneous Φ-space of order k with the standard metric g. Then all canonical almost product structures P on G/H are compatible with g, i.e. $(G/H, g, P)$ is a pseudo-Riemannian almost product naturally reductive space.*

4. Examples

a) Pseudo-Riemannian 4-symmetric spaces.

General properties of the canonical structures of classical type on homogeneous Φ-spaces of order 4 were obtained in [BD]. In particular, we need the following result.

Theorem 5 [BD]. *Let G/H be a homogeneous Φ-space of order 4, $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ the corresponding reductive decomposition of $\mathfrak{g}$, $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2$ the decomposition with respect to the canonical almost product structure $P = \theta^2$. Then we have the relations*

$$[\mathfrak{m}_1, \mathfrak{m}_1] \subset \mathfrak{h}, \; [\mathfrak{m}_1, \mathfrak{m}_2] \subset \mathfrak{m}_2, \; [\mathfrak{m}_2, \mathfrak{m}_2] \subset \mathfrak{m}_1 \oplus \mathfrak{h}.$$

Now one can characterize the distributions of the canonical structure P in the Riemannian case. More precisely, we prove the following

Theorem 6. *Let G be a semisimple Lie group, $(G/H, g, P)$ a homogeneous Φ-space of order 4 with the standard metric g and the canonical structure $P = \theta^2$. Then G/H belongs to the class $(\mathbf{TGF}, \mathbf{AF})$. Moreover, $(G/H, g, P)$ is $(\mathbf{TGF}, \mathbf{TGF})$ if and only if G/H is locally symmetric ($[\mathfrak{m}, \mathfrak{m}] \subset \mathfrak{h}$) and $[\mathfrak{m}_1, \mathfrak{m}_2] = 0$.*

The proof is an immediate application of theorems 2 and 5.

Remark 3. Riemannian 4-symmetric spaces of the classical compact groups have been classified and geometrically described as fiber bundles by J.A.Jimenez (see [J]).

b) The sphere S^5.

It is well-known (see [Sa], [Ko]) that the sphere $S^5 \cong SU(3)/SU(2)$ admits the structure of a Riemannian homogeneous Φ-space of order 4 with the standard metric g. This representation of S^5 is not symmetric. The canonical reductive supplement $\mathfrak{m}$ corresponding to the outer automorphism Φ of the group $SU(3)$ is indicated in [BS]. It follows from theorem 6 that the canonical structure P on S^5 is exactly of type $(\mathbf{TGF}, \mathbf{AF})$. More precisely, the foliation mentioned is one-dimensional.

c) The 6-dimensional generalized Heisenberg group.

We breafly formulate some notions and results related to the 6-dimensional generalized Heisenberg group (N, g). As to details, we refer to [Ka1], [Ka2], [TV].

The Lie algebra $\mathfrak{n}$ of the 6-dimensional generalized Heisenberg group can be defined as follows. Consider the linear space

$$\mathfrak{n} := V \oplus Z = L(x_1, x_2, x_3, x_4) \oplus L(a_1, a_2)$$

with the brackets:

$$[x_1, x_2] = a_1, \quad [x_1, x_3] = a_2,$$
$$[x_2, x_4] = -a_2, \quad [x_3, x_4] = a_1,$$

all the other brackets are zero.

It is a 2-step nilpotent Lie algebra with center Z. The simply connected, connected Lie group with the Lie algebra $\mathfrak{n}$ is exactly the 6-dimensional generalized Heisenberg group N. Besides, the Lie group N has a left invariant metric g induced by an inner product on $\mathfrak{n}$.

It is known (see [TV], p.112) that (N, g) is a Riemannian homogeneous Φ-space of order 4. More exactly, the automorphism Φ is determined by means of the isometric automorphism φ of the Lie algebra $\mathfrak{n}$ such that $\varphi^4 = id$. For convenience, we consider φ written in the form

$$\varphi : (x_1, x_2, x_3, x_4, a_1, a_2) \to (-x_4, -x_3, x_2, x_1, -a_1, -a_2).$$

By our notations we have $\theta = \varphi$. It follows that for the canonical structure $P = \theta^2$ we have

$$\mathfrak{m} = \mathfrak{n} = Z \oplus V = \mathfrak{m}_1 \oplus \mathfrak{m}_2,$$

i.e. Z is vertical and V is horizontal with respect to P. It is easy to show that (g, P) is a Riemannian almost product structure.

It is important to note that (N, g) is not a naturally reductive homogeneous space (see [Ka2], [TV]). It means that we cannot use the theorems above and have to directly calculate. The Levi-Civita connection ∇ of the metric g was indicated in [Ka1]:

$$\left\{ \begin{array}{c} \nabla_x y = \frac{1}{2}[x, y] \\ \nabla_a x = \nabla_x a = -\frac{1}{2} j(a) x \\ \nabla_a b = 0, \end{array} \right. , \tag{6}$$

where $a, b \in Z, x, y \in V$ and $x_2 = j(a_1)x_1, x_3 = j(a_2)x_1, x_4 = j(a_1)j(a_2)x_1$.

Now directly using (6), we get $\nabla_a P = 0$ for any $a \in Z$. This means that the invariant distribution on (N, g) defined by Z is of type **TGF**. In the same manner one can show that $\nabla_x(P)x = 0$ for any $x \in V$, i.e. V belongs to **AF**.

To sum up, we have

Theorem 7. *The 6-dimensional generalized Heisenberg group (N, g) belongs to the class $(\mathbf{TGF}, \mathbf{AF})$ with respect to the canonical Riemannian almost product structure $P = \theta^2$ of a homogeneous Φ-space of order 4.*

References

[B] Balashchenko, V. V., *Riemannian geometry of canonical structures on regular Φ-spaces*, Preprint No.174/1994, Fakultät für Mathematik der Ruhr-Universität Bochum, 1994, 1-19.

[BD] Balashchenko, V. V., Dashevich, O. V., *Geometry of canonical structures on homogeneous Φ-spaces of order 4*, Russian Math. Surveys, **49**(1994), no.4, 149-150.

[BS] Balashchenko, V. V., Stepanov, N. A., *Canonical affinor structures of classical type on regular Φ-spaces*, Sbornik: Mathematics, **186**(1995), no.11, 1551-1580.

[GH] Gray, A., Hervella, L. M., *The sixteen classes of almost Hermitian manifolds and their linear invariants*, Ann. Mat. Pura Appl., **123**(1980), 35-58.

[GM] Gil-Medrano, O., *Geometric properties of some classes of Riemannian almost-product manifolds*, Rend. Circ. Mat. Palermo, Ser.2, **32**(1983), no.3, 315-329.

[GMN] Gil-Medrano, O., Naveira, A. M., *Some remarks about the Riemannian curvature operator of a Riemannian almost-product manifold*, Rev. Roumaine Mat. Pures Appl., **30**(1985), no.8, 647-658.

[J] Jimenez, J. A., *Riemannian 4-symmetric spaces*, Trans. Amer. Math. Soc., **306**(1988), no.2, 715-734.

[Ka1] Kaplan, A., *Riemannian nilmanifolds attached to Clifford modules*, Geom. Dedicata, **11**(1981), 127-136.

[Ka2] Kaplan, A., *On the geometry of groups of Heisenberg type*, Bull. London Math. Soc., **15**(1983), 35-42.

[Ko] Kowalski, O., *Generalized symmetric spaces*, LN in Math., V. 805, Berlin, Heidelberg, New York: Springer-Verlag, 1980.

[KN] Kobayashi, S., Nomizu, K., *Foundations of differential geometry*, V.2, Intersc. Publ. J.Wiley&Sons, New York - London, 1969.

[Mi] Miquel, V., *Some examples of Riemannian almost-product manifolds*, Pacif. J. Math., **111**(1984), no.1, 163-178.

[Mo] Montesinos, A., *On certain classes of almost-product structures*, Michigan Math. J., **30**(1983), 31-36.

[Na] Naveira, A. M., *A classification of Riemannian almost-product manifolds*, Rend. Mat. (7), **3**(1983), no.3, 577-592.

[No] Nomizu, K., *Invariant affine connections on homogeneous spaces*, Amer. J. Math., **76**(1954), no.1, 33-65.

[R] Reinhart, B. L., *Foliated manifolds with bundle-like metrics*, Ann. of Math., **69**(1959), 119-131.

[Sa] Sanchez, C. U., *Regular s-structure on spheres*, Indiana Univ. Math. J., **37**(1988), no.1, 165-180.

[St] Stepanov, N. A., *Basic facts of the theory of φ-spaces*, Izv. Vyssh. Uchebn. Zaved. Mat., 1967, no.3, 88-95; English transl. in Soviet Math. (Iz. VUZ), **11**(1967), no.3.

[TV] Tricerri, F., Vanhecke, L., *Homogeneous structures on Riemannian manifolds*, London Math. Soc., LN Ser., no.83, 1983.

Faculty of Mathematics and Mechanics, Belarussian State University
F. Scorina av. 4, Minsk, 220080, BELARUS
E-mail: vitaly@mmf.bsu.unibel.by