Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 497–508

Degenerate time-dependent Lagrangians of second order: the fourth order differential equation problem*

Manuel de León   Juan C. Marrero

Abstract. We develop a constraint algorithm for degenerate time-dependent Lagrangians of second order and solve the so-called fourth order differential equation problem. The study is inspired in the Gotay and Nester constraint algorithm and the notion of intermediate spaces introduced by Gràcia, Pons and Román-Roy.

Keywords. Degenerate, time-dependent, higher order Lagrangians, constraint algorithm.

MS classification. 58F05, 53C15, 53C05, 70H35.

1 Introduction

Several physical theories use higher order Lagrangians. A classical example of a second order regular Lagrangian appears in the elastic theory of straight beams [1, 18]. Also, the study of classical and relativistic spinning particles involved second order Lagrangians [37, 38, 9, 2]. Degenerate higher order Lagrangians appear in the study of the Einstein-Hilbert's action for gravitation (in fact, the Lagrangian of the Einstein theory of relativity does in fact contains second order derivatives of the metric field [10]), the Polyakov's formulation of the string [36] by adding a term dependent on the extrinsic curvature (see also [31]), the Podolski's generalized electrodynamics [35, 23] and its generalizations to Yang-Mills theories [11, 12], or the Lagrangian of two-points particles interacting gravitationally [39] (see also [19]). Also, the Hamiltonian formulation of the non-linear system described by the Korteweg-de Vries equations involved higher order Lagrangians [1, 32, 33]. We remit to [30] for a recent and interesting survey about the problems in the formulation of theories involving higher order derivatives.

The global geometric formulation of higher order Lagrangian systems was developed in the framework of higher order tangent bundles by using its underlying canonical structure, the almost tangent structure of higher order [22, 23, 24, 4]. An alternative procedure by using the higher order variational calculus on fibred manifolds is due to García and Muñoz [13]. A recent approach based on the notion of Lepagean forms introduced by Krupka is due to Krupková [20] (see also the references therein). Notice that the local study of these systems is due to Ostrogradskii and dates back to the past century [42].

*Supported by DGICYT-SPAIN, Proyecto PB91-0142 and Consejería de Educación del Gobierno de Canarias-SPAIN.
This paper is in final form and no version of it will be submitted for publication elsewhere.

Recently, Gràcia et al. [17] have developed a geometric constraint algorithm for degenerate Lagrangians of higher order by using the so called intermediate spaces (see [3]for the order 2). This technique permit us to decompose the Legendre-Ostrogradskii transformation in several steps and consider the constraint algorithm in each step. Further, on each step the corresponding higher order differential equation problem is considered. To do this, they used the notion of intermediate evolution operators. Another approach to this problem is due to Cariñena and López [6]. They have developed a constraint algorithm by extending the formulation of Skinner and Rusk [40, 41]. Also they introduce a time-evolution operator which passes Hamiltonian to Lagrangian constraints.

The aim of this paper is to extend the Gotay and Nester algorithm [14, 15, 16] to higher order degenerate time-dependent Lagrangians. We follow the ideas introduced in [7, 8] for degenerate time-dependent Lagrangians of order 1. Our approach is different to that of Gràcia, Pons and Román-Roy and we do not use the notion of evolution operators. However the notion of intermediate spaces will be widely used. In fact, our purpose is to follow the spirit of the Gotay and Nester algorithm. Since we deal with time-dependent Lagrangians, then the symplectic formalism is not adequate. Thus, we must use the cosymplectic formalism developed in [25] (see [26, 27, 5] for the order 1). We notice that only the second order case is considered, but the results may be easily extended to the general case. The paper is structured as follows. In section 2 we recall the cosymplectic formalism for regular time-dependent second order Lagrangians; a first constraint algorithm is considered. In section 3 we introduce the intermediate spaces for time-dependent Lagrangians and develop a constraint algorithm for each step. The fourth order differential equation problem is solved in section 4. We end the paper with an application to linear Lagrangians on the accelerations (see [29] for the autonomous case).

We thank P. Pitanga for several helpful conversations.

2 Time-dependent second order Lagrangians

The generalized evolution space of order k of an n-dimensional manifold Q is the $(k+1)n+1$ -dimensional manifold $J^k(R,Q)$ of all k-jets $j^k\sigma$ of differentiable curves in Q (see [28]). We may identify in a canonical way $J^k(R,Q)$ with the product manifold $R \times T^kQ$. If (q^i) are coordinates on Q, then we denote by $(t, q_0^i, q_1^i, \cdots, q_k^i)$ the induced coordinates on $J^k(R,Q)$. We denote by $\tilde{\rho}_r^k : J^k(R,Q) \longrightarrow J^r(R,Q)$, $r < k$ the canonical projection. From the natural identification $J^k(R,Q) \cong R \times T^kQ$ we may transport all the geometric structures on T^kQ to $J^k(R,Q)$. For instance, the canonical almost tangent structure J_1 of order k on T^kQ can be extended to a tensor field of type (1, 1) also denoted by J_1 and locally given by $J_1 = \sum_{r=0}^{k-1}(r+1)\frac{\partial}{\partial q_{r+1}^i} \otimes dq_r^i$. For each $r, 2 \le r \le k$, we set $J_r = J_1^r$, the composition of J_1 r-times. Also, let C_1 be the Liouville vector field on T^kQ may be translated to $J^k(R,Q)$. The translated vector field is also denoted by C_1 and it is locally given by $C_1 = \sum_{r=1}^{k} rq_r^i \frac{\partial}{\partial q_r^i}$. We define $C_r = J_{r-1}C_1, 2 \le r \le k$. We define the following $(1,1)$-tensor field: $\bar{J}_1 = J_1 + \frac{\partial}{\partial t} \otimes dt$. $\bar{J}_1$ is the canonical almost stable tangent structure of order k on $J^k(R,Q)$ (see [21]).

Recall that a vector field ξ on $J^k(R,Q)$ is said to be a $(k+1)$-th differential equation if $\bar{J}_1\xi = (\partial/\partial t) + C_1$, or equivalently, if $dt(\xi) = 1$ and $J_1\xi = C_1$. A solution of ξ is a curve σ in Q such that $j^k\sigma$ is an integral curve of ξ.

We deduce that a $(k+1)$-th differential equation ξ on $J^k(R,Q)$ is locally expressed by $\xi = \frac{\partial}{\partial t} + q_1^i \frac{\partial}{\partial q_0^i} + q_2^i \frac{\partial}{\partial q_1^i} + \cdots + q_k^i \frac{\partial}{\partial q_{k-1}^i} + \xi^i \frac{\partial}{\partial q_k^i}$, where $\xi^i = \xi^i(t, q_0^j, q_1^j, \cdots, q_k^j)$. Then a curve $\sigma(t) = (q^i(t))$ in Q is a solution of ξ if and only if it satisfies the following system of non-autonomous differential equations of order $k+1$:

$$\frac{d^{k+1}q^i}{dt^{k+1}} = \xi^i\Big(t, q(t), \frac{dq}{dt}, \cdots, \frac{d^k q}{dt^k}\Big), \quad 1 \leq i \leq n.$$

Let $L : J^2(R,Q) \longrightarrow R$ be a Lagrangian of order 2 and define the 1-form $\alpha_L = d_{J_1}L - \frac{1}{2}d_T d_{J_2}L$, the 2-form $\omega_L = -d\alpha_L$ and the function $E_L = C_1 L - \frac{1}{2}d_T C_2 L - L$ on $J^3(R,Q)$. E_L is the energy associated to L and then the Poincaré-Cartan 2-form on $J^3(R,Q)$ is given by $\Omega_L = -d\alpha_L + dE_L \wedge dt$. Let us recall that the generalized momenta are given by $\hat{p}_0^i = \frac{\partial L}{\partial q_1^i} - d_T(\frac{\partial L}{\partial q_2^i})$, $\hat{p}_1^i = \frac{\partial L}{\partial q_2^i}$. Then we have $E_L = \hat{p}_0^i q_1^i + \hat{p}_1^i q_2^i - L(t, q_0^i, q_1^i, q_2^i)$ and $\Omega_L = dq_0^i \wedge d\hat{p}_0^i + dq_1^i \wedge d\hat{p}_1^i + dE_L \wedge dt$.

The Legendre-Ostrogradskii transformation is the fibred mapping $Leg : J^3(R,Q) \longrightarrow R \times T^*TQ$ over TQ locally given by $Leg(t, q_0^i, q_1^i, q_2^i, q_3^i) = (t, q_0^i, q_1^i, \hat{p}_0^i, \hat{p}_1^i)$.

The Lagrangian L is said to be regular if the Hessian matrix $(W_{ij} = \frac{\partial^2 L}{\partial q_2^i \partial q_2^j})$ is non-singular. Otherwise L is said to be singular or degenerate. The following assertions are equivalent: (i) L is regular; (ii) (Ω_L, dt) is a cosymplectic structure; and (iii) the Legendre-Ostrogradskii transformation $Leg : J^3(R,Q) \longrightarrow R \times T^*TQ$ is a local diffeomorphism. If Leg is a global diffeomorphism then L is said to be hyperregular.

Now consider the global equations of motion :

$$\iota_\xi \Omega_L = 0, \quad i_\xi dt = 1. \tag{1}$$

If L is regular, then there exists a unique vector field ξ_L on $J^3(R,Q)$ satisfying (1). Furthermore ξ_L is a fourth order differential equation such that its solutions are the Euler-Lagrange equations for L:

$$\frac{d^2}{dt^2}\Big(\frac{\partial L}{\partial q_2^i}\Big) - \frac{d}{dt}\Big(\frac{\partial L}{\partial q_1^i}\Big) + \frac{\partial L}{\partial q_0^i} = 0.$$

In fact, ξ_L is the Reeb vector field of the cosymplectic structure (Ω_L, dt).

For the sake of simplicity, we suppose that L is hyperregular. Then we have the Hamiltonian function $H : R \times T^*TQ \longrightarrow R$, $H \circ Leg = E_L$ (if L is not hyperregular, then H is only locally defined). Let $\lambda_{TQ} = p_0^i dq_0^i + p_1^i dq_1^i$ be the Liouville 1-form on T^*TQ (here considered on $R \times T^*TQ$) and $\omega_{TQ} = -d\lambda_{TQ}$ the canonical symplectic 2-form on T^*TQ (also considered on $R \times T^*TQ$). Then we have a cosymplectic structure $(\Omega_H = \omega_{TQ} + dH \wedge dt, dt)$. Let X_H be the Reeb vector field of the structure (Ω_H, dt). X_H is the Hamiltonian vector field for H, i.e., $i_{X_H}\Omega_H = 0, i_{X_H}dt = 1$. Moreover $TLeg(\xi_L) = X_H$ and the integral curves of X_H satisfying the Hamilton equations for H:

$$\frac{dq_0^i}{dt} = \frac{\partial H}{\partial p_0^i}, \quad \frac{dp_0^i}{dt} = -\frac{\partial H}{\partial q_0^i}, \quad \frac{dq_1^i}{dt} = \frac{\partial H}{\partial p_1^i}, \quad \frac{dp_1^i}{dt} = -\frac{\partial H}{\partial q_1^i}. \tag{2}$$

From the local expression of X_H we can check that X_H satisfies a second order differential equation condition: $J(T\gamma_2(X_H)) = C$, where J is the canonical tensor field on $J^1(R,Q)$, $\gamma_2 : R \times T^*TQ \longrightarrow R \times TQ$ the canonical projection and C the Liouville vector field on TQ, here considered in $R \times TQ$. In fact (2) becomes:

$$\frac{dq_0^i}{dt} = \frac{\partial H}{\partial p_0^i}, \quad \frac{dp_0^i}{dt} = -\frac{\partial H}{\partial q_0^i}, \quad \frac{d^2 q_0^i}{dt^2} = \frac{\partial H}{\partial p_1^i}, \quad \frac{dp_1^i}{dt} = -\frac{\partial H}{\partial q_1^i}.$$

If the Lagrangian L is singular, then (Ω_L, dt) is not longer a cosymplectic structure and hence the equation (1) has no necessarily one solution; even, if there exists one solution, say ξ, then ξ is not necessarily unique nor a fourth order differential equation. Furthermore, in general there will not be a Hamiltonian counterpart.

A first attempt to solve this problem consists in to develop a constraint algorithm for the equations (1). In fact, we can extend the algorithm for time-dependent Lagrangians of order 1 [7, 8] and hence we obtain a sequence of submanifolds

$$\cdots S_3 \overset{g_3}{\hookrightarrow} S_2 \overset{g_2}{\hookrightarrow} S_1 = J^3(R,Q).$$

One of three possibilities may occur: (i) there exists a positive integer r such that $S_r = \emptyset$; (ii) we obtain a submanifold $S_r \neq \emptyset$ such that dim $S_r = 0$; (iii) there exists a positive integer r such that $S_r = S_{r+1}$, and dim $S_r \neq 0$. In the first case $S_r = \emptyset$ implies that (1) have no solution. In the second case the final submanifold S_r consists of isolated points and the solution of (1) is $\xi = 0$. In the third case we have a final submanifold $S_F = S_r$ and there exists a vector field ξ on S_F such that $(i_\xi \Omega_L = 0,\ i_\xi dt = 1)_{|S_F}$. In such a case the submanifold S_F will be called the final constraint submanifold.

Next, we consider the problem of the existence of a Hamiltonian formulation of the dynamics. We need an additional hypothesis on the Lagrangian. We suppose that $M_1 = Leg(R \times T^3Q)$ is a submanifold of $R \times T^*TQ$ and denote by $j_1 : M_1 \longrightarrow R \times T^*TQ$ the corresponding inmersion. Hence we may consider the map $Leg_1 : R \times T^3Q \longrightarrow M_1$; in other words, the map Leg_1 is the restriction of the map Leg onto its image. We say that L almost regular if Leg is a submersion onto M_1, and the fibers $Leg^{-1}(Leg\ (z))$ of Leg are connected for all $z \in J^3(R,Q)$. The manifold M_1 is called the primary constraint submanifold. Since Ker TLeg = Ker $\Omega_L \cap$Im J_2 then the manifold M_1 may be canonically identified with the leaf space of the foliation on $J^3(R,Q)$ determined by the involutive distribution Ker $TLeg$, i. e. $M_1 \cong R \times TQ/$Ker $TLeg$. Now we can check that the energy E_L is constant on the fibers of the submersion Leg and then we may define a Hamiltonian function $H_1 : M_1 \longrightarrow R$ such that $E_L = H_1 \circ Leg_1$. We put $\omega_1 = j_1^*\omega_{TQ}$, $\eta_1 = j_1^*(dt)$, $\Omega_{H_1} = \omega_1 + dH_1 \wedge \eta_1$. Then we have $Leg_1^*\ \omega_1 = \omega_L$, $Leg_1^*\ \eta_1 = dt$ and $Leg_1^*\ \Omega_{H_1} = \Omega_L$. Hence we may apply the constraint precosymplectic algorithm in both sides of the above diagram.

On the Hamiltonian side, the algorithm gives another sequence of submanifolds

$$\cdots M_4 \overset{j_4}{\hookrightarrow} M_3 \overset{j_3}{\hookrightarrow} M_2 \overset{j_2}{\hookrightarrow} M_1,$$

in such a way that $Leg_1(S_l) = M_l$, for all l. Consequently, Leg_1 induces a sequence of submersions $Leg_l : S_l \longrightarrow M_l$ such that the following diagrams are commutative

$$\begin{tikzcd} S_{l-1} \arrow[r, "Leg_{l-1}"] & M_{l-1} \\ S_l \arrow[u, "g_l"] \arrow[r, "Leg_l"] & M_l \arrow[u, "j_l"'] \end{tikzcd}$$

Therefore, the behaviour of the algorithm on both sides is the same. Thus, if we obtain a final constraint submanifold $S_{F'}$ on the Lagrangian side then we have a final constraint submanifold $M_{F'}$ on the Hamiltonian side and conversely (see [6] for the autonomous case).

Suppose now that X is a vector field on $S_{F'}$ such that satisfies (1); then, if X is $Leg_{F'}$-projectable to a vector field $TLeg_{F'}(X)$ on $M_{F'}$ we have $(i_{TLeg_{F'}(X)}\Omega_{H_1} = 0,\ i_{TLeg_{F'}(X)}\eta_1 = 1)_{|M_{F'}}$. Conversely, if Y is a vector field on $M_{F'}$ such that $(i_Y\Omega_{H_1} = 0,\ i_Y\eta_1 = 1)_{|M_{F'}}$, then every $X \in (TLeg_{F'})^{-1}(Y)$ satisfies $(i_X\Omega_L = 0,\ i_X dt = 1)_{|S_{F'}}$. Thus, if L is almost regular then there exists a Hamiltonian formulation of the dynamics, and the Lagrangian and Hamiltonian formulations are equivalent. Notice that from the construction procedure the submanifolds $S_{F'}$ and $M_{F'}$ are maximal and that for all $x \in M_{F'}$, $Leg^{-1}(x) \subseteq S_{F'}$. Therefore the involutive distribution Ker $TLeg$ restricts to the submanifold $S_{F'}$.

In order to solve the fourth order differential equation problem this formulation is not adequate. In fact, following Gotay and Nester [14, 15, 16] and Chinea, de León and Marrero [7, 8] we must construct a submanifold S of $S_{F'}$ and a vector field X on S such that $(i_X\Omega_L = 0,\ i_X dt = 1,\ J_1X = C_1)_{|S}$. For this purpose we must prove that if X is a $Leg_{F'}$-projectable vector field on $S_{F'}$ satisfying (1) then $J_1X - C_1 \in$ Ker $TLeg$. But, in general, it is not true. Thus this procedure does not work. In fact, we must incorporate to the constraint algorithm the condition $J_3X = C_3$ on the Lagrangian side and the condition of second order differential equation on the Hamiltonian side.

3 The constraint algorithms

Let $Leg : J^3(R,Q) \longrightarrow R\times T^*TQ$ be the Legendre-Ostrogradskii transformation defined by a Lagrangian L of order 2. Following Gràcia, Pons and Román-Roy [17] we first define the intermediate space P_1 that connects $J^3(R,Q)$ and $R \times T^*TQ$, through a decomposition of Leg

$$P_0 = J^3(R,Q) \xrightarrow{\alpha_0} P_1 \xrightarrow{\alpha_1} P_2 = R \times T^*TQ,$$

where $\alpha_0(t, q_0^i, q_1^i, q_2^i, q_3^i) = (t, q_0^i, q_1^i, q_2^i, \hat{p}_0^i)$, and $\alpha_1(t, q_0^i, q_1^i, q_2^i, p_0^i) = (t, q_0^i, q_1^i, p_0^i, \hat{p}_1^i)$. Then $Leg = \alpha_1 \circ \alpha_0$. Let us briefly describe the intermediate space P_1 (for an alternative description see [6]). Define $\bar{P}_1 = (R\times T^2Q)\times_{R\times TQ}(R\times T^*TQ)$. Let λ_{TQ} be the Liouville 1-form on T^*TQ and set $\phi = ({}^tJ)^*\lambda_{TQ}$, where ${}^tJ : T^*TQ \longrightarrow T^*TQ$ is the adjoint operator of the canonical almost tangent structure J on TQ and $({}^tJ)^*\lambda_{TQ}$ is the pull-back of λ_{TQ} via the mapping ${}^tJ$. On the other hand, let $\tilde{\phi} = J_1^*(d_{J_1}L - \frac{1}{2}d_T d_{J_2}L)$; $\tilde{\phi}$ is a 1-form on $R \times T^3Q$ which projects onto a 1-form $\bar{\phi}$ on $R \times T^2Q$. Then $P_1 = \{y = ((t,v),(t,\alpha)) \in \bar{P}_1 \mid \bar{\phi}(y) = \phi(y)\}$. Since $\phi = p_1^i dq_0^i$ and $\bar{\phi} = \hat{p}_1^i dq_0^i$ we deduce that $P_1 = \{y = ((t, q_0^i, q_1^i, q_2^i, p_0^i, p_1^i = \hat{p}_1^i) \in \bar{P}_1\}$. P_1 is a $(4n+1)$-dimensional submanifold of $\bar{P}_1$. We know that P_1 is fibred over $J^2(R,Q) \cong R \times T^2Q$, the canonical projection being denoted by $\gamma_1 : P_1 \longrightarrow R \times T^2Q$. We also define the vector subbundle V_1 of TP_1 by $V_1(y) = \{Y \in T_yP_1 \mid \bar{J}_1(T\gamma_1 Y) = 0\}$. On P_1 we define the partial energy function $E_1 = p_0^i q_1^i + \hat{p}_1^i q_2^i - L$ and the partial Poincaré-Cartan 2-form $\Omega_{E_1} = \alpha_1^*\omega_{TQ} + dE_1 \wedge \tau_1$, where $\tau_1 = \alpha_1^* dt$.

From a direct computation we prove that rank $\alpha_0 =$ rank $\alpha_1 = 3n + 1 +$ rank W_{ij}. Then L is regular iff any (and hence both) α_l, $l = 0, 1$ is a local diffeomorphism. Then the following assertions are equivalent: (i) L is regular; (ii) (Ω_L, dt) is cosymplectic; and (iii) (Ω_{E_1}, τ_1) is cosymplectic. In such a case we obtain three vector fields $\xi_0 = \xi_L$, ξ_1 and $\xi_2 = X_H$ determined by the three motion equations $i_{\xi_0}\Omega_L = 0, i_{\xi_0}dt = 1$; $i_{\xi_1}\Omega_{E_1} = 0, i_{\xi_1}\tau_1 = 1$; and $i_{\xi_2}\Omega_H = 0, i_{\xi_2}dt = 1$, respectively. A direct computation shows that $J_1\xi_0 = C_1$, $J_1(T\gamma_1(\xi_1)) = C_1$, and $J(T\gamma_2(\xi_2)) = C$. Notice that in the second equation J_1 is the canonical tensor field on $J^2(R,Q)$. In other words, ξ_0 is a fourth order differential equation and ξ_1 (resp. ξ_2) satisfies a third (resp. second) order differential equation condition.

We set $P_{11} = \alpha_0(P_0)$, $P_{21} = \alpha_1(P_1)$, and $\alpha_{01} : P_0 \longrightarrow P_{11}$, $\alpha_{11} : P_1 \longrightarrow P_{21}$ the corresponding restrictions. Further we set $P_{22} = \alpha_{11}(P_{11})$ and $\alpha_{12} : P_{11} \longrightarrow P_{22}$ the restriction of α_{11}. Fron now on, we are assuming the following hypotheses: (i) P_{11}, P_{21} and P_{22} are submanifolds; and (ii) α_{01}, α_{11} and Leg_1 are submersions with connected fibres. Then we obtain that α_{12} is also a submersion. Moreover, we have the following identifications:

$$P_{11} \cong \frac{J^3(R,Q)}{\mathrm{Ker}\ T\alpha_0}, \quad P_{21} \cong \frac{P_1}{\mathrm{Ker}\ T\alpha_1}, \quad M_1 = P_{22} \cong \frac{J^3(R,Q)}{\mathrm{Ker}\ TLeg}.$$

We remark that if the Legendre-Ostrogradskii transformation and the Hessian matrix $(W_{ij} = \frac{\partial^2 L}{\partial q_2^i \partial q_2^j})$ of L have constant rank, then P_{11}, P_{21} and P_{22} are, locally, submanifolds and α_{01}, α_{11} and Leg_1 are submersions.

We can easily check that the partial energy function E_1 is constant along the fibres of the submersion $\alpha_{11} : P_1 \longrightarrow P_{21}$. Hence it induces a function $\tilde{H}_1 : P_{21} \longrightarrow R$ such that $\tilde{H}_1 \circ \alpha_{11} = E_1$. Furthermore we have $\tilde{H}_1 \circ j_{22} = H_1$, where $j_{22} : P_{22} \longrightarrow P_{21}$ is the canonical inclusion. We set $\Omega_{\tilde{H}_1} = j_{21}^*\omega_{TQ} + d\tilde{H}_1 \wedge \tilde{\eta}_1$, being $j_{21} : P_{21} \longrightarrow P_2$ the inclusion and $\tilde{\eta}_1 = j_{21}^* dt$.

Next we apply the constraint algorithm to the three following equations:

$$\begin{aligned} &i_X\Omega_L = 0, \quad i_X dt = 1, \quad J_3X = C_3, \\ &i_X\Omega_{E_1} = 0, \quad i_X\tau_1 = 1, \quad \text{with } X \text{ tangent to } P_{11}, \\ &i_X\Omega_{\tilde{H}_1} = 0, \quad i_X\tilde{\eta}_1 = 1, \quad \text{with } X \text{ tangent to } P_{22}. \end{aligned}$$

In the first case, we obtain a sequence of submanifolds

$$\cdots P_{03} \overset{j_{03}}{\hookrightarrow} P_{02} \overset{j_{02}}{\hookrightarrow} P_{01} \overset{j_{01}}{\hookrightarrow} P_0 = J^3(R,Q),$$

where

$$P_{01} = \{x \in P_0 \mid \exists X \in T_xP_0 \text{ such that } i_X\Omega_L = 0, i_X dt = 1, J_3X = C_3\}.$$

On P_{01} there exists a solution but in general it is not tangent to P_{01} and then we must consider a new submanifold P_{02} on which there exists a solution tangent to P_{01}, and so on.

In the second case, we obtain a sequence

$$\cdots P_{13} \overset{j_{13}}{\hookrightarrow} P_{12} \overset{j_{12}}{\hookrightarrow} P_{11} \overset{j_{11}}{\hookrightarrow} P_1,$$

where

$$P_{12} = \{y \in P_{11} \mid \exists Y \in T_yP_{11} \text{ such that } \langle T_yP_1, i_Y\Omega_{E_1}\rangle = 0, i_Y\tau_1 = 1\}.$$

As above, on P_{12} there exists a solution but in general it is not tangent to P_{12} and then we must consider a new submanifold P_{13} on which there exists a solution tangent to P_{12}, and so on.

Finally, in the third case we obtain a sequence of submanifolds

$$\cdots P_{23} \overset{j_{23}}{\hookrightarrow} P_{22} \overset{j_{22}}{\hookrightarrow} P_{21} \overset{j_{21}}{\hookrightarrow} P_2 = R \times T^*TQ,$$

where

$$P_{23} = \{z \in P_{22} \mid \exists Z \in T_z P_{22} \text{ such that } \langle T_z P_{21}, i_Z \Omega_{\tilde{H}_1} \rangle = 0, i_Z \tilde{\eta}_1 = 1\}.$$

By a similar device we obtain P_{24}, P_{25}, and so on.

A direct computation shows that $\alpha_0(P_{01}) = P_{12}$ and the induced mapping $\alpha_{02} : P_{01} \longrightarrow P_{12}$ is also a submersion. Proceeding by induction we obtain a sequence of submersions $\alpha_{0,r+1} : P_{0r} \longrightarrow P_{1,r+1}$. Furthermore, we can check that $\alpha_{12}(P_{12}) = P_{23}$ and the induced mapping $\alpha_{13} : P_{12} \longrightarrow P_{23}$ is also a submersion. Then we obtain a sequence of submersions $\alpha_{1,r+1} : P_{1r} \longrightarrow P_{2,r+1}$. Therefore the three algorithms are equivalent. If one of these stops in some level, then the other two stop in the same level. In such a case we obtain three final constraint submanifolds $P_{0F}, P_{1,F+1}$ and $P_{2,F+2}$ which are connected by the partial Legendre-Ostrogradskii transformations:

$$P_{0F} \stackrel{\alpha_{0,F+1}}{\longrightarrow} P_{1,F+1} \stackrel{\alpha_{1,F+2}}{\longrightarrow} P_{2,F+2}.$$

The following diagram summarizes the above results:

$$\begin{tikzcd}
P_0 = J^3(R,Q) \arrow[r, "\alpha_0"] \arrow[dr, "\alpha_{01}"] \arrow[rr, bend left, "Leg"] & P_1 \arrow[r, "\alpha_1"] \arrow[dr, "\alpha_{11}"] & P_2 = R \times T^*TQ \\
P_{01} \arrow[u, "j_{01}"] \arrow[dr, "\alpha_{02}"] & P_{11} \arrow[u, "j_{11}"] \arrow[dr, "\alpha_{12}"] & P_{21} \arrow[u, "j_{21}"] \\
P_{02} \arrow[u, "j_{02}"] \arrow[dr, "\alpha_{03}"] & P_{12} \arrow[u, "j_{12}"] \arrow[dr, "\alpha_{13}"] & M_1 = P_{22} \arrow[u, "j_{22}"] \\
{} \arrow[u] & P_{13} \arrow[u, "j_{13}"] \arrow[dr, "\alpha_{14}"] & P_{23} \arrow[u, "j_{23}"] \\
 & {} \arrow[u] & P_{24} \arrow[u, "j_{24}"] \\
 & & {} \arrow[u] \\
{} & & \\
P_{0F} \arrow[u, "j_{0F}"] \arrow[dr, "\alpha_{0,F+1}"] & & \\
 & P_{1,F+1} \arrow[u, "j_{1,F+1}"] \arrow[dr, "\alpha_{1,F+2}"] & \\
 & & P_{2,F+2} \arrow[u, "j_{2,F+2}"]
\end{tikzcd}$$

It is clear that $P_{0F} \subseteq S_{F'}$ and $P_{2,F+2} \subseteq M_{F'}$.

4 The fourth order differential equation problem

A direct computation shows that $\dim \operatorname{Ker} T\alpha_0 \leq \dim \operatorname{Ker} TLeg \leq 2 \dim \operatorname{Ker} T\alpha_0$. In this section we shall assume that $\dim \operatorname{Ker} TLeg = 2 \dim \operatorname{Ker} T\alpha_0$. Thus, we have

Ker $T\alpha_1 = T\alpha_0(\text{Ker } TLeg)$, which implies that the involutive distribution Ker $T\alpha_1$ restricts to the submanifold P_{11}. Therefore for all $p_1 \in P_{11}$ we deduce that $\alpha_1^{-1}(\alpha_1(p_1)) \subseteq P_{11}$.

Suppose that we have obtained three final constraint submanifods $P_{0F}, P_{1,F+1}$ and $P_{2,F+2}$ which are connected by the partial Legendre–Ostrogradskii $\alpha_{0,F+1}$ and $\alpha_{1,F+2}$.

Proceeding by induction we have that for all $p_1 \in P_{1,F+1}$, $\alpha_1^{-1}(\alpha_1(p_1)) \subseteq P_{1,F+1}$ which implies that the involutive distribution Ker $T\alpha_1$ restricts to the submanifold $P_{1,F+1}$.

Let ξ_2 be a vector field on $P_{2,F+2}$ such that $(i_{\xi_2}\Omega_{\tilde{H}_1} = 0, i_{\xi_2}\tilde{\eta}_1 = 1)_{|P_{2,F+2}}$ and (t, q_0, q_1, q_2, p_0) local coordinates on P_1. Since $\alpha_{11}^*\Omega_{\tilde{H}_1} = \Omega_{E_1}, (T\alpha_1)(\frac{\partial}{\partial p_0}) = \frac{\partial}{\partial p_0}$ and $i_{\frac{\partial}{\partial p_0}}\Omega_{E_1} = q_1 dt - dq_0$, we obtain that ξ_2 satisfies the second order differential equation condition, i.e., $J(T\gamma_2(\xi_2)) = C$.

Next, let Y be a vector field on $P_{1,F+1}$ such that $(i_Y\Omega_{E_1} = 0, i_Y\tau_1 = 1)_{|P_{1,F+1}}$. Then there exists a unique vector field Y^* on $P_{1,F+1}$ such that $Y^* \in (\text{Ker } T_{\alpha_1})_{|P_{1,F+1}}$ and $(T\gamma_1)(Y^*) = J_1((T\gamma_1)Y) - C_1$. Suppose that Y is α_1-semiprojectable, i.e., $Y = Z + U$, where Z is α_1-projectable and $U \in V_1$. Denote by $\tilde{y'}$ the leaf of the distribution Ker $T\alpha_1$ passing through y'. By a similar technique to that used in [7, 8] we can choose an unique point y on each leaf $\tilde{y'}$ of $(\text{Ker } T\alpha_1)_{|P_{1,F+1}}$ such that $Y^*(y) = 0$, or equivalently $J_1(T\gamma_1(Y(y))) = C_1(\gamma_1(y))$, i.e., Y satisfies the third order differential equation condition at y. In fact, if $y' = (z, \beta) \in (R \times T^2Q) \times_{(R\times TQ)} (R \times T^*TQ)$ then $y = (T(\tilde{\rho}_1^2 \circ \gamma_1)(Y(y')), \beta)$. Notice that $T(\tilde{\rho}_1^2 \circ \gamma_1)(Y(y')) \in R \times T^2Q$ since Y satisfies the second order differential equation condition and $i_Y\tau_1 = 1$. Denote $y = W_Y$. If Y' is another α_1-semiprojectable solution such that $Y' - Y \in V_1$, then $W_{Y'} = W_Y$. Put $y = W_{[Y]}$, where $[Y]$ denotes the equivalence class of Y. Let $S_{[Y]}$ the set of these points $W_{[Y]}$ and define the canonical inclusion $j_1 : P_{2,F+2} \cong \frac{P_{1,F+1}}{(\text{Ker } T\alpha_1)_{|P_{1,F+1}}} \longrightarrow P_{1,F+1}$, $\tilde{y} \longrightarrow W_{[Y]}(\tilde{y})$, where $\tilde{y}$ denotes the leaf through y and $W_{[Y]}(\tilde{y})$ is the unique point on the leaf $\tilde{y}$ such that $J_1(T\gamma_1(Y(y))) = C_1(\gamma_1(y))$. Then $S_{[Y]} = \text{Im } j_1$ and the inclusion j_1 is a section of the submersion $\alpha_{1,F+2} : P_{1,F+1} \longrightarrow P_{2,F+2}$ in such a way that $S_{[Y]} \cong P_{2,F+2} \cong \frac{P_{1,F+1}}{(\text{Ker } T\alpha_1)_{|P_{1,F+1}}}$. Furthermore, along $S_{[Y]}$ we have $i_Y\Omega_{E_1} = 0, i_Y\tau_1 = 1, J_1(T\gamma_1(Y)) = C_1 \circ \gamma_1$, but Y is not necessarily tangent to $S_{[Y]}$. Hence we proceed as follows. Let ξ_2 be a vector field on $P_{2,F+2}$ such that $(i_{\xi_2}\Omega_{\tilde{H}_1} = 0, i_{\xi_2}\tilde{\eta}_1 = 1)_{|P_{2,F+2}}$ and consider a vector field Y on $P_{1,F+1}$ such that $T\alpha_{1,F+2}(Y) = \xi_2$. Then, since $\alpha_{1,F+2} \circ j_1 = id_{P_{2,F+2}}$ and using that $V_1 \subseteq$ Ker $T\alpha_1$ and the fact that $\alpha_{1,F+2} : S_{[Y]} \longrightarrow P_{2,F+2}$ is a diffeomorphism, we deduce that $\xi_1 = Tj_1(\xi_2)$ is the unique vector field on $S_{[Y]}$ which verifies $(i_{\xi_1}\Omega_{E_1} = 0, i_{\xi_1}\tau_1 = 1, J_1(T\gamma_1(\xi_1)) = C_1 \circ \gamma_1)_{|S_{[Y]}}$.

Therefore there exists a submanifold S_1 of $P_{1,F+1}$ and an unique vector field ξ_1 on it such that ξ_1 is a solution of the partial motion equations and satisfies the third order differential equation condition.

From the construction of section 3 and proceeding by induction we deduce that for all $p_0 \in P_{0F}$, $\alpha_0^{-1}(\alpha_0(p_0)) \subseteq P_{0F}$. Thus, $\tilde{S}_0 = \alpha_0^{-1}(S_1)$ is a submanifold of P_{0F} in such a way that the involutive distribution Ker $T\alpha_0$ restricts to $\tilde{S}_0$ and the induced mapping $\tilde{\alpha}_0 : \tilde{S}_0 \to S_1$ is a submersion. Suppose that $\tilde{\xi}_0$ is an $\tilde{\alpha}_0$−projectable vector field onto ξ_1. Then $(i_{\tilde{\xi}_0}\Omega_L = 0, i_{\tilde{\xi}_0}dt = 1, J_2\tilde{\xi}_0 = C_2)_{|\tilde{S}_0}$ and $J_1\tilde{\xi}_0 - C_1 \in$ Ker $T\alpha_0$. Denote by $\tilde{x}$ the leaf of the distribution Ker $T\alpha_0$ passing through x. By a similar procedure to that used above we can find an unique point $W_{[\tilde{\xi}_0]}(\tilde{x})$ on each leaf $\tilde{x}$ on which $\tilde{\xi}_0$ satisfies the fourth order differential equation condition. In fact, $W_{[\tilde{\xi}_0]}(\tilde{x}) = (T\tilde{\rho}_2^3)(\tilde{\xi}_0(x))$. We remark that $W_{[\tilde{\xi}_0]}(\tilde{x}) \in R \times T^3Q$ since $\tilde{\xi}_0$ satisfies the third order differential equation condition and $i_{\tilde{\xi}_0}dt = 1$. Set $S_{\tilde{\xi}_0} = \{W_{[\tilde{\xi}_0]}(\tilde{x})\}$. Notice that $\frac{\tilde{S}_0}{(\mathrm{Ker}\, T\alpha_0)_{|\tilde{S}_0}} \cong S_{[\xi_1]} = S_1$. Consider the canonical inclusion $j_0 : S_1 \cong \frac{\tilde{S}_0}{(\mathrm{Ker}\, T\alpha_0)_{|\tilde{S}_0}} \longrightarrow \tilde{S}_0$, $\tilde{x} \longrightarrow W_{[\tilde{\xi}_0]}(\tilde{x})$. Then $S_{\tilde{\xi}_0} = \mathrm{Im}\, j_0$ and $S_{\tilde{\xi}_0}$ is a submanifold of $\tilde{S}_0$ diffeomorphic to S_1. Thus we have $(i_{\tilde{\xi}_0}\Omega_L = 0, i_{\tilde{\xi}_0}dt = 1, J_1\tilde{\xi}_0 = C_1)_{|S_{\tilde{\xi}_0}}$. But $\tilde{\xi}_0$ is not necessarily tangent to $S_{\tilde{\xi}_0}$. Now, it is sufficient to define $\xi_0 = Tj_0(\xi_1)$. In fact, since $\alpha_{0,F+1} \circ j_0 = id_{S_1}$ and using that the mapping $\alpha_{0,F+1} : S_{\tilde{\xi}_0} \longrightarrow S_1$ is a diffeomorphism, we deduce that ξ_0 is the unique vector field tangent to $S_{\tilde{\xi}_0}$ which verifies $(i_{\xi_0}\Omega_L = 0, i_{\xi_0}dt = 1, J_1\xi_0 = C_1)_{|S_{\tilde{\xi}_0}}$.

Therefore there exists a submanifold S_0 of P_{0F} and an unique vector field ξ_0 on it such that ξ_0 is a solution of the motion equations and satisfies the fourth order differential equation condition.

5 Linear Lagrangians on the accelerations

Let f, h two functions on $R\times TQ$. We consider a Lagrangian $L : R\times T^2Q \longrightarrow R$ defined by $L = d_T f + (\tilde{\rho}_1^2)^* h$. We suppose that f does not depends on t, since in this case then the term $\frac{\partial f}{\partial t}$ will be included in h. In local coordinates we have $L = \frac{\partial f}{\partial q_0} q_1 + \frac{\partial f}{\partial q_1} q_2 + h(t, q_0, q_1)$, i.e., L is linear on the accelerations (see Pais, Uhlenbeck [34] and Müller-Hoissen [30]). The Ostrogradskii generalized momenta are given by $\hat{p}_0 = \frac{\partial f}{\partial q_0} + \frac{\partial h}{\partial q_1}$ and $\hat{p}_1 = \frac{\partial f}{\partial q_1}$. Hence $E_L = (\tilde{\rho}_1^3)^* E_h$, where $E_h = Ch - h$.Thus $\Omega_L = (\tilde{\rho}_1^3)^*\Omega_h$, $\Omega_h = -dd_J h + dE_h \wedge dt$. The intermediate space P_1 is:

$$P_1 = \{(t, q_0, q_1, q_2, p_0, p_1) \in (R \times T^2Q)_{\times(R\times TQ)}(R \times T^*TQ) \mid p_1 = \hat{p}_1 = \frac{\partial f}{\partial q_1}\}.$$

Further $\alpha_0(t, q_0, q_1, q_2, q_3) = (t, q_0, q_1, q_2, \frac{\partial f}{\partial q_0} + \frac{\partial h}{\partial q_1}, \frac{\partial f}{\partial q_1})$. Therefore dim Ker $T\alpha_0 = n$. If we define a 1-form on $R\times TQ$ by $\theta = df + d_J h$, then $P_{11} = (R\times T^2Q)_{\times(R\times TQ)}\tilde{\theta}(R\times TQ)$, where $\tilde{\theta}(t, v) = (t, \theta(t, v)) \in R \times T^*TQ$. Notice that θ is a 1-form on TQ depending on t. Then P_{11} is a $(3n + 1)$-dimensional submanifold of P_1. Moreover, the projection $\gamma_1 : P_1 \longrightarrow R \times T^2Q$ induces a diffeomorphism $\gamma_1 : P_{11} \longrightarrow R \times T^2Q$ in such a way that $\gamma_1 \circ \alpha_{01} = \tilde{\rho}_2^3$. Thus α_{01} is a submersion. On the other hand we have $\alpha_1(t, q_0, q_1, q_2, p_0) = (t, q_0, q_1, p_0, \frac{\partial f}{\partial q_1})$. Then $P_{21} =$ Im α_1 is a $(3n+1)$-dimensional submanifold of $R\times T^*TQ$ and $\alpha_{11} : P_1 \longrightarrow P_{21}$ is also a submersion. Finally $P_{22} = M_1 = \tilde{\theta}(R\times TQ)$ and then it is a $(2n+1)$-dimensional submanifold of $R\times T^*TQ$. The projection $\gamma_2 : R\times T^*TQ \longrightarrow R\times TQ$ induces a diffeomorphism $\gamma_2 : P_{22} \longrightarrow R \times TQ$. Since $\gamma_2 \circ Leg_1 = \tilde{\rho}_1^3$ then we deduce that Leg_1 is also a submersion. Thus, we obtain that dim Ker $TLeg = 2n$. Furthermore we have $\gamma_2^*\Omega_h = \Omega_{H_1}$, $E_h \circ \gamma_2 = H_1$.

Consequently the set of motion equations becomes: $i_X\Omega_L = 0$, $i_X dt = 1$, $J_3X = C_3$, on $R \times T^3Q$; $i_Y\tilde{\Omega}_1 = 0$, $i_Y dt = 1, J_2Y = C_2$,on $R \times T^2Q$ with $\tilde{\Omega}_1 = (\tilde{\rho}_1^2)^*\Omega_h$ and $i_Z\Omega_h = 0$, $i_Z dt = 1$, $JZ = C$,on $R \times TQ$.

We remark that $\gamma_1^*\tilde{\Omega}_1 = j_{11}^*\Omega_{E_1}$. We have an equivalence between the corresponding solutions through the canonical projections $\tilde{\rho}_2^3$ and $\tilde{\rho}_1^2$:

$$
\begin{tikzcd}
P_0 = R \times T^3 Q \arrow[r] & \tilde{P}_1 = R \times T^2 Q \arrow[r] & \tilde{P}_2 = R \times TQ \\
P_{01} \arrow[u] \arrow[r] & \tilde{P}_{11} \arrow[u] \arrow[r] & \tilde{P}_{21} \arrow[u] \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u] \\
{} & {} & {} \\
P_{0r} \arrow[u] \arrow[r] & \tilde{P}_{1r} \arrow[u] \arrow[r] & \tilde{P}_{2r} \arrow[u] \\
{} \arrow[u] & {} \arrow[u] & {} \arrow[u]
\end{tikzcd}
$$

Next we discuss the fourth order differential equation problem. Suppose that $h : R \times TQ \longrightarrow R$ is a regular Lagrangian of order 1. Then (Ω_h, dt) is a cosymplectic structure and denote by ξ_h its Reeb vector field. Notice that $J\xi_h = C$. Thus (Ω_{H_1}, η_1) is a cosymplectic structure on P_{22} with Reeb vector field ξ_2 such that $T\gamma_2(\xi_2) = \xi_h$. Hence ξ_2 is the unique vector field on P_{22} such that $(i_{\xi_2}\Omega_{\tilde{H}_1} = 0, i_{\xi_2}\tilde{\eta}_1 = 1)_{|P_{22}}$. Now, take on P_{11} a α_1-projectable vector field Y onto ξ_2. A direct computation shows that $S_1 = S_{[Y]} = (\xi_h \times \tilde{\theta})(R \times TQ)$. Furthermore we have that $\xi_h \circ \gamma_2 = \gamma_1 \circ j_1$. Thus, if we put $\xi_1 = Tj_1(\xi_2)$ then $T\tilde{\gamma}_1(\xi_1) = (\xi_h^C)_{|\xi_h(R \times TQ)}$, where ξ_h^C is the complete lift of ξ_h and $\tilde{\gamma}_1 : S_1 \longrightarrow \xi_h(R \times TQ)$ is the diffeomorphism induced by γ_1. Notice that $\xi_h(R \times TQ) \subset R \times T^2Q$ since $J\xi_h = C$ and $i_{\xi_h} dt = 1$.

Next we put $\tilde{S}_0 = \alpha_0^{-1}(S_1)$. Hence $\tilde{S}_0 = (\tilde{\rho}_2^3)^{-1}(\xi_h(R \times TQ))$. Let $\tilde{\xi}_0$ be a vector field on $\tilde{S}_0$ which is $\tilde{\rho}_2^3$-projectable onto $(\xi_h^C)_{|\xi_h(R \times TQ)}$ (or equivalently, $\tilde{\alpha}_0$-projectable onto ξ_1). A direct computation shows that $S_0 = S_{\tilde{\xi}_0} = \xi_h^C(\xi_h(R \times TQ))$. Furthermore, since $\xi_h^C \circ \tilde{\gamma}_1 = j_0$, we obtain that the the vector field ξ_0 is precisely $(\xi_h^C)^C_{|\xi_h^C(\xi_h(R \times TQ))}$.

We conclude that linear Lagrangians on the accelerations are in fact equivalent to Lagrangians of order 1.

6 Outlook

In a next paper we shall extend these results to the arbitrary higher order, say k. For a singular Lagrangian $L : J^k(R, Q) \longrightarrow R$ the Hamiltonian equations must be verify a k-th order differential equation condition which may be geometrically expressed as $J_1(T\gamma_k(Y)) = C_1$. Here J_1 is the canonical almost tangent structure and C_1 the Liouville vector field on $T^{k-1}Q$ and $\gamma_k : R \times T^*T^{k-1}Q \longrightarrow R \times T^{k-1}Q$ is the canonical projection. Then we must consider on the Lagrangian side the equations

$$i_X \Omega_L = 0, \; i_X \, dt = 1, \; J_{k+1}X = C_{k+1}.$$

An extension of the formalism of Skinner and Rusk following Cariñena and López [6] can be useful. Also time-evolution and intermediate evolution operators may be constructed for higher order degenerate time-dependent Lagrangians.

References

[1] R. Abraham, J. E. Marsden: *Foundations of Mechanics*, The Benjamin-Cummings Publ. Co. , Reading, Massachusetts, 1978.

[2] L. C. de Andrés, M. de León, P. R. Rodrigues: *Connections on tangent bundles of higher order associated to regular Lagrangians*, Geometriae Dedicata, 39, (1991), 17-28.

[3] C. Batle, J. Gomis, J. M. Pons, N. Román-Roy: *Lagrangian and Hamiltonian constraints for second-order singular Lagrangians*, J. Phys. A: Math. Gen., **21** (1988), 2693-2703.

[4] F. Cantrijn, M. Crampin, W. Sarlet: *Higher order differential equations and higher order Lagrangian mechanics*, Proc. Camb. Phil. Soc. 99 (1986), 565-587.

[5] F. Cantrijn, M. de León, E. A. Lacomba: *Gradient vector fields on cosymplectic manifolds*, J. Phys. A: Math. Gen. **25**, (1992), 175-188.

[6] J. F. Cariñena, C. López: *The time-evolution operator for higher-order singular Lagrangians*, Int. J. Modern Phys. A **7** 11 (1992), 2447-2468.

[7] D. Chinea, M. de León, J. C. Marrero: *The constraint algorithm for degenerate non-autonomous Lagrangian systems*, Proceedings Workshop Recent Topics in Differential Geometry, Puerto de la Cruz (Tenerife, Spain), Publ. Univ. La Laguna, Serie Informes, 32, (1991), 13-29.

[8] D. Chinea, M. de León, J. C. Marrero: *The constraint algorithm for time-dependent Lagrangians*, Preprint UM-1991.

[9] G. C. Constantelos: *On the Hamilton-Jacobi Theory with Derivatives of Higher Order*, Nuovo Cim. , **84 B**, 1 (1984), 91-101.

[10] M. Dresden, S. K. Dutt: *Pure gravity as a constrained second-order system*, New York, Preprint ITP-SB-86-32.

[11] C. A. P. Galvão, B. M. Pimentel: *The canonical structure of Podolsky generalized electrodynamics*, Can. J. Phys. 66 (1988), 460-466.

[12] C. A. P. Galvão: *A generalized Yang-Mills theory I : general aspects of the classical theory*, Preprint ISSN 0029-3865.

[13] P. L. García, J. Muñoz-Masqué: *Higher order Analytical Mechanics*, Dynamical systems and partial differential equations (Caracas, 1984), pp. 19-47, Ed. equinoccio, Caracas, 1986.

[14] M. J. Gotay: *Presymplectic Manifolds, Geometric Constraint Theory and the Dirac-Bergmann Theory of Constraints*, Dissertation, Center for Theoretical Physics, University of Maryland, 1979.

[15] M. J. Gotay, J. M. Nester: *Presymplectic Lagrangian systems I: the constraint algorithm and the equivalence theorem*, Ann. Inst. H. Poincaré, **A30** (1979), 129-142.

[16] M. J. Gotay, J. M. Nester: *Presymplectic Lagrangian systems II: the second-order differential equation problem*, Ann. Inst. H. Poincaré, **A32** (1980), 1-13.

[17] X. Grácia, J. M. Pons, N. Román-Roy: *Higher order Lagrangian systems: Geometric structures, dynamics, and constraints*, J. Math. Phys., **32** 10 (1991), 2744-2763.

[18] Th. v. Kármán, M. A. Biot: *Mathematical Methods in Engineering*, McGraw-Hill, New York, 1940.

[19] E. H. Kerner: *Note on the representation of many-particle dynamics as higher-order single-particle dynamics, with a means to relativistic elevation of Newtonian dynamics*, J. Math. Phys., **33** 5 (1992), 1675-1684.

[20] O. Krupková: *Variational analysis on fibered manifolds over one-dimensional bases*, Thesis, Silesian University at Opava, March, 1992.

[21] M. de León, J. A. Oubiña, P. R. Rodrigues, M. Salgado: *Almost s-Tangent Manifolds of Higher-Order*, Pacif. J. of Math. **154** 2 (1992), 201-213.

[22] M. de León, P. R. Rodrigues: *Formalisme hamiltonien symplectique sur les fibrés tangents d'ordre supérieur*, C. R. Acad. Sci. Paris, série II, 301 (1985) 103-106.

[23] M. de León, P. R. Rodrigues: *Generalized Classical Mechanics and Field Theory*, North-Holland Math. Ser. 112, Amsterdam, l985.

[24] M. de León, P. R. Rodrigues: *Almost tangent geometry and higher order mechanical systems*, Proceedings of the Conference on Differential Geometry and Its Applications, August, 24-30, 1986, Brno, Reidel and J. E. Purkyně, Brno (1987) 179-195.

[25] M. de León, P. R. Rodrigues: *Higher order almost tangent geometry and non-autonomous Lagrangian dynamics*, Proceedings of the Winter School on Geometry and Physics, Srni, 10-17 January, 1987, Rend. Circ. Mat. Palermo, II, 16 (1987) 157-171.

[26] M. de León, P. R. Rodrigues: *Dynamical connections and non-autonomous Lagrangian systems*, Annales de la Faculté des Sciences de Toulouse, **IX** 2 (1988), 171-181.

[27] M. de León, P. R. Rodrigues: *Almost contact structures and time-dependent Lagrangian systems*, Portugaliae Mathematica, **47**, 2 (1990), 115-130.

[28] M. de León, M. H. Mello, P. R. Rodrigues: *Reduction of singular Lagrangians of higher order*, 5th International Conference on Differential Geometry and Its Applications, Opava (Czechoslovakia), August 24-28, 1992.

[29] C. López-Lacasta: *Estudio geométrico de sistemas con ligaduras*, Tesis Doctoral, Departamento de Física Teórica, Universidad de Zaragoza, 1989.

[30] F. Müller-Hoissen: *Higher Derivative versus Second Order Field Equations*, Annalen der Physik, **48** 8 (1991), 543-557.

[31] D. G. C. McKeon: *Classical motion of a point particle with extrinsic curvature*, Can. J. Phys. **69** (1991), 830-832.

[32] Y. Nutku: *Hamiltonian formulation of the KdV equation*, J. Math. Phys. **25** (1984), 2007-2008.

[33] F. Neyzi, Y. Nutku: *Canonical structures for dispersive waves in shallow water*, J. Math. Phys. **28** (1987), 1499-1504.

[34] A. Pais, G. E. Uhlenbeck: *On field theories with non-localized action*, Physical Review, **79** 1 (1950), 145-165.

[35] B. Podolski, P. Schwed: *Review of a Generalized Electrodynamics*, Rev. Mod. Phys. 20 (1948), 40-50.

[36] A. Polyakov: *Fine structure of strings*, Nucl. Phys. B 268 (1986), 406-412.

[37] F. Riewe: *Generalized Mechanics of a Spinning Particle*, Lett. Nuovo Cimento, **1** 20 (1971), 807-808.

[38] F. Riewe: *Relativistic Classical Spinning-Particle Mechanics*, Nuovo Cim., **B8** 1 (1972), 271-277.

[39] Y. Saito, R. Sugano, T. Ohta, T. Kimura: *A dynamical formalism of singular Lagrangian system with higher derivatives*, J. Math. Phys. **30** 5 (1989), 1122-1132..

[40] R. Skinner, R. Rusk: *Generalized Hamiltonian Dynamics I.- Formulation on $T^*Q \oplus TQ$*, J. Math. Phys. **24** 11 (1983), 2589-2594.

[41] R. Skinner, R. Rusk: *Generalized Hamiltonian Dynamics II.- Gauge Transformations*, J. Math. Phys. **24** 11 (1983), 2595-2601.

[42] E. T. Whittaker: *A treatise on the Analytical Dynamics of Particle and Rigid Bodies*, Cambridge Univ. Press, Cambridge, 1988.

M. de León: Instituto de Matemáticas y Física Fundamental, Consejo Superior de Investigaciones Científicas, Serrano 123, 28006 Madrid, SPAIN.
e-mail: CEEML02@CC.CSIC.ES

J. C. Marrero: Departamento de Matemática Fundamental, Facultad de Matemáticas, Universidad de La Laguna, La Laguna, Tenerife, Canary Islands, SPAIN.
e-mail: JSC@IAC.ES

Received 29 September 1992