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REDUCTION OF SINGULAR LAGRANGIANS OF HIGHER-ORDER

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Abstract. Given a singular non-autonomous Lagrangian of higher order we construct a reduced evolution space of the same order and a local regular Lagrangian on it which is gauge equivalent to the original one. The general geometrical framework used is the theory of almost stable tangent geometry of higher order introduced in [17]. We show that higher order almost stable manifols can be endowed with a local Poincaré-Cartan form via the inverse problem of Lagrangian Dynamics.
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1. Introduction

Let Q be an n-dimensional manifold, with local coordinates (q^i) and $J^k(I\!R, Q)$ the stable tangent bundle of order k of Q, with local coordinates $(t_0, q_0^i, q_1^i, \cdots, q_k^i)$ where t is the Cartesian coordinate in $I\!R$, $q_r^i = (d^r/dt^r)_{t_0}(q^i \circ \sigma), 1 \leq r \leq k, 1 \leq i \leq n$. Sometimes we shall put $q_0^i = q^i$. Consider a non-autonomous Lagrangian of order k on Q, i.e., a function $L : J^k(I\!R, Q) \to I\!R$. We know (see [19]) that the dynamics of L is developed in the stable tangent bundle of order $2k-1$ of Q, say the jet manifold $J^{2k-1}(I\!R, Q)$. In fact we may construct a 2-form Ω_L (the Poincaré-Cartan 2-form) and the energy function E_L on $J^{2k-1}(I\!R, Q)$. This construction is obtained by translating the canonical geometric structures (higher order almost tangent geometry and the Liouville vector field) on the higher order tangent bundle T^kQ to $J^k(I\!R, Q)$ via the canonical identification $J^k(I\!R, Q) \cong I\!R \times T^kQ$. Then we obtain a canonical geometry on $J^k(I\!R, Q)$, called *almost stable tangent geometry of higher order* (see [17]). If L is regular, then (Ω_L, dt) is a *cosymplectic structure* and hence there exists a unique vector field ξ_L on $J^{2k-1}(I\!R, Q)$ such that

$$i_{\xi_L} \Omega_L = 0, \quad i_{\xi_L} dt = 1. \tag{1.1}$$

The vector field ξ_L, called *Euler-Lagrange vector field*, is a $2k$-th order non-autonomous differential equation on $J^{2k-1}(I\!R, Q)$ and its solutions are precisely the solutions of the

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Euler-Lagrange equations and they may be considered as the intrinsic version of the motion equations.

If L is singular (or degenerate) then the pair (Ω_L, dt) is not cosymplectic and thus these equations has no solution in general, and even if it exists it will not be unique nor a $2k$-th order differential equation. The aim of this paper is to reduce the evolution space $J^{2k-1}(\mathbb{R}, Q)$ to an integrable almost stable-tangent manifold of the same order on which exists a local regular Lagrangian gauge equivalent to the original one, having the same dynamical information. To do this we use the notion of almost stable-tangent structure of higher order. This reduction procedure may be compared with the Marsden-Weinstein one [23, 24] by means of the constraint algorithm developed in [6] (see [5] for the order one, [9] for the autonomous case of order one and [10] for the autonomous case of higher order).

The autonomous (resp. non-autonomous) case of order one was studied by Cantrijn *et al.* [3] (resp. de León *et al.* [16] (see also [14]). The autonomous higher order situation was studied in [15]. In the present article we are assuming that a global dynamics exists - i.e., only *primary Lagrangian constraints* are involved, and thus the constraint algorithm is not used. Indeed, "the constraint algorithm is of completely different nature as the geometric structures on evolution space and to make complatible both processes requires further refinements of the theory" [14]. Concerning the constraint algorithm we notice a recent study of higher-order autonomous Lagrangians by Gráxia *et al* [10] using what they call *intermediate spaces*.

The paper is structured as follows. In Section 2 we prove that higher order almost stable tangent manifolds may serve as evolution spaces. By the way the inverse problem of Lagrangian dynamics is examined. The reduction procedure is presented in Section 3. An Appendix gives some basic results.

2. Higher order Lagrangians on Integrable Almost Stable Tangent Manifolds. The Inverse Problem

This section is devoted to show that *almost s-tangent manifolds endowed with an appropriate cosymplectic structure admit locally a Lagrangian function given by a Poincaré-Cartan 2-form*. Moreover, underlying the geometric procedure it will be clear that the inverse problem for higher order Lagrangian dynamics is also developed – *given a $2k$-th order differential equation ξ, does there exist a Lagrangian function L such that ξ is precisely the Euler-Lagrange vector field for L?* (see [7, 8, 12, 16, 22]).

Let L be a non-autonomous Lagrangian with energy E_L. Consider the 1-form $\alpha_L = \Delta(L)$, where

$$\Delta = \sum_{r=1}^{k} (-1)^{r-1} \frac{1}{r!} d_T^{r-1} d_{J_r}, \tag{2.1}$$

(for a definition of d_T and d_{J_r} see [18] and for J_r see below). The *Poincaré-Cartan 1-form* (resp. *2-form*) is $\theta_L = \alpha_L - E_L dt$ (resp. $\Omega_L = \omega_L + dE_L \wedge dt$ if we set $\omega_L = -d\alpha_L$). Its local expressions are

$$\alpha_L = \sum_{r=1}^{k} p_i^r dq_{r-1}^i, \quad \omega_L = \sum_{r=1}^{k} dq_{r-1}^i \wedge dp_i^r, \quad p_i^r = \sum_{s=0}^{k-r} (-1)^s \frac{d^s}{dt^s}(\frac{\partial L}{\partial q_{r+s}^i}).$$

Notice that p_i^r depends at most in q_{2k-r}, i.e., $p_i^r = p_i^r(q_0, q_1, \cdots, q_{2k-r})$, and we have $(\partial p_i^r / \partial q_{2k-r}^j) = (-1)^{k-r}(\partial^2 L / \partial q_k^i \partial q_k^j)$. If we set $\theta_r^i = dq_r^i - q_{r+1}^i dt\,;\ 0 \leq r \leq k-1$, then we can writte $\theta_L = \sum_{r=1}^{k} p_i^r\, \theta_{r-1}^i + Ldt$, and so $\Omega_L = \sum_{r=1}^{k}\{dp_i^r \wedge \theta_{r-1}^i + p_i^r\, d\theta_{r-1} + dL \wedge dt\}$, from which we deduces

$$i_{\frac{\partial}{\partial q_s^j}}\, \Omega_L = -\frac{\partial p_i^r}{\partial q_s^j}\, \theta_{r-1}^i\,, \quad if\ k \leq s \leq 2k-1$$

$$i_{\frac{\partial}{\partial q_s^j}}\, \Omega_L = -\frac{\partial p_i^r}{\partial q_s^j}\, \theta_{r-1}^i + dp_j^{s+1} + p_j^s\, dt - \frac{\partial L}{\partial q_s^j}\, dt\,, \quad if\ 0 \leq s \leq k-1\,.$$

Proposition 2.1. *Let ξ be a $2k$-th order differential equation on $J^{2k-1}(\mathbb{R},Q)$ and Ω be a closed 2-form on $J^{2k-1}(\mathbb{R},Q)$ such that: (1) (Ω, dt) is a cosymplectic structure; (2) $\mathcal{L}_\xi \Omega = 0$; (3) $(J^k)^*\Omega = 0$. Then there exists a local Lagrangian $L : J^k(\mathbb{R},Q) \longrightarrow \mathbb{R}$ which is a solution of the inverse problem for ξ.*

Proof. $(J^k)^*\Omega_L = 0$ implies that Ω is semibasic with respect to the fibration $\rho_{k-1}^{2k-1} : J^{2k-1}(\mathbb{R},Q) \longrightarrow J^{k-1}(\mathbb{R},Q)$. Then there exists a ρ_{k-1}^{2k-1} -semibasic 1-form α on $J^{2k-1}(\mathbb{R},Q)$ such that $\Omega = -d\alpha$. Hence α is locally expressed as follows: $\alpha = Ddt + \sum_{r=0}^{k-1} B_r^i\, (\partial/\partial q_{r+1}^i)$. Thus we obtain $\alpha(\xi) = D + \sum_{r=0}^{k-1} B_r^i q_{r+1}^i$. Also, since $\mathcal{L}_\xi\, \Omega = -d\mathcal{L}_\xi\, \alpha = 0$, we deduce that there exists a locally defined function L on $J^{2k-1}(\mathbb{R},Q)$ such that $\mathcal{L}_\xi\, \alpha = dL$, from which we deduce, by applying to d/dt, $\partial/\partial q_s^i$, $1 \leq s \leq 2k-1$, that $(\partial L/\partial q_s^i) = 0\,,\ k+1 \leq s \leq 2k-1$, which imply that L is in fact locally defined on $J^k(\mathbb{R},Q)$ and

$$\xi(B_0^i) = \frac{\partial L}{\partial q_0^i}; \quad B_s^i = \frac{\partial L}{\partial q_{s+1}^i} - \xi(B_{s+1}^i), \quad 0 \leq s \leq k-2; \quad B_{k-1}^i = \frac{\partial L}{\partial q_k^i}\,.$$

Hence $p_i^r = B_{r+1}^i$, for all r, $1 \leq r \leq k-1$. Therefore we have $E_L = \alpha(\xi) - D - L$, $\Delta(L) = \alpha - D\, dt$. Define $h = L - \alpha(\xi)$. Then we have $\alpha_L = \alpha + hdt$. Thus $\Omega_L = \Omega - dh \wedge dt$ and hence $i_\xi \Omega_L = i_\xi \Omega - (\xi h)dt + dh = -dh - (\xi h)dt + dh = -(\xi h)dt = 0$, since $i_\xi d\alpha = dh$ implies $\xi h = i_\xi dh = i_\xi i_\xi d\alpha = 0$. We remark that L is a regular function since (Ω_L, dt) is a cosymplectic structure. Thus, we have $\xi = \xi_L$. □

Corollary 2.2. *A necessary and sufficient condition for a $2k$-th order differential equation ξ on $J^{2k-1}(\mathbb{R},Q)$ be a regular Euler-Lagrange vector field is the existence of a closed 2-form on $J^{2k-1}(\mathbb{R},Q)$ like in Proposition 2.1.*

Next, we shall extend the above result for almost s-tangent manifolds of order $2k-1$. First of all let us recall that from the natural identification $J^k(\mathbb{R},Q) \cong \mathbb{R} \times T^kQ$ (using the mapping $j_t^k\sigma \rightarrow (t, j_0^k\tau)$, where $\tau(s) = \sigma(s+t)$) we may transport all the geometric structures on T^kQ to $J^k(\mathbb{R},Q)$. For instance, the set of canonical tensor fields J_r of type (1,1) can be extended to $J^k(\mathbb{R},Q)$ (and also denoted by J_r), with local expression

$$J_r = \sum_{s=0}^{k-r} \frac{(r+s)!}{s!}\, \frac{\partial}{\partial q_{r+s}^i} \otimes dq_s^i; \quad 1 \leq r \leq k.$$

For $r = 1$ one has the canonical almost tangent structure J_1 of order k of T^kQ Also, we have the family of vector fields C_r, locally given by

$$C_r = \sum_{s=1}^{k-r+1} \frac{(r+s-1)!}{(s-1)!}\, q_s^i\, \frac{\partial}{\partial q_{r+s-1}^i}; \quad 1 \leq r \leq k,$$

(for $k = 1$ one has the canonical Liouville vector field C_1). We define the following $(1,1)$-tensor fields:

$$\bar{J}_1 = J_1 + \frac{\partial}{\partial t} \otimes dt\,, \quad \tilde{J}_1 = J_1 - C_1 \otimes dt\,,$$

and one has: $\bar{J}_r = \bar{J}_1^r$ and $\tilde{J}_r = \tilde{J}_1^r$, i.e.,

$$\bar{J}_r = J_r + \frac{\partial}{\partial t} \otimes dt, \quad \tilde{J}_r = J_r - C_r \otimes dt \; (\tilde{J}_r = \bar{J}_r - (C_r + \frac{\partial}{\partial t}) \otimes dt)\,.$$

Now, $(\bar{J}_1, dt, \partial/\partial t)$ is an almost s-tangent structure on $J^k(I\!R, Q)$ according to the following definition.

Definition 2.1. ([17])- Let V be an $(k+1)n+1$-dimensional manifold. If there is a triple $(\bar{F}, \tau, T)$, where $\bar{F}$ is a tensor field of type (1.1), τ is a 1-form and T a vector field on V such that (i) $\tau(T) = 1$; (ii) $\bar{F}^{k+1} = \tau \otimes T$; (iii) $rank\ \bar{F} = kn+1$, then we say that V is endowed with an *almost s-tangent structure of order k*. In such a case V is called an *almost s-tangent manifold of order k*.

In [17] an almost s-tangent structure $(\bar{F}, \tau, T)$ of order k was interpreted as a G-structure. Then $(\bar{F}, \tau, T)$ is integrable as a G-structure if and only if there exist local coordinates $(t, q_0^i, q_1^i, \cdots, q_k^i)$ around each point of V such that

$$\bar{F} = \sum_{r=0}^{k} (r+1)\, \frac{\partial}{\partial q_{r+1}^i} \otimes dq_r^i + \frac{\partial}{\partial t} \otimes dt, \quad \tau = dt, \quad T = \frac{\partial}{\partial t}\,.$$

(we refer [27] for an application of stable tangent bundles to the unified field theory). Hence the canonical almost s-tangent structure $(\bar{J}_1, dt, \partial/\partial t)$ is integrable. Furthermore *every almost s-tangent structure of order k is integrable if and only if is locally isomorphic to the canonical one.*

We also recall that a vector field ξ on $J^k(I\!R, Q)$ is said to be a $(k+1)$-*th order differential equation* if $\bar{J}_1 \xi = (\partial/\partial t) + C_1$. We deduce that a $(k+1)$-th order differential equation ξ on $J^k(I\!R, Q)$ is locally expressed by

$$\xi = \frac{\partial}{\partial t} + q_1^i \frac{\partial}{\partial q_0^i} + q_2^i \frac{\partial}{\partial q_1^i} + \cdots + q_k^i \frac{\partial}{\partial q_{k-1}^i} + \xi^i \frac{\partial}{\partial q_k^i}\,,$$

where $\xi^i = \xi^i(t, q_0, q_1, \cdots, q_k)$.

Definition 2.2. A *semispray of type r* on $J^k(I\!R, Q)$ is a vector field ξ_r verifying $\bar{J}_r(\xi_r) = (\partial/\partial t) + C_r$.

Locally:

$$\xi_r = \frac{\partial}{\partial t} + q_1^i \frac{\partial}{\partial q_0^i} + q_2^i \frac{\partial}{\partial q_1^i} + \cdots + q_{k-r+1}^i \frac{\partial}{\partial q_{k-r}^i} + \xi_{k-r+1}^i \frac{\partial}{\partial q_{k-r+1}^i} + \cdots + \xi_k^i \frac{\partial}{\partial q_k^i}\,.$$

It is obvious that a $(k+1)$-th differential equation on T^kQ is in fact a semispray of type 1. Locally the action of J_{r-1} on ξ_r is given by:

$$J_{r-1}(\xi_r) - C_{r-1} = \frac{k!}{(k-r+1)!} \left[\xi_{k-r+1}^i - q_{k-r+2}^i \right] \frac{\partial}{\partial q_k^i}.$$

It is easy to see that a vector field ξ is a $(k+1)$-th differential equation iff $\langle \xi, \theta_r^i \rangle = 0$ and $\langle \xi, dt \rangle = 1$.

Definition 2.3. A *dynamical connection* on $J^k(I\!R,Q)$ is a tensor field Γ of type (1, 1) on $J^k(I\!R,Q)$ such that (*i*) $J_1\Gamma = \tilde{J}_1$; (*ii*) $\tilde{J}_1\Gamma = \tilde{J}_1$; (*iii*) $\Gamma\tilde{J}_k = -\tilde{J}_k$; (*iv*) $\Gamma J_k = -J_k$.

Then the local expressions of Γ are

$$\Gamma(\frac{\partial}{\partial t}) = -\sum_{r=1}^{k} q_r^i \frac{\partial}{\partial q_{r-1}^i} + \Gamma^i \frac{\partial}{\partial q_k^i},$$

$$\Gamma(\frac{\partial}{\partial q_r^i}) = \frac{\partial}{\partial q_r^i} + \sum_{r=1}^{k} (\Gamma^{r-1})_j^i \frac{\partial}{\partial q_k^j}, \quad 0 \leq r \leq k-1, \quad \Gamma(\frac{\partial}{\partial q_k^i}) = -\frac{\partial}{\partial q_k^i}.$$

The vector field $\xi = \Gamma^2(\partial/\partial t)$ is locally expressed by

$$\xi = \frac{\partial}{\partial t} + \sum_{r=0}^{k-1} q_r^i \frac{\partial}{\partial q_{r-1}^i} + (\Gamma^i + \sum_{r=1}^{k} q_r^j (\Gamma^{r-1})_j^i) \frac{\partial}{\partial q_k^i}$$

i.e. it is a $(k+1)$-th order differential equation, called *canonically associated to* Γ. It is possible to show ([19]) that *if ξ is a given $(k+1)$-th order differential equation on $J^k(I\!R,Q)$ then Γ defined by*

$$\Gamma_\xi = -\frac{2}{k+1}\mathcal{L}_\xi \tilde{J}_1 + \frac{k-1}{k+1}(Id - \xi \otimes dt)$$

is a dynamical connection on $J^k(I\!R,Q)$ whose associated $(k+1)$-th order differential equation is precisely ξ. It follows that *if L is a regular Lagrangian and ξ_L the Euler-Lagrange vector field, then there exists a dynamical connection Γ_L on ρ_{2k-2}^{2k-1} : $J^{2k-1}(I\!R,Q) \longrightarrow J^{2k-2}(I\!R,Q)$ whose paths are solutions of the motion equations*. This connection is given by $\Gamma_L = -(1/k)\mathcal{L}_{\xi_L} \tilde{J}_1 + (k-1/k)(Id - \xi_L \otimes dt)$ and $\tilde{\xi}$ is the $2k$-th order differential equation associated to Γ.

We show now how to extend Proposition 2.1 to an almost integrable s-tangent manifold of higher-order. Let X be a vector field on a manifold V endowed with an integrable almost s-tangent structure $(\bar{F},\tau,T)$ of order $2k-1$. We set $C_X = \bar{F}(X) - T$, $F = \bar{F} - T\otimes\tau$, $\tilde{F} = F - C_X \otimes \tau$.

In adapted coordinates $(t, q_0^i, q_1^i, \cdots, q_{2k-1}^i)$ we have

$$\tilde{F}(\frac{\partial}{\partial t}) = -C_X\,; \quad \tilde{F}(\frac{\partial}{\partial q_r^i}) = (r+1)\frac{\partial}{\partial q_{r+1}^i}, \quad 0 \leq r \leq 2k-2\,;$$

$$\tilde{F}(\frac{\partial}{\partial q_{2k-1}^i}) = 0\,; \quad F(\frac{\partial}{\partial t}) = 0\,;$$

$$F(\frac{\partial}{\partial q_r^i}) = (r+1)\frac{\partial}{\partial q_{r+1}^i}, \quad 0 \leq r \leq 2k-2; \quad F(\frac{\partial}{\partial q_{2k-1}^i}) = 0\,; \tag{2.2}$$

Now, let us define the following operator:

$$\Gamma_X = -\frac{1}{k}\mathcal{L}_X \tilde{F} + \frac{k-1}{k}(Id - X\otimes\tau).$$

We suppose that Γ_X behaves as a dynamical connection in the fibration $\rho_{2k-2}^{2k-1} : J^{2k-1}(I\!R,Q) \to J^{2k-2}(I\!R,Q)$, i.e. ,

(2.3)

(*i*) $F\Gamma_X = \tilde{F}$; (*ii*) $\tilde{F}\Gamma_X = \tilde{F}$; (*iii*) $\Gamma_X \tilde{F}^{2k-1} = -\tilde{F}^{2k-1}$; (*iv*) $\Gamma_X F^{2k-1} = -F^{2k-1}$, as well as

$$(v)\ F[X,T]=0. \tag{2.4}$$

Let us express X locally as $X = u\,(\partial/\partial t) + \sum_{r=0}^{2k-1} X_r^i\,(\partial/\partial q_r^i)$, $1 \leq i \leq n$. Then, from (2.2) one has $C_X = (u-1)\,(\partial/\partial t) + \sum_{r=0}^{2k-2}(r+1)\,X_r^i\,(\partial/\partial q_{r+1}^i)$

From the conditions (i - iv) one obtains

$$u-1=0\,, \quad \frac{\partial X_0^i}{\partial q_1^j} = \delta_j^i\,;$$

$$\frac{\partial X_0^i}{\partial q_{s+1}^j} = 0\,, \quad 1 \leq s \leq 2k-2\,, \quad \frac{\partial X_s^i}{\partial q_{2k-11}^j} = 0\,, \quad 0 \leq s \leq 2k-3\,;$$

$$\frac{\partial X_{2k-2}^i}{\partial q_{2k-1}^j} = \delta_j^i\,;$$

$$\frac{\partial X_s^i}{\partial q_r^j} = \frac{r+1}{s+1}\frac{\partial X_{s+1}^i}{\partial q_{r+1}^j}\,, \quad 0 \leq s \leq 2k-3\,,\ 0 \leq r \leq 2k-2\,,\ r \neq s+1\,;$$

$$\frac{\partial X_s^i}{\partial q_{s+1}^j} = \frac{\partial X_{s+1}^i}{\partial q_{s+2}^j}\,, \quad i \neq j\,;$$

$$-\frac{1}{s+1}\frac{\partial X_s^i}{\partial q_{s+1}^i} + \frac{1}{s+2}\frac{\partial X_{s+1}^i}{\partial q_{s+2}^j} = 1\,, \quad 0 \leq s \leq 2k-3\,.$$

From these equalities we deduce that:

$$u=1\,; \quad \frac{\partial X_s^i}{\partial q_{s+1}^j} = \delta_j^i\,, \quad 0 \leq s \leq 2k-2\,;$$

$$\frac{\partial X_s^i}{\partial q_r^j} = 0\,, \quad 0 \leq s \leq 2k-3\,,\ s+2 \leq r\,;$$

$$\frac{s+1}{r+1}\frac{\partial X_s^i}{\partial q_r^j} = \frac{\partial X_{s+1}^i}{\partial q_{r+1}^j}\,, \quad r \neq s+1\,,\ 0 \leq s \leq 2k-3\,,\ 0 \leq r \leq 2k-2\,;$$

and from (v) we obtain $(\partial X_s^i/\partial t) = 0$, $0 \leq s \leq 2k-2$. Thus the functions X_s^i take the following form: $X_s^i = q_{s+1}^i + g_s(q_0^j, q_1^j, \cdots, q_s^j)$. For instance: $X_0^i = q_1^i + g_0(q_0^j)$, $X_1^i = q_2^i + g_1(q_0^j, q_1^j)$, and so on. Define a new coordinate system $(\bar{t}, \bar{q}_0^i, \bar{q}_1^i, \cdots, \bar{q}_{2k-1}^i)$ by setting $\bar{t} = t\,, \bar{q}_0^i = q_0^i\,,\ \bar{q}_r^i = \sum_{s=0}^{r-1} X_s^j\,(\partial \bar{q}_{r-1}^i/\partial q_s^j)\,,\ 1 \leq r \leq 2k-1$, where $X_s^j = q_{s+1}^j + g_s(q_0, q_1, \cdots, q_s)$. A direct computation shows that $(\bar{t}, \bar{q}_0^i, \bar{q}_1^i, \cdots, \bar{q}_{2k-1}^i)$ is such that the vector field X *reads locally as a $2k$-th order differential equation*, say

$$X = \frac{\partial}{\partial \bar{t}} + \bar{q}_1^i\frac{\partial}{\partial \bar{q}_0^i} + \bar{q}_2^i\frac{\partial}{\partial \bar{q}_1^i} + \cdots + \bar{q}_{2k-1}^i\frac{\partial}{\partial \bar{q}_{2k-2}^i} + \xi^i\frac{\partial}{\partial \bar{q}_{2k-1}^i}\,,$$

and such coordinate system is also adapted to the integrable almost s-tangent structure $(\bar{F}, T, \tau)$.

Definition 2.4. A vector field X verifying the above $(i)-(v)$ conditions is called a *non-autonomous regular Lagrangian dynamical system of order* $2k$ on an integrable almost s-tangent manifold V of order $2k-1$ if there exists a closed 2-form Ω such that:

(1) (Ω, τ) is a cosymplectic structure; (2) $\mathcal{L}_X\Omega = 0$; (3) $(F^k)^*\Omega = 0$.

As the arguments used in Proposition 2.1 (and the Corollary 2.2) are local, we use a similar procedure to show the following

Theorem 2.3. *Let X be a non-autonomous regular Lagrangian dynamical system of order $2k$ on an integrable almost s-tangent manifold V of order $2k-1$. Then there exists a local function $L : U \subset V \longrightarrow \mathbb{R}$ such that $\Omega = \Omega_L$ and X is the Euler-Lagrange vector field associated to L.*

3. Reduction of Singular Lagrangians of Higher-order

As we know the singularity of L implies that Ω_L is no longer of maximal rank. Also, (Ω_L, dt) is not in general a precosymplectic structure. Nevertheless, we need some *weak regularity* condition. In the autonomous case, the geometrical structure which support the Lagrangian dynamics is a presymplectic structure. Thus, for each $t \in \mathbb{R}$ let us define the autonomous Lagrangian $L_t : T^kQ \longrightarrow \mathbb{R},\ L_t(x) = L(t,x)$. Suppose that the rank of ω_{L_t} is constant, say $2a$ and independent of t. If we set $\varpi_L = \omega_L - \Delta(\partial L/\partial t) \wedge dt$ then we have $\Omega_L = \varpi_L + (\Delta(\partial L/\partial t) + dE_L) \wedge dt$. We see that that the restriction of ϖ_L to each level $\{t\} \times T^{2k-1}Q$ is precisely ω_{L_t}. Then our assumption is equivalent to say that ϖ_L is of constant rank $2a$. Notice that ϖ_L is not presymplectic since it is not closed. Then we have

$$\Omega_L^b = \varpi_L^b + b\varpi_L^{b-1} \wedge (\Delta(\frac{\partial L}{\partial t}) + dE_L) \wedge dt\,,\ \forall b \geq 1.$$

Since $\varpi_L^a \neq 0,\ \varpi_L^{a+1} = 0$, we obtain $(\Omega_L)^a \wedge dt \neq 0,\ (\Omega_L)^{a+1} \wedge dt = 0,\ (\Omega_L)^{a+2} = 0$, and so $2a \leq rank\ \Omega_L \leq 2a+2$. According to [16] we know that *the rank of Ω_L at a point $x \in J^{2k-1}(\mathbb{R}, Q)$ is $2a$ if and only if there exists a tangent vector $\xi \in T_x(J^{2k-1}(\mathbb{R}, Q))$ such that $i_\xi(\Omega_L)_x = 0,\ i_\xi(dt)_x = 1$.*

If $L : J^k(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ is a degenerate Lagrangian, then the dynamical equations (1.1) will not posses a globally defined solution in general, and even if it exists it will not be unique. However, we confine ourselves to those Lagrangians for which (1.1) posseses a global solution (see the introduction).

Definition 3.1. A degenerate non-autonomous Lagrangian $L : J^k(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ admits a *global dynamics* if there exists a globally defined vector field X on $J^{2k-1}(\mathbb{R}, Q)$ satisfying $i_X\Omega_L = 0,\ i_X dt = 1$.

In such a case, Ω_L has constant rank $2a$ and further (Ω_L, dt) is a precosymplectic structure on $J^{2k-1}(\mathbb{R}, Q)$ (of rank $2a$). The converse is also true.

Definition 3.2. The *gauge distribution* $\mathcal{K}$ of L is defined by $\mathcal{K} = Ker\ \Omega_L \cap\ Ker\ dt$.

Proposition 3.1. *If $L : J^k(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ admits a global dynamics and $J_r(\mathcal{K}) = \mathcal{K} \cap Im\ J_r$, for all r, $1 \leq r \leq 2k-1$, then there exists a $2k$-th order differential equation ξ such that $i_\xi\Omega_L = 0,\ i_\xi dt = 1$.*

Proof. Suppose that X is a vector field on $J^{2k-1}(\mathbb{R}, Q)$ such that $i_X\,\Omega_L = 0,\ i_X\,dt = 1$. Then X is locally expressed by $X = (\partial/\partial t) + A_0^i(\partial/\partial q_0^i) + \cdots + A_{2k-1}^i\,(\partial/\partial q_{2k-1}^i)$ and we have

$$(i_X\Omega_L)(\frac{\partial}{\partial q_{2k-1}^j}) = \frac{\partial p_i^1}{\partial q_{2k-1}^j}(A_0^i - q_1^i)\,,$$

which implies $(\partial p_i^k/\partial q_{2k-1}^j)\,(A_0^i - q_1^i) = 0$. On the other hand we have

$$J_{2k-1}X - C_{2k-1} = (2k-1)!\,(A_0^i - q_1^i)\frac{\partial}{\partial q_{2k-1}^i}\,.$$

Hence

$$i_{J_{2k-1}-C_{2k-1}}\,\Omega_L = (2k-1)!(A_0^i - q_1^i)i_{\frac{\partial}{\partial q_{2k-1}^i}}\,\Omega_L = -(2k-1)!(A_0^i - q_1^i)\frac{\partial p_i^1}{\partial q_{2k-1}^j} = 0\,.$$

Thus we deduce that $J_{2k-1}X - C_{2k-1} \in \; Ker\;\Omega_L \cap \; Ker\; dt = \mathcal{K}$. Also, $J_{2k-1}X - C_{2k-1} \in Im\; J_{2k-1} = J_{2k-1}(\mathcal{K})$. Then there exists a vector field $Y \in \mathcal{K}$ such that $J_{2k-1}X - C_{2k-1} = J_{2k-1}Y$. If we put $\xi_{2k-1} = X - Y$ we have $J_{2k-1}\xi_{2k-1} = (\partial/\partial t) + C_{2k-1}$, i.e., ξ_{2k-1} is a semispray of type $2k-1$ and $i_{\xi_{2k-1}}\,\Omega_L = 0,\; i_{\xi_{2k-1}}dt = 1$.

Next we proceed by induction. Suppose that there exists a semispray of type r, $k+1 \leq r \leq 2k-1$ such that $i_{\xi_r}\,\Omega_L = 0,\; i_{\xi_r}\,dt = 1$. Then ξ_r is locally expressed by

$$\xi_r = \frac{\partial}{\partial t} + q_1^i\frac{\partial}{\partial q_0^i} + \cdots + q_{2k-r}^i\frac{\partial}{\partial q_{2k-r-1}^i} + \cdots + B_{2k-r}^i\frac{\partial}{\partial q_{2k-r}^i} + \cdots + B_{2k-1}^i\frac{\partial}{\partial q_{2k-1}^i}\,.$$

We obtain

$$J_{r-1}\xi_r - C_{r-1} = \frac{(2k-1)!}{(2k-r)!}\,(B_{2k-r}^i - q_{2k-r+1}^i)\frac{\partial}{\partial q_{2k-1}^i}\,,$$

from which we deduce

$$i_{J_{r-1}\xi_r - C_{r-1}}\,\Omega_L = -\frac{(2k-1)!}{(2k-r)!}\,(B_{2k-r}^i - q_{2k-r+1}^i)\frac{\partial p_j^1}{\partial q_{2k-1}^i}\theta_0^j\,.$$

On the other hand, we have

$$(i_{\xi_r}\,\Omega_L)(\frac{\partial}{\partial q_{r-1}^j}) = \sum_{s=1}^{k}\frac{\partial p_i^s}{\partial q_{r-1}^j}\theta_{s-1}^i(\xi_r)$$

$$= \frac{\partial p_i^{2k-r+1}}{\partial q_{r-1}^j}\theta_{2k-r}^i(\xi_r) = (-1)^{2k-r+1}\frac{\partial p_i^1}{\partial q_{2k-1}^j}(B_{2k-r}^i - q_{2k-r+1}^i)\,,$$

Thus, $J_{r-1}\xi_r - C_{r-1} \in \mathcal{K} \cap \; Im\; J_{r-1} = J_{r-1}(\mathcal{K})$. Then there exists a vector field $Y \in \mathcal{K}$ such that $J_{r-1}\xi_r - C_{r-1} = J_{r-1}Y$. If we put $\xi_{r-1} = \xi_r - Y$ we deduce that ξ_{r-1} is a semispray of type $r-1$ and $i_{\xi_{r-1}}\,\Omega_L = 0,\; i_{\xi_{r-1}}dt = 1$. Consequently, there exists a semispray ξ_{k-1} of type $k-1$ on $J^{2k-1}(I\!R, Q)$ such that $i_{\xi_{k-1}}\,\Omega_L = 0,\; i_{\xi_{k-1}}dt = 1$.

Next, we shall apply another time an induction on the type of semispray. Suppose that there exists a semispray $\xi_r, 1 \leq r \leq k-1$ of type r on $J^{2k-1}(I\!R, Q)$ such that $i_{\xi_r}\,\Omega_L = 0,\; i_{\xi_r}dt = 1$. Then ξ_r is locally expressed by

$$\xi_r = \frac{\partial}{\partial t} + q_1^i\frac{\partial}{\partial q_0^i} + \cdots + q_{2k-r}^i\frac{\partial}{\partial q_{2k-r-1}^i} + \cdots + D_{2k-r}^i\frac{\partial}{\partial q_{2k-r}^i} + \cdots + D_{2k-1}^i\frac{\partial}{\partial q_{2k-1}^i}\,.$$

and we obtain

$$J_{r-1}\xi_r - C_{r-1} = \frac{(2k-1)!}{(2k-r)!}\,(D_{2k-r}^i - q_{2k-r+1}^i)\frac{\partial}{\partial q_{2k-1}^i}\,,$$

From $i_{\xi_r}\Omega_L = 0$, we have

$$(i_{\xi_r}\Omega_L)(\frac{\partial}{\partial q^j_{r-1}}) = (dp^r_j + p^{r-1}_j dt - \frac{\partial L}{\partial q^j_{r-1}} dt)(\xi_r)$$

$$= \xi_r(p^r_j) + p^{r-1}_j - \frac{\partial L}{\partial q^j_{r-1}}$$

$$= (D^i_{2k-r} - q^i_{2k-r+1})\frac{\partial p^r_j}{\partial q^i_{2k-r}} = (-1)^{2k-r+1}(D^i_{2k-r} - q^i_{2k-r+1})\frac{\partial p^1_j}{\partial q^i_{2k-1}} .$$

Hence $J_{r-1}\xi_r - C_{r-1} \in \mathcal{K} \cap \ Im\ J_{r-1} = J_{r-1}\mathcal{K}$ and there exists a vector field $Y \in \mathcal{K}$ such that $J_{r-1}\xi_r - C_{r-1} = J_{r-1}Y$. If we put $\xi_{r-1} = \xi_r - Y$ we deduce that ξ_{r-1} is a semispray of type $r-1$ and $i_{\xi_{r-1}}\Omega_L = 0$, $i_{\xi_{r-1}}dt = 1$. Consequently, there exists a semispray $\xi = \xi_1$ of type 1 on $J^{2k-1}(\mathbb{R}, Q)$ such that $i_\xi\,\Omega_L = 0$, $i_\xi dt = 1$. □

In what follows we shall consider a degenerate non-autonomous Lagrangian L satisfying the following conditions:

(D1) ϖ_L has constant rank, say $2a$;
(D2) L admits a global dynamics;
(D3) $J_r(\mathcal{K}) = \mathcal{K} \cap\ Im\ J_r$, for all r, $1 \leq r \leq 2k-1$;
(D4) the foliation defined by $\mathcal{K} = Ker\ \Omega_L \cap\ Ker\ dt$ is a fibration.

The assumptions (D1) and (D2) are equivalent to say that (Ω_L, dt) is a precosymplectic structure. The assumption (D3) implies that there exists a $2k$-th order differential equation ξ such that $i_\xi\,\Omega_L = 0$, $i_\xi\,dt = 1$. Further, (D3) implies that $dim\ \mathcal{K} = 2k\ dim\ (\mathcal{K} \cap\ Im\ J_{2k-1}) = 2kd$, where $d = dim\ \mathcal{K} \cap\ Im\ J_{2k-1}$. We denote by $\pi_L : J^{2k-1}(\mathbb{R}, Q) \longrightarrow J^{2k-1}(\mathbb{R}, Q)/\mathcal{K}$ the canonical projection. For what follows we need the following result (see [3] for a proof).

Proposition 3.2. *Let D be an integrable distribution on a manifold M such that the foliation defined by D is a fibration and let M/D be the leaf space.*

(1) A vector field X on M projects onto M/D iff $[X, Z] \in D$, for all $Z \in D$.

(2) A p-form ω projects onto M/D iff $i_Z\omega = 0$, $\mathcal{L}_Z\omega = 0$, for all $Z \in D$. When ω is closed, then it projects onto M/D iff $i_Z\omega = 0$, for all $Z \in D$.

(3) A tensor field R of type (1,1) on M projects onto M/D iff $R(D) \subset D$ and $Im\ (\mathcal{L}_Z R) \subset D$, for all $Z \in D$.

In our case, it is clear that, under the given assumptions (D1)-(D4), Ω_L and dt project onto $(J^{2k-1}(\mathbb{R}, Q))/\mathcal{K}$, i.e. there exist a closed 2-form $\tilde{\Omega}_L$ and a closed 1-form τ on $(J^{2k-1}(\mathbb{R}, Q))/\mathcal{K}$ such that $\pi^*_L\tilde{\Omega}_L = \Omega_L$, $\pi^*_L\tau = dt$. We set $\bar{J}^{2k-1}(\mathbb{R}, Q) = J^{2k-1}(\mathbb{R}, Q)/\mathcal{K}$. Then we deduce that $\bar{J}^{2k-1}(\mathbb{R}, Q)$ has dimension $2kc+1$, where $c = n-d$, $a = kc$, and further $\tilde{\Omega}^a_L \wedge \tau \neq 0$, $\tilde{\Omega}^{a+1}_L = 0$. Thus $(\tilde{\Omega}_L, \tau)$ is a cosymplectic structure on $\bar{J}^{2k-1}(\mathbb{R}, Q)$.

Proposition 3.3. *(1) $\bar{J}_1$ projects onto $\bar{J}^{2k-1}(\mathbb{R}, Q)$ iff $(L_Z\,\bar{J}_1)(\mathcal{K}) \subset \mathcal{K}$, for all $Z \in \mathcal{K}$;*
(2) $\partial/\partial t$ projects onto $\bar{J}^{2k-1}(\mathbb{R}, Q)$ iff $[\partial/\partial t, Z] \in \mathcal{K}$, for all $Z \in \mathcal{K}$.

Theorem 3.4. *Suppose that $\bar{J}_1$ and $\partial/\partial t$ project onto $\bar{J}^{2k-1}(\mathbb{R}, Q)$. Then $(\bar{F}, \tau, T)$ is an integrable almost s-tangent structure of order $2k-1$ on $\bar{J}^{2k-1}(\mathbb{R}, Q)$, where $\bar{F}$, and T are the projections of $\bar{J}_1$ and $\partial/\partial t$, respectively.*

Proposition 3.5. *(1) Each vector field X such that $i_X\Omega_L = 0$, $i_X dt = 1$ projects onto $\bar{J}^{2k-1}(\mathbb{R}, Q)$. (2) If $\bar{J}_1$ is projectable onto $\bar{F}$, then $\bar{J}_r$ is also projectable, for all r, $2 \leq r \leq 2r-1$ and further its projection is $\bar{F}^r$. (3) If $\bar{J}_1$ and $\partial/\partial t$ are projectable, then the Liouville vector field C_1 on $J^{2k-1}(\mathbb{R}, Q)$ is also projectable. (4) Hence if $\bar{J}_1$ and $\partial/\partial t$ are projectable onto $\bar{F}$ and T, respectively, then $\tilde{J}_1$ is projectable and its projection is precisely $\tilde{F} = F - C \otimes \tau$, where C denotes the projection of C_1.*

Suppose that $\bar{J}_1$ and $\partial/\partial t$ project onto $\bar{J}^{2k-1}(\mathbb{R}, Q)$ and let $(\bar{F}, \tau, T)$ be the projected integrable almost s-tangent structure of order $2k-1$ on $\bar{J}^{2k-1}(\mathbb{R}, Q)$. From Proposition 3.1 one has a $2k$-th order differential equation ξ and Proposition 3.5 shows that ξ also projects onto $\bar{\xi}$. Therefore we have $C_{\bar{\xi}} = C$.

Define a tensor field of type (1, 1) on $\bar{J}^{2k-1}(\mathbb{R}, Q)$ by

$$\Gamma_{\bar{\xi}} = -\frac{1}{k}\mathcal{L}_{\bar{\xi}}\tilde{F} + \frac{k-1}{k}(Id - \bar{\xi} \otimes \tau).$$

Proposition 3.6. *Γ_ξ projects onto $\Gamma_{\bar{\xi}}$. Further, $\Gamma_{\bar{\xi}}$ satisfies the following properties:*

$$(1)\ F\Gamma_{\bar{\xi}} = \bar{F}; \quad (2)\ \tilde{F}\Gamma_{\bar{\xi}} = \tilde{F}; \quad (3)\ \Gamma_{\bar{\xi}}\tilde{F}^{2k-1} = -\tilde{F}^{2k-1}; \quad (4)\ \Gamma_{\bar{\xi}}F^{2k-1} = -F^{2k-1};$$

$$(5)\ \mathcal{L}_{\bar{\xi}}\tilde{\Omega}_L = 0; \quad (6)\ (F^k)^*\tilde{\Omega}_L = 0; \quad (7)\ F[\bar{\xi}, T] = 0.$$

The next Theorem can be shown using the results stated above and the properties about equivalence of Lagrangians (for the equivalence problem we remit to the Appendix).

Theorem 3.7. *$\bar{\xi}$ is a non-autonomous regular Lagrangian dynamical system of order $2k$ on $\bar{J}^{2k-1}(\mathbb{R}, Q)$. Then there exists a local function $\bar{L} : U \subset \bar{J}^{2k-1}(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ such that $\Omega = \Omega_{\bar{L}}$ and $\bar{\xi}$ is the Euler-Lagrange vector field associated to $\bar{L}$. If we set $\tilde{L} = \bar{L} \circ \pi_L$, then $\tilde{L}$ and L are geometrically equivalent (that is, the corresponding Poincaré-Cartan 2-form are the same) and hence equivalent (their gauge distributions are equal) and gauge equivalent, i.e, there exists a closed 1-form α on $J^{k-1}(\mathbb{R}, Q)$ such that $L = \tilde{L} + \hat{\alpha}$.*

Next we characterize a class of singular Lagrangians giving a condition for the projections of $\bar{J}_1$ and $\partial/\partial t$. Denote by $j_k : J^k(\mathbb{R}, Q) \cong \mathbb{R} \times T^kQ \longrightarrow T^k(\mathbb{R} \times Q) \cong \mathbb{R}^{k+1} \times T^kQ$ the canonical injection; in local coordinates we have $j_k(t, q_0^i, \cdots, q_k^i) = (t, q_0^i, 0, q_1^i, \cdots, 0, q_k^i)$. Let s be a section of $\pi : \mathbb{R} \times TQ \longrightarrow J^1(\mathbb{R}, Q)$. If s is locally given by $s(t, q_0^i) = (t, q_0^i, s^i(t, q_0))$, then $\hat{s} = j_k \circ s$ is a vector field on $\mathbb{R} \times Q$ locally given by $\hat{s}(t, q_0^i) = s^i(\partial/\partial q_0^i)$ i.e. , $\hat{s}$ is a time-dependent vector field on Q. Thus we deduce that the restriction of the r-th lift $\hat{s}^r$ of $\hat{s}$ to $T^k(\mathbb{R} \times Q)$ is not tangent in general to the submanifold $J^k(\mathbb{R}, Q)$. Consider the natural identification $T^k(\mathbb{R} \times Q) \cong \mathbb{R}^{k+1} \times T^kQ$ locally given by $(t, q_0^i, \tau_1, q_0^i, \cdots, \tau_k, q_k^i) = (t, \tau_1, \cdots, \tau_k, q_0^i, q_1^i, \cdots, q_k^i)$, in such a way that t and $\tau_1, \cdots, \tau_k$ are global coordinates on $T^k(\mathbb{R} \times Q)$. We set $\tilde{s}^r = \hat{s}^r - \sum_{a=1}^k d\tau_a(\hat{s}^r)(\partial/\partial\tau_a)$. Then $\tilde{s}^r$ is tangent to $J^k(\mathbb{R}, Q)$ and it is called the r-th *lift* of s to $J^k(\mathbb{R}, Q)$.

Now, let $\mathcal{D}$ be a distribution on $\mathbb{R} \times Q$ which admits a local basis in a neighbourhood of each point of the form $\{\hat{s_1}, \cdots, \hat{s_a}\}$, where $\{s_1, \cdots, s_a\}$ are a linearly independent local sections of $\mathbb{R} \times TQ \longrightarrow \mathbb{R} \times Q$, $1 \leq a \leq n+1$. Hence $\{\hat{s_1}, ..., \hat{s_a}\}$ are linearly independent vector fields on $\mathbb{R} \times Q$. We may construct a new $(k+1)a$ -dimensional distribution $\tilde{D}^k$ on $J^k(\mathbb{R}, Q)$ as follows: $\tilde{\mathcal{D}}^k$ is locally spanned by the local bases of the form $\{\tilde{s}_1^r, \cdots, \tilde{s}_a^r,\ 0 \leq r \leq k\}$.

Definition 3.3. A distribution $\mathcal{K}$ on $J^k(\mathbb{R}, Q)$ is said to be an *s-tangent distribution* if there exists a distribution $\mathcal{D}$ on $\mathbb{R} \times Q$ such that $\mathcal{K} = \tilde{\mathcal{D}}^k$ and $[\partial/\partial t, \mathcal{D}] \subset \mathcal{D}$.

Theorem 3.8. *Suppose that $L : J^k(I\!R, Q) \longrightarrow I\!R$ is a non-autonomous singular Lagrangian such that its gauge distribution $\mathcal{K} = Ker\ \Omega_L \cap Ker\ dt$ is an s-tangent distribution on $J^{2k-1}(I\!R, Q)$. Then $\bar{J}_1$, C_1 and $\partial/\partial t$ project onto $\bar{J}^{2k-1}(I\!R, Q)$.*

Appendix A. Auxiliary Results

A.1. r-lifts of vector fields. If X is a vector field on a manifold Q, then X may be lifted to a vector field X^r, $0 \leq r \leq k$ to T^kQ; X^r is called the r-th *lift* of X. In particular, X^k is the *complete lift* of X. If ϕ_t is the flow generated by X then $T^k\phi_t$ is the flow generated by X^k. Then we have $X^r = J_{k-r}X^k$. Further we see that X^r, $0 \leq r \leq k-1$ is vertical with respect to the canonical fibration $T^kQ \longrightarrow T^{r-1}Q$.

A.2. Paths of a dynamical connection. From the equations given in definition 2.3 of a dynamical connection we deduce that $\Gamma^3 - \Gamma = 0$, i.e. Γ is an f-structure on $J^k(I\!R, Q)$ (see [20]). The associated complementary projectors are $m = Id - \Gamma^2$ and $l = \Gamma^2$. If we set $\mathcal{N} = l(J^k(I\!R, Q))$ and $\mathcal{M} = m(J^k(I\!R, Q))$, then they are complementary distributions. We deduce that $\mathcal{N}$ (resp. $\mathcal{M}$) is $(k+1)n$-dimensional (resp. one-dimensional), locally spanned by $\{\partial/\partial q^i_r,\ 0 \leq r \leq k\}$ (resp. globally spanned by $m(\partial/\partial t)$). If we now set $\mathcal{H} = (1/2)(Id + \Gamma)(\mathcal{N})$, $\mathcal{V} = (1/2)(Id - \Gamma)(\mathcal{N})$, then $\mathcal{H}$ and $\mathcal{V}$ are complementary distributions in $\mathcal{N}$, that is $\mathcal{N} = \mathcal{H} \oplus \mathcal{V}$. Moreover the distribution $\mathcal{V}$ is locally spanned by $\{\partial/\partial q^i_k\}$ and then it is precisely the vertical distribution of the fibration $\rho^k_{k-1} : J^k(I\!R, Q) \longrightarrow J^{k-1}(I\!R, Q)$. Hence Γ defines in fact a connection in $\rho^k_{k-1} : J^k(I\!R, Q) \longrightarrow J^{k-1}(I\!R, Q)$ which horizontal distribution is $\mathcal{M} \oplus \mathcal{H}$.

A *path* of a dynamical connection Γ on $J^k(I\!R, Q)$ is a curve s in Q such that the k-jet s^k of s is a horizontal curve in $J^k(I\!R, Q)$, that is the k-jet $s^k(t)$ belongs to $(\mathcal{M} \oplus \mathcal{H})_{s^k(t)}$ for all t. It is possible to show that a dynamical connection and its associated semispray have the same paths.

A.3. Equivalence of higher-order Lagrangians. Let α be a 1-form on $J^{k-1}(I\!R, Q)$. Then we may define a function $\hat{\alpha}$ on $J^k(I\!R, Q)$ by $\hat{\alpha}(j^k_0\sigma) = \langle T_k(j^k_0\sigma), \alpha(j^{k-1}_0\sigma)\rangle$. If $\alpha = Adt + \sum_{r=0}^{k-1} (A_r)_i\, dq^i_r$, then we have $\hat{\alpha}(t, q^i_0, q^i_1, \cdots, q^i_k) = A + \sum_{r=0}^{k-1} (A_r)_i\, q^i_{r+1}$. Notice that Ω_L is semibasic with respect to the fibration $\rho^{2k-1}_{k-1} : J^{2k-1}(I\!R, Q) \longrightarrow J^{k-1}(I\!R, Q)$.

We propose several kinds of equivalence of regular non-autonomous higher order Lagrangians. For those having the same dynamics:

Definition A.1. We say that two regular non-autonomous Lagrangians L_1 and L_2 are *equivalent* if $\xi_{L_1} = \xi_{L_2}$.

For those determinating the same cosymplectic structure:

Definition A.2. We say that two regular non-autonomous Lagrangians L_1 and L_2 are *geometrically equivalent* if $\Omega_{L_1} = \Omega_{L_2}$.

From the above Definitions one has: *If L_1 and L_2 are geometrically equivalent then they are equivalent.*

Since $\Omega_{aL_1+bL_2} = a\Omega_{L_1} + b\Omega_{L_2}$, for all $a, b \in I\!R$, we deduce that L_1 and L_2 are geometrically equivalent iff $\Omega_L = 0$, where $L = L_1 - L_2$. Also, it is not hard to show the following

Proposition A.1. *A Lagrangian $L : J^k(I\!R, Q) \longrightarrow I\!R$ is such that $\Omega_L = 0$ iff there exists a closed 1-form α on $J^{k-1}(I\!R, Q)$ such that $L = \hat{\alpha}$.*

Corollary A.2. *Two regular non-autonomous Lagrangians L_1 and L_2 are geometrically equivalent iff $L_2 = L_1 + \hat{\alpha}$, where α is a closed 1-form on $J^{k-1}(\mathbb{R}, Q)$.*

Definition A.3. Two regular non-autonomous Lagrangians $L_1, L_2 : J^k(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ are said to be *gauge equivalent* if $L_2 = L_1 + \hat{\alpha}$, where α is a closed 1-form on $J^{k-1}(\mathbb{R}, Q)$.

According to the Corollary, we deduce that *two regular non-autonomous Lagrangians L_1 and L_2 are gauge equivalent iff they are geometrically equivalent.*

Now, we consider degenerate non-autonomous Lagrangians. We introduce the following definitions.

Definition A.4. Let $L_1, L_2 : J^k(\mathbb{R}, Q) \longrightarrow \mathbb{R}$ be two degenerate non-autonomous Lagrangians. We say that L_1 and L_2 are :

1. *equivalent* if their gauge distributions are equal: $Ker\ \Omega_{L_1} \cap\ Ker\ dt = Ker\ \Omega_{L_2} \cap Ker\ dt$.
2. *geometrically equivalent* if $\Omega_{L_1} = \Omega_{L_2}$.
3. *gauge equivalent* if $L_2 = L_1 + \hat{\alpha}$, where α is a closed 1-form on $J^{k-1}(\mathbb{R}, Q)$.

It is clear that, if L_1 and L_2 are regular, these definitions coincide with the previous ones. Since Corollary A.2 holds true, we deduce the following properties:

(1) *If L_1 and L_2 are geometrically equivalent, then they are equivalent.*

(2) *L_1 and L_2 are geometrically equivalent iff they are gauge equivalent.*

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