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Topics in the calculus of variations: Finite order variational sequences

D. Krupka

Abstract: It is known that there exists a mapping Λ assigning to a first order lagrangian λ its Lepagean equivalent $\Lambda(\lambda)$ in such a way that $d\Lambda(\lambda) = 0$ if and only if the Euler-Lagrange form E_λ vanishes identically, i.e., $E_\lambda = 0$. In this paper we discuss within the theory of finite order variational sequences an analogue of Λ for higher order lagrangians. It turns out that $\Lambda(\lambda)$ is a class of forms rather than a differential form defined on the same domain as λ. The main use of Λ is to define the order of a lagrangian in a more adequate way than the usual one. We show how the order of a lagrangian can be determined. We find the order of a variationally trivial lagrangian. Applying these results to the classification problem of symmetry transformations we obtain a higher order analogue of the Noether-Bessel-Hagen equation.

Keywords: Fibered manifold, r-jet, lagrangian, contact form, variational sequence, order of a lagrangian, Euler-Lagrange mapping, Helmholtz-Sonin mapping, variationally trivial lagrangians.

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1. Introduction

Let X be an n-dimensional manifold, Y a *fibered manifold* of dimension $n+m$ with projection $\pi : Y \to X$. The *r-jet prolongation* of Y will be written as J^rY, and $\pi_r : J^rY \to X, \pi_{r,s} : J^rY \to J^sY$, where $0 \le s \le r$, will denote the *canonical jet projections*.

If $W \subset Y$ is an open set, we write $W_r = \pi_{r,0}^{-1}(W)$, and denote by $\Omega_r^0 W$ the ring of smooth functions defined on W_r. For $k \ge 1$ we denote by $\Omega_r^k W$ the $\Omega_r^0 W$-module of smooth k-forms on W_r. The submodule of $\Omega_r^k W$ consisting of π_r-horizontal (resp. $\pi_{r,0}$-horizontal, resp. $\pi_{r,s}$-horizontal) forms will be written as $\Omega_{r,X}^k W$ (resp. $\Omega_{r,Y}^k W$ resp. $\Omega_{r,Z}^k W$, where $Z = J^sY$).

The standard *horizontalization* mapping is denoted by h; h maps $\Omega_r^k W$ to $\Omega_{r+1,X}^k W$, where $1 \le k \le n$. Notice that restricting h to $\Omega_{r,Z}^k W \subset \Omega_r^k W$ where $Z = J^rY$, we obtain a mapping $h : \Omega_{r,Z}^k W \to \Omega_r^k W \subset \Omega_{r+1,X}^k W$. We also denote h by p_0 and write $p_i\rho$ for

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the i-contact component of a form ρ (the definitions of these concepts will be recalled in Sec. 2).

Let E_λ denote the Euler-Lagrange form of a lagrangian $\lambda \in \Omega^n_{1,X}Y$. The following result concerns the *first order* variational calculus, and is taken from [8].

There exists an R-linear mapping $\Lambda : \Omega^n_{1,X}Y \to \Omega^n_{1,Y}Y$ *satisfying the following three conditions :*

(1) *For every lagrangian* $\lambda \in \Omega^n_{1,X}Y$,

$$h(\Lambda(\lambda)) = \lambda. \tag{1.1}$$

(2) *If* $\rho \in \Omega^n_1 Y$ *is a* $\pi_{1,0}$*-projectable form, then*

$$\Lambda(h\rho)) = \rho. \tag{1.2}$$

(3) *For every lagrangian* $\lambda \in \Omega^n_{1,X}W$ *the form* $E_\lambda = p_1 d\Lambda(\lambda) \in \Omega^{n+1}_2 Y$ *is* $\pi_{2,0}$*-horizontal, and* $E_\lambda = 0$ *if and only if*

$$d\Lambda(\lambda) = 0. \tag{1.3}$$

The first part of (3) means that $\Lambda(\lambda)$ is a Lepagean form; condition (1) then means that $\Lambda(\lambda)$ is a Lepagean equivalent of λ. The second part of (3) shows that Euler-Lagrange form E_λ vanishes if and only if $d\Lambda(\lambda) = 0$; in this sense $\Lambda(\lambda)$ has a remarkable property to stand "closer" to the exterior derivative operator d as the well-known Poincaré-Cartan equivalent $\Theta(\lambda)$.

The mapping Λ was discovered by Krupka in connection with the study of lagrangians $\lambda \in \Omega^n_{1,X}Y$ of the form $h(\rho)$, where $\rho \in \Omega^n_0 Y$ (see [12]).

Recall that a π-projectable vector field ξ on Y is said to generate *generalized invariant* transformations of a lagrangian $\lambda \in \Omega^n_{1,X}Y$, if

$$\partial_{J^2\xi} E_\lambda = 0, \tag{1.4}$$

i.e., if the Lie derivative of the Euler-Lagrange form E_λ by the 2-jet prolongation $J^2\xi$ of ξ vanishes.

The following additional properties of Λ have been proved (see [11] and [10]):

(4) *If* $\lambda \in \Omega^n_{1,X}Y$ *is such that* $E_\lambda = 0$*, then the form* $\Lambda(\lambda)$ *is a unique form such that* $h(\Lambda(\lambda)) = \lambda$ *and* $d\Lambda(\lambda) = 0$*. This form is* $\pi_{1,0}$*-projectable.*

(5) *A* π*-projectable vector field* ξ *on* Y *generates generalized invariant transformations of a lagrangian* $\lambda \in \Omega^n_{1,X}Y$ *if and only if there exists an* n*-form* ρ *on* Y *such that*

$$\partial_{J^2\xi}\lambda = h(\rho), \quad d\rho = 0. \tag{1.5}$$

Clearly, applying (1.2) to (1.5) we get $\rho = \Lambda(\partial_{J^2\xi}\lambda)$, and (1.5) reduces to the condition

$$d\Lambda(\partial_{J^2\xi}\lambda) = 0. \tag{1.6}$$

Condition (1.6) which may be used to determine the generators ξ of the generalized invariant transformation, is a generalization of the classical *Noether-Bessel-Hagen equation.*

Practically the same theory as above was developed by Betounes [5] (see also [6]) who obtained, moreover, the following result [4]:

(6) *For every isomorphism* $\alpha : Y \to Y$ *of the fibered manifold* Y,

$$(J^1\alpha)^*\Lambda(\lambda) = \Lambda((J^1\alpha)^*\lambda), \tag{1.7}$$

where $J^1\alpha : J^1Y \to J^1Y$ *is the first jet prolongation of* α.

Combining this result with (1.6) we obtain $\partial_{J^1\xi} d\Lambda(\lambda) = 0$ which is another version of the Noether-Bessel-Hagen equation.

A comparison of the papers [8] and [4] from the point of view of the Lepagean forms can be found in [9].

Denote

$$\omega_0 = dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n, \quad \omega^\sigma = dy^\sigma - y_i^\sigma dx^i. \tag{1.8}$$

If λ is expressed by

$$\lambda = L\omega_0, \tag{1.9}$$

then

$$\Lambda(\lambda) = L\omega_0 + \sum_{q=1}^{n} \frac{\partial^q L}{\partial y_{j_1}^{\sigma_1} \partial y_{j_2}^{\sigma_2} \ldots \partial y_{j_q}^{\sigma_q}} \, dx^1 \wedge \ldots \wedge dx^{j_1 - 1} \wedge \omega^{\sigma_1} \wedge dx^{j_1+1} \wedge \ldots \wedge dx^{j_q - 1} \wedge \omega^{\sigma_q} \wedge dx^{j_q + 1} \wedge \ldots \wedge dx^n. \tag{1.10}$$

The aim of this paper is to discuss a possible extension of the mapping Λ to the higher order variational calculus. Our discussion is going on within the framework of the theory of finite order variational sequences [13].

Since the calculations in the general case of r-th order jet spaces are extremely difficult and long instead of presenting the details of the proofs we shall explain the main ideas of how the mapping Λ is introduced for the first order case; a complete exposition including proofs will be given elsewhere.

Let $W \subset Y$ be an open set, and $\rho \in \Omega_0^n W$. Write in a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, where $V \subset W$,

$$\begin{aligned} h\rho &= \frac{1}{n!} \, B_{i_1 i_2 \ldots i_n} dx^{i_1} \wedge \ldots \wedge dx^{i_n}, \\ p_j\rho &= \sum \frac{1}{j!(n-j)!} \, B_{\sigma_1 \ldots \sigma_j i_{j+1} \ldots i_n} \omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_j} \wedge dx^{i_{j+1}} \wedge \ldots \wedge dx^{i_n}, \\ &\qquad 1 \le j \le n-1, \\ p_n\rho &= \frac{1}{n!} \, B_{\sigma_1 \sigma_2 \ldots \sigma_n} \omega^{\sigma_1} \wedge \omega^{\sigma_2} \wedge \ldots \wedge \omega^{\sigma_n}. \end{aligned} \tag{1.11}$$

Using the decomposition

$$\pi_{1,0}^* \rho = h\rho + p_1\rho + \ldots + p_n\rho$$

and the formula $d\pi_{1,0}^*\rho = \pi_{1,0}^* d\rho$, we obtain $hd\rho + p_1 d\rho + \ldots + p_{n+1} d\rho = dh\rho + dp_1\rho + \ldots + dp_n\rho$. Hence

$$\begin{aligned} \pi_{2,1}^* hd\rho &= hdh\rho, \\ \pi_{2,1}^* p_j d\rho &= p_j dp_{j-1}\rho + p_j dp_j\rho, \quad 1 \le j \le n, \\ \pi_{2,1}^* p_{n+1} d\rho &= p_{n+1} dp_n\rho \end{aligned} \tag{1.12}$$

(here $p_0 = h$). Express both sides in terms of the basis of linear forms $dx^i, \omega^\sigma, \omega^\sigma_j, dy^\sigma_{ij}$ on $V_2 \subset J^2Y$. Then the forms on the left contain the exterior factors dx^i, ω^σ only, while the right-hand side expressions depend also on ω^σ_j. The sum of such terms should therefore vanish identically. We shall find the corresponding identities.

The first or the conditions (1.12) reduces to the identity $0 = 0$. Consider the case $1 \le j \le n$. We get

$$\begin{aligned}
&\frac{1}{(j-1)!(n-j+1)!}\frac{\partial B_{\sigma_1\ldots\sigma_{j-1}i_j\ldots i_n}}{\partial y^\nu_p}\ \omega^\nu_p \wedge \omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_{j-1}} \wedge dx^{i_j} \wedge \ldots \wedge dx^{i_n}\\
&\quad + \frac{1}{j!(n-j)!}\ j \cdot (-1)^j B_{\sigma_1\ldots\sigma_j i_{j+1}\ldots i_n}\omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_{j-1}} \wedge \omega^{\sigma_j}_{i_j}\\
&\qquad \wedge dx^{i_j} \wedge dx^{i_{j+1}} \wedge \ldots\ldots \wedge dx^{i_n}\\
&= \Big(\frac{1}{(j-1)!(n-j+1)!}\frac{\partial B_{\sigma_1\ldots\sigma_{j-1}i_j\ldots i_n}}{\partial y^\nu_p} - \frac{1}{(j-1)!(n-j)!}\delta^p_{i_j} B_{\sigma_1\ldots\sigma_{j_1}\nu i_{j+1}\ldots i_n}\Big)\\
&\qquad \cdot \omega^\nu_p \wedge \omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_{j-1}} \wedge dx^{i_j} \wedge \ldots \wedge dx^{i_n}. \qquad (1.13)
\end{aligned}$$

Therefore

$$\frac{1}{n-j+1}\frac{\partial B_{\sigma_1\ldots\sigma_{j-1}i_j\ldots i_n}}{\partial y^\nu_p} - B_{\sigma_1\ldots\sigma_{j-1}\nu i_j\ldots i_n}\delta^p_{i_j} = 0. \qquad (1.14)$$

Contracting the left-hand side in p, i_j we finally obtain

$$\frac{1}{j}\frac{\partial B_{\sigma_1\ldots\sigma_{j-1}q i_{j+1}\ldots i_n}}{\partial y^\nu_q} - B_{\sigma_1\ldots\sigma_{j-1}\nu i_{j+1}\ldots i_n} = 0, \quad 1 \le j \le n. \qquad (1.15)$$

If $j = n+1$, we get the term

$$\frac{1}{n!}\frac{\partial B_{\sigma_1\sigma_2\ldots\sigma_n}}{\partial y^\nu_p}\ \omega^\nu_p \wedge \omega^{\sigma_1} \wedge \omega^{\sigma_2} \ldots \wedge \omega^{\sigma_n}, \qquad (1.16)$$

which gives

$$\frac{\partial B_{\sigma_1\sigma_2\ldots\sigma_n}}{\partial y^\nu_p} = 0. \qquad (1.17)$$

Conditions (1.15), (1.16) and (1.17) determine $p_j\rho$ with the help of $h\rho$.

In particular, writing

$$B_{i_1 i_2\ldots I_n} = L\varepsilon_{i_1 i_2\ldots i_n}, \qquad (1.18)$$

we get

$$\begin{aligned}
p_j\rho = {} & \frac{1}{j!(n-j)!}\ \frac{1}{j!}\ \frac{\partial^j L}{\partial y^{\sigma_1}_{q_1}\partial y^{\sigma_2}_{q_2}\ldots\partial y^{\sigma_j}_{q_j}}\ \varepsilon_{q_1 q_2\ldots q_j i_{j+1}\ldots i_n}\omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_j}\\
& \wedge dx^{i_{j+1}} \wedge \ldots \wedge dx^{i_n}. \qquad (1.19)
\end{aligned}$$

Denote by $(p_1, \ldots, p_j)$ the unique sequence of indices between 1 and n, complementary to $(i_{j+1}, \ldots, i_n)$, such that $p_1 < \ldots < p_j$. Then

$$\begin{aligned}
\omega^{\sigma_1} \wedge \ldots \wedge \omega^{\sigma_j} \wedge dx^{i_{j+1}} \wedge \ldots \wedge dx^{i_n} = {} & \varepsilon^{p_1\ldots p_j i_{j+1}\ldots i_n} dx^1 \wedge \ldots \wedge dx^{p_1-1} \wedge \omega^{\sigma_1}\\
& \wedge dx^{p_1+1} \wedge \ldots \wedge dx^{p_j-1} \wedge \omega^{\sigma_j} \wedge dx^{p_j+1} \wedge \ldots \wedge dx^n. \qquad (1.20)
\end{aligned}$$

Since

$$\varepsilon_{q_1\ldots g_j i_{j+1}\ldots i_n}\varepsilon^{p_1\ldots p_j i_{j+1}\ldots i_n} = j!(n-j)!\delta^{p_1}_{q_1}\ldots\delta^{p_j}_{q_j} \quad \text{antisym}(p_1\ldots p_j), \tag{1.21}$$

we easily obtain the above expression (1.10).

We note that this paper is not intended to compare the techniques and result of the theory of finite order variational sequences and the variational bicomplexes which are based on finite jet constructions. For the theory of variational bicomplexes the reader is referred to [1–3], [7], and [14–16], where further references can be found.

2. Finite order variational sequences

In this section we recall basic notions of the theory of differential forms on jet prolongations of a fibered manifold which are necessary for next sections. (see [9], [12], [13]). We use the same notation as in Introduction.

Let $r \geq 1$ be an integer. To every tangent vector $\xi \in TJ^{r+1}Y$ at a point $J^{r+1}_x\gamma \in J^{r+1}Y$one can assign a tangent vector $h\xi \in TJ^rY$ at the point $\pi_{r+1,r}(J^{r+1}_x\gamma) = J^r_x\gamma$, and a tangent vector $p\xi \in TJ^rY$ at $J^r_x\gamma$ by

$$h\xi = T_xJ^r\gamma \cdot T\pi_{r+1}\cdot\xi, \quad p\xi = T\pi_{r+1,r}\cdot\xi - h\xi. \tag{2.1}$$

Clearly, $p\xi$ is always a π_r-vertical vector. ξ is π_{r+1}-vertical if and only if $h\xi = 0$; ξ is $\pi_{r+1,r}$-vertical if and only if $h\xi = 0,\ p\xi = 0$.

The mappings $\xi \to h\xi, \xi \to p\xi$ are vector bundle homomorphisms of $TJ^{r+1}Y$ to TJ^rY over the jet projection $\pi_{r+1,r}$. We call h the *horizontalization.*

The horizontalization $h : TJ^{r+1}Y \to TJ^rY$ induces a decomposition of the modules of differential forms on J^rY in the following sense. Let $W \subset Y$ be an open set, and let $\rho \in \Omega^k_rW,\ k \geq 1$. Let $J^{r+1}_x\gamma \in W_{r+1}$ be a point, $\xi_1, \xi_2, \ldots, \xi_k \in TJ^{r+1}Y$ tangent vectors at this point. Write for each $i = 1, 2, \ldots, k$

$$T\pi_{r+1,r}\cdot\xi_i = h\xi_i + p\xi_i, \tag{2.2}$$

and denote

$$\begin{aligned} &p_0\rho(J^{r+1}_x\gamma)(\xi_1,\xi_2,\ldots,\xi_k) = \rho(J^r_x\gamma)(h\xi_1,h\xi_2,\ldots,h\xi_k),\\ &p_1\rho(J^{r+1}_x\gamma)(\xi_1,\xi_2,\ldots,\xi_k) = \rho(J^r_x\gamma)(p\xi_1,h\xi_2,\ldots,h\xi_k)\\ &\qquad\qquad + \rho(J^r_x\gamma)(h\xi_1,p\xi_2,h\xi_3,\ldots,h\xi_k) + \ldots + \rho(J^r_x\gamma)(h\xi_1,\ldots,h\xi_{k-1},p\xi_k),\\ &\ldots\\ &p_k\rho(J^r_x\gamma)(\xi_1,\xi_2,\ldots,\xi_k) = \rho(J^r_x\gamma)(p\xi_1,p\xi_2,\ldots,p\xi_k). \end{aligned} \tag{2.3}$$

$p_i\rho$ is a k-form on W_{r+1}, called the *i-contact component* of ρ. Sometimes we denote

$$h\rho = p_0\rho, \quad p\rho = \sum_{i=1}^{k} p_i\rho \tag{2.4}$$

and call $h\rho$ (resp. $p\rho$) the *horizontal* (resp. *contact*) *component* of ρ.

Since by the definition of the pull-back,

$$\begin{aligned} &\pi^*_{r+1,r}\rho(J^{r+1}_x\gamma)(\xi_1,\xi_2,\ldots,\xi_k)\\ &\qquad = \rho(J^r_x\gamma)(T\pi_{r+1,r}\cdot\xi_1, T\pi_{r+1,r}\cdot\xi_2,\ldots,T\pi_{r+1,r}\cdot\xi_k), \end{aligned} \tag{2.5}$$

we easily obtain a decomposition formula

$$\pi_{r+1,r}^{*}\rho = \sum_{i=0}^{k} p_i\rho = h\rho + p\rho \cdot \tag{2.6}$$

ρ is called 0-*contact*, or *horizontal*, if $p\rho = 0$; ρ is called *contact*, if $h\rho = 0$. We say that ρ is q-*contact*, if $\pi_{r+1,r}^{*}\rho = p_q\rho$, or, which is the same, if $p_j\rho = 0$ for $j \neq q$. We say that ρ is of *order of contactness* $\geq q$, if $h\rho = 0$, $p_1\rho = 0, \ldots, p_{q-1}\rho = 0$.

In what follows we often use *multiindices* $I = (i_1 i_2 \ldots i_p)$, where $1 \leq i_1, i_2, \ldots, i_p \leq n$. For such a multiindex I we denote $p = |I|$, and call this number the *length* of I. It will be convenient to denote by Ii the multiindex $(i_1 i_2 \ldots i_p i)$.

Let (V, ψ), $\psi = (x^i, y^\sigma)$, be a fibered chart on Y. Denote for every multiindex I such that $0 \leq |I| \leq r$, and every σ, $1 \leq \sigma \leq m$,

$$\omega_I^\sigma = dy_I^\sigma - y_{Ii}^\sigma dx^i. \tag{2.7}$$

Then on V_{r+1}, with the obvious notational convention,

$$hdx^i = dx^i, \quad hdy_I^\sigma = y_{Ii}^\sigma dx^i, \quad pdx^i = 0, \quad pdy_I^\sigma = \omega_I^\sigma. \tag{2.8}$$

Obviously, the forms

$$dx^i, \omega_I^\sigma, dy_J^\sigma, \quad 0 \leq |I| \leq r, \ |J| = r+1, \tag{2.9}$$

define a basis of linear forms on V_{r+1}. We have

$$\begin{aligned} d\omega_I^\sigma &= -\omega_{Ii}^\sigma \wedge dx^i, \quad 0 \leq |I| \leq r-1, \\ d\omega_I^\sigma &= -dy_{Ii}^\sigma \wedge dx^i, \quad |I| = r. \end{aligned} \tag{2.10}$$

If $J_x^{r+1}\gamma \in V_{r+1}$ is a point and $\xi \in TJ^{r+1}Y$ a tangent vector at $J_x^{r+1}\gamma$,

$$\xi = \xi^i \frac{\partial}{\partial x^i} + \sum_{0 \leq |I| \leq r+1} \Xi_I^\sigma \frac{\partial}{\partial y_I^\sigma}, \tag{2.11}$$

then

$$\begin{aligned} &dx^i(J_x^{r+1}\gamma)(\xi) = \xi^i, \\ &\omega_I^\sigma(J_x^{r+1}\gamma)(\xi) = \Xi_I^\sigma - y_{I_i}^\sigma \xi^i, \quad 0 \leq |I| \leq r, \\ &dy_J^\sigma(J_x^{r+1}\gamma)(\xi) = \Xi_J^\sigma, \quad |J| = r+1. \end{aligned} \tag{2.12}$$

If $f : V_r \to R$ is a smooth function, we have

$$\pi_{r+1,r}^{*} df = hdf + pdf, \tag{2.13}$$

where

$$hdf = d_i f \cdot dx^i, \quad d_i f = \frac{\partial f}{\partial x^i} + \sum_{|I| \leq r} \frac{\partial f}{\partial y_I^\sigma} y_{Ii}^\sigma, \tag{2.14}$$

$$pdf = \sum_{|I| \leq r} \frac{\partial f}{\partial y_I^\sigma} \omega_I^\sigma. \tag{2.15}$$

In particular,

$$d_i y_I^\sigma = y_{Ii}^\sigma. \tag{2.16}$$

We also use the notation

$$d'f = \frac{df}{dx^i}dx^i + \sum_{|I|\leq r-1} \frac{df}{dy_I^\sigma}dy_I^\sigma, \tag{2.17}$$

and define a function $d_i'f : V_r \to R$ by

$$hd'f = d_i'f \cdot dx^i. \tag{2.18}$$

Then

$$df = d'f + \sum_{|I|=r} \frac{df}{dy_I^\sigma}dy_I^\sigma, \tag{2.19}$$

$$d_i f = d_i'f + \sum_{|I|=r} \frac{df}{dy_I^\sigma}y_{Ii}^\sigma. \tag{2.20}$$

Let $r \geq 1, k \geq 0$ be fixed. We denote by Ω_r^k the direct image of the sheaf of k-forms over J^rY by the projection $\pi_{r,0} : J^rY \to Y$. If $p_i : \Omega_r^k \to \Omega_{r+1}^k$ is considered as a sheaf morphism, we denote

$$\Omega_{r(c)}^k = \begin{cases} \ker h, & 1 \leq k \leq n, \\ \ker p_{k-n}, & k \geq n+1. \end{cases} \tag{2.21}$$

Then we set

$$d\Omega_{r(c)}^k = \operatorname{im} d_k, \tag{2.22}$$

where d_k is the sheaf morphism defined by the exterior derivative d of k-forms, and

$$\Theta_r^k = d\Omega_{r(c)}^{k-1} + \Omega_{r(c)}^k. \tag{2.23}$$

We get a diagram

$$\begin{tikzcd}
 & & & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
 & & 0 \arrow[r] \arrow[d] & \Theta_r^1 \arrow[r, "d"] \arrow[d] & \Theta_r^2 \arrow[r, "d"] \arrow[d] & \Theta_r^3 \arrow[r] \arrow[d] & \dots \\
0 \arrow[r] & R \arrow[r] & \Omega_r^0 \arrow[r, "d"] \arrow[dr] & \Omega_r^1 \arrow[r, "d"] \arrow[d] & \Omega_r^2 \arrow[r, "d"] \arrow[d] & \Omega_r^3 \arrow[r] \arrow[d] & \dots \\
 & & & \Omega_r^1/\Theta_r^1 \arrow[r, "E_1"] \arrow[d] & \Omega_r^2/\Theta_r^2 \arrow[r, "E_2"] \arrow[d] & \Omega_r^3/\Theta_r^3 \arrow[r] \arrow[d] & \dots \\
 & & & 0 & 0 & 0 &
\end{tikzcd} \tag{2.24}$$

in which the sequence

$$0 \to R \to \Omega_r^0 \to \Omega_r^1/\Theta_r^1 \to \Omega_r^2/\Theta_r^2 \to \Omega_r^3/\Theta_r^3 \to \dots \tag{2.25}$$

is an acyclic resolution of the constant sheaf R over Y. This sequence is called the *variational sequence of order* r over Y. The sheaf morphism E_n (resp. E_{n+1}) is the *Euler-Lagrange mapping* (resp. the *Helmholtz-Sonin mapping*).

If $\rho \in \Omega_r^n$ is a germ and $[\rho] = \lambda = f_0\omega_0$ its class in Ω_r^n/Θ_r^n, then

$$E_n(\lambda) = \varepsilon_\sigma \omega^\sigma \wedge \omega_0, \tag{2.26}$$

where

$$\varepsilon_\sigma = \sum_{k=0}^{r+1} (-1)^k d_{j_1} \dots d_{j_k} \frac{\partial f_0}{\partial y^\sigma_{j_1 \dots j_k}}. \tag{2.27}$$

If $\rho \in \Omega_r^{n+1}$ is a germ and $[\rho] = \varepsilon = \varepsilon_\sigma \omega^\sigma \wedge \omega_0$ its class in $\Omega_r^{n+1}/\Theta_r^{n+1}$, where ε_σ are defined on $V_s \subset J^s Y$, then

$$\begin{aligned} E_{n+1}(\varepsilon) = \frac{1}{2} \sum_{i=1}^{s} \Bigg(& \frac{\partial \varepsilon_\nu}{\partial y^\sigma_{j_1 \dots j_i}} - (-1)^i \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 \dots j_i}} \\ & - \sum_{k=i+1}^{s} (-1)^k \binom{k}{i} d_{j_{i+1}} \dots d_{j_k} \frac{\partial \varepsilon_\sigma}{\partial y^\nu_{j_1 \dots j_k}} \Bigg) \omega^\sigma_{j_1 \dots j_i} \wedge \omega^\nu \wedge \omega_0. \end{aligned} \tag{2.28}$$

A well-known conclusion of these results for the calculus of variations consists in a cohomology description of the difference between the *local* and *global triviality* of global sections of Ω_r^n (or Ω_r^n/Θ_r^n), and between the *local* and *global variationality* of global sections of Ω_r^{n+1} (or $\Omega_r^{n+1}/\Theta_r^{n+1}$) (see [1], [2], [13–16]).

3. Projectable and π-decomposable forms

We begin by presenting three auxiliary algebraic assertions which can be used in the proofs of our main theorems. If in the formulas below a symbol like antisym$(i_1 i_2 \dots i_s)$ (resp. sym$(j_1 j_2 \dots j_r)$) is used, then the expression preceding this symbol should be considered as antisymmetrized in $i_1, i_2, \dots, i_s$ (resp. symmetrized in $j_1, j_2, \dots, j_r$).

Lemma 1. *Let r, s, and n be positive integers, and let $A^{j_1 \dots j_r}{}_{i_1 \dots i_s}$, $1 \le j_1, \dots, j_r$, $i_1, \dots, i_s \le n$, be a system of real numbers symmetric in the superscripts and antisymmetric in the subscripts. Put for every p, q, $1 \le p, q \le n$,*

$$Q^{j_1 \dots j_r p}{}_{i_1 \dots i_s q} = A^{j_1 \dots j_r}{}_{i_1 \dots i_s} \delta^p_q \quad \text{antisym}(i_1 \dots i_s q) \ \text{sym}(j_1 \dots j_r p), \tag{3.1}$$

$$B^{j_2 \dots j_r}{}_{i_2 \dots i_s} = \frac{rs}{h + r - s} A^{p j_2 \dots j_r}{}_{p i_2 \dots i_s}. \tag{3.2}$$

(a) *The system $A^{j_1 \dots j_r}{}_{i_1 \dots i_s}$ obeys the identity*

$$\begin{aligned} Q^{j_1 \dots j_r p}{}_{i_1 \dots i_s p} = & \frac{n + r - s}{(r+1)(s+1)} (A^{j_1 \dots j_r}{}_{i_1 \dots i_s} - B^{j_2 \dots j_r}{}_{i_2 \dots i_s} \delta^{j_1}_{i_1}) \\ & \text{antisym}(i_1 i_2 \dots i_s) \ \text{sym}(j_1 j_2 \dots j_r). \end{aligned} \tag{3.3}$$

(b) *The system* $C^{j_1\dots j_r}{}_{i_1\dots i_s}$ *defined by the decomposition*

$$A^{j_1\dots j_r}{}_{i_1\dots i_s} = B^{j_2\dots j_r}{}_{i_2\dots i_s}\delta^{j_1}_{i_1} + C^{j_1\dots j_r}{}_{i_1\dots i_s}$$
$$\text{antisym}(i_1 i_2 \dots i_s)\ \text{sym}(j_1 j_2 \dots j_r) \tag{3.4}$$

is traceless, i.e., for all $j_2, \dots j_r,\ i_2, \dots, i_s$

$$C^{pj_2\dots j_r}{}_{pi_2\dots i_s} = 0. \tag{3.5}$$

(c) *Equation*

$$Q^{j_1\dots j_r p}{}_{i_1\dots i_s q} = 0 \tag{3.6}$$

is satisfied if and only if there exists a system $B^{j_2\dots j_r}{}_{i_2\dots i_s}$ *symmetric in the superscripts and antisymmetric in the subscripts such that*

$$A^{j_1\dots j_r}{}_{i_2\dots i_s} = B^{j_2\dots j_r}{}_{i_2\dots i_s}\delta^{j_1}_{i_1} \quad \text{antisym}(i_1 i_2 \dots i_s)\ \text{sym}(j_1 j_2 \dots j_r). \tag{3.7}$$

The system $B^{j_2\dots j_r}{}_{i_2\dots i_s}$ *is unique, and is traceless, i.e.*

$$B^{pj_3\dots j_r}{}_{pi_3\dots i_s} = 0. \tag{3.8}$$

Lemma 2. *Let* n, k, j *be integers such that* $2 \le k \le n$, $1 \le j \le k-1$. *Let* $r_1, r_2, \dots, r_j$ *be positive integers, and let* I_1 *(resp.* $I_2, \dots, I_j$*) be a multiindex of length* r_1 *(resp.* $r_2, \dots, r_j$*) whose entries are arbitrary indices belonging to the set* $\{1, 2, \dots, n\}$. *Consider the system of linear, homogeneous equations*

$$A^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2}\dots i_k}\delta^{p_1}_{i_1}\delta^{p_2}_{i_2}\dots\delta^{p_j}_{i_j} = 0$$
$$\text{antisym}(i_1 i_2 \dots i_k)\ \text{sym}(I_1 p_1)\ \text{sym}(I_2 p_2)\ \dots\ \text{sym}(I_j p_j), \tag{3.9}$$

where $1 \le p_1, p_2, \dots, p_j, i_1, i_2, \dots, i_k \le n$, *and the unknown system* $A^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2}\dots i_k}$ *is antisymmetric in the subscripts, and symmetric in the superscripts entering every of the multiindices* $I_1, I_2, \dots, I_j$.

Then every solution of this system is of the form

$$A^{I_1 I_2 \dots I_j}{}_{i_{j+1} i_{j+2}\dots i_k}$$
$$= B^{Q_1 I_2 \dots I_j}{}_{i_{j+2}\dots i_k}\delta^{q_1}_{i_{j+1}} + B^{I_1 Q_2 I_3 \dots I_j}{}_{i_{j+2}\dots i_k}\delta^{q_2}_{i_{j+1}} + B^{I_1 \dots I_{j-1} Q_j}{}_{i_{j+2}\dots i_k}\delta^{q_j}_{i_{j+1}}$$
$$\text{antisym}(i_{j+1} i_{j+2} \dots i_k)\ \text{sym}(Q_1 q_1)\ \dots\ \text{sym}(Q_j q_j), \tag{3.10}$$

where $I_1 = Q_1 q_1, \dots, I_j = Q_j q_j$, *and* $B^{Q_1 I_2 \dots I_s}{}_{i_{j+2}\dots i_k}, \dots, B^{I_1 \dots I_{j-1} Q_j}{}_{i_{j+2}\dots i_k}$ *are arbitrary constants antisymmetric in the subscripts, and symmetric in the superscripts entering every of the multiindices* $Q_1, \dots, Q_j, I_1, \dots, I_j$.

Lemma 3. *Let* n, k, j *be integers such that* $1 \le j \le n-1, k \ge n+1$. *Let* r *be a positive integer, and let* $I_1, I_2, \dots, I_{n-k+j}$ *be multiindices of length* r *whose entries are arbitrary indices from the set* $\{1, 2, \dots, n\}$. *Consider the system of linear, homogeneous equations*

$$A^{I_1 \dots I_{k-n} I_{k-n+1} \dots I_{k-n+j}}{}_{i_{j+1}\dots i_n}\delta^{p_{k-n+1}}_{i_1}\dots\delta^{p_{k-n+j}}_{i_j} = 0$$
$$\text{antisym}(i_1 i_2 \dots i_n)\ \text{sym}(I_{k-n+1} p_{k-n+1})\ \dots\ \text{sym}(I_{k-n+j} p_{k-n+j}) \tag{3.11}$$

for the unknowns $A^{I_1\dots I_{k-n}I_{k-n+1}\dots I_{k-n+j}}{}_{i_{j+1}\dots i_n}$, *antisymmetric in the subscripts and in the multiindices* $I_1,\dots,I_{k-n+j}$, *and symmetric in the superscripts entering every of the multiindices* $I_1,\dots,I_{k-n+j}$, *where* $1\le p_1,\dots,p_j,i_1,\dots,i_n\le n$. *Every solution of this system has the form*

$$\begin{aligned}
&A^{I_1\dots I_{k-n}I_{k-n+1}\dots I_{k-n+j}}{}_{i_{j+1}\dots i_n}\\
&\quad = C^{Q_1\dots Q_{k-n}Q_{k-n+1}I_{k-n+2}\dots I_{k-n+j}}{}_{i_{j+k-n+2}\dots i_n}\delta^{p_1}_{i_{j+1}}\dots\delta^{p_{k-n}}_{i_{j+k-n}}\delta^{p_{k-n+1}}_{i_{j+k-n+1}}\\
&\qquad +\dots+C^{I_1\dots I_{j-1}Q_jQ_{j+1}\dots Q_{k-n+j}}{}_{i_{j+k-n+2}\dots i_n}\delta^{p_n}_{i_{j+1}}\dots\delta^{p_{k-n+j-1}}_{i_{j+k-n}}\delta^{p_{k-n+j}}_{i_{j+k-n+1}}\\
&\quad \mathrm{antisym}(i_{j+1}\dots i_n)\ \mathrm{sym}(Q_1p_1)\ \dots\ \mathrm{sym}(Q_{k-n+j}p_{k-n+j}),\\
&\quad j\le 2n-k-1,\\
&A^{I_1\dots I_{k-n}I_{k-n+1}\dots I_{k-n+j}}{}_{i_{j+1}\dots i_n}=0,\quad j>2n-k-1,
\end{aligned}\tag{3.12}$$

where $I_1=Q_1p_1,\dots,I_{k-n+j}=Q_{k-n+j}p_{k-n+j}$, *and* $C^{Q_1\dots Q_{k-n+1}Q_{k-n+2}\dots I_{k-n+j}}{}_{i_{j+k-n+2}\dots i_n},\dots,\ C^{I_1\dots I_{n-1}Q_n\dots Q_{k-n+j}}{}_{i_{j+k-n+2}\dots i_n}$ *are arbitrary constants antisymmetric in the subscripts and symmetric in the superscripts entering every of the multiindices* $Q_1,\dots,Q_{k-n+j}$, $I_1,\dots,I_{k-n+j}$.

Let $\rho\in\Omega^K_rW$ be a form, and consider the pull-back $\pi^*_{r+1,r}\rho$. We have an invariant decomposition of $\pi^*_{r+1,r}\rho$ into the summands $p_i\rho$ (2.6). In the following theorem we derive some identities for the chart expressions of the coefficients in $p_i\rho$.

Theorem 1. *Let* $k\ge 1$, *and let* $\rho\in\Omega^k_rW$ *be a form. Let for every* j, $0\le j\le k$, $p_j\rho$ *be expressed by*

$$\begin{aligned}
p_j\rho=\sum_{s=0}^{j}\frac{1}{s!(j-s)!(k-j)!}&B^{J_1\ \ J_sI_{s+1}\ \ I_j}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_j i_{j+1}\dots i_k}\omega^{\sigma_1}_{J_1}\wedge\dots\wedge\omega^{\sigma_s}_{J_s}\\
&\wedge\omega^{\sigma_{s+1}}_{I_{s+1}}\wedge\dots\wedge\omega^{\sigma_j}_{I_j}\wedge dx^{i_{j+1}}\wedge\dots\wedge dx^{i_k},
\end{aligned}\tag{3.13}$$

where $J_1,\dots,J_s$ *(resp.* $I_{s+1},\dots,I_j$*) are multiindices of length* $<r$ *(resp.* $=r$*). Denote for* $j<k$

$$\begin{aligned}
&C^{J_1\ \ J_sI_{s+1}\ \ I_{j-1}p_2\dots p_r}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_{j-1}\sigma_j\quad i_{j+2}\dots i_k}\\
&\quad=\frac{r(k-j)}{n+r-k+j}B^{J_1\ \ J_sI_{s+1}\ \ I_{j-1}pp_2\dots p_r}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_{j-1}\sigma_j\quad pi_{j+2}\dots i_k}.
\end{aligned}\tag{3.14}$$

Then the coefficients $B^{J_1\ \ J_sI_{s+1}\ \ I_j}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_j\sigma_{j+1}\dots i_k}$ *obey the following conditions:*

$$\frac{\partial B^{J_1\ \ J_sI_{s+1}\ \ I_k}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_k}}{\partial y^\sigma_L}=0,\quad 0\le s\le k,\quad |L|=r+1,$$

$$\begin{aligned}
&B^{J_1\ \ J_sI_{s+1}\ \ I_{j-1}I_j}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_{j-1}\sigma_j i_{j+1}\dots i_k}-C^{J_1\ \ J_sI_{s+1}\ \ I_{j-1}p_2\dots p_r}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_{j-1}\sigma_j\quad i_{j+2}\dots i_k}\delta^{p_1}_{i_{j+1}}\\
&\quad=\frac{r+1}{n+r-k+j}\frac{\partial B^{J_1\ \ J_sI_{s+1}\ \ I_{j-1}}_{\sigma_1\dots\sigma_s\sigma_{s+1}\dots\sigma_{j-1}qi_{j+1}\dots i_k}}{\partial y^{\sigma_j}_{I_jq}}\quad\mathrm{antisym}(i_{j+1}\dots i_k)\\
&\quad\mathrm{sym}(p_1\dots p_r),\ I_j=(p_1\dots p_r),\ 1\le j\le k,\ 0\le s\le j-1,
\end{aligned}$$

$$\frac{\partial B^{J_1 \quad J_s I_{s+1} \quad I_{j-1}}_{\sigma_1 \ldots \sigma_s \sigma_{s+1} \ldots \sigma_{j-1} i_j \ldots i_k}}{\partial y^{\sigma_j}_{I_j q}} - \frac{(r+1)(k-j+1)}{n+r-k+j} \frac{\partial B^{J_1 \quad J_s I_{s+1} \quad I_{j-1}}_{\sigma_1 \ldots \sigma_s \sigma_{s+1} \ldots \sigma_{j-1} q i_{j+1} \ldots i_k}}{\partial y^{\sigma_j}_{I_j q}} \delta^p_{i_j} = 0$$
$$\text{antisym}(i_j \ldots i_k) \ \ \text{sym}(I_j p), \quad 1 \le j \le k, \quad 0 \le s \le j-1. \tag{3.15}$$

The proof of Theorem 1 is based on analogous identities as (1.12), and on Lemma 1,(a).

Let ρ be a k-form on an open set $W \subset J^r Y$. Let $1 \le k \le n$. We say that ρ is *π-decomposable* if its horizontal (i.e. 0-contact) component $h\rho$ is $\pi_{r+1,r}$-projectable. If $1 \le k \le \dim J^r Y$ we say that ρ is *π-decomposable* if its $(k-n)$-contact component $p_{k-n}\rho$ is $\pi_{r+1,r}$-projectable.

If $1 \le k \le n$ and ρ is π-decomposable, then

$$\rho = \rho_1 + \rho_2, \tag{3.16}$$

where ρ_1 is a π_r-horizontal form, and ρ_2 is a contact form; in fact, ρ_1 is the $\pi_{r+1,r}$-projection of $h\rho$. Analogous decompositions arise for k-forms with $k \ge n+1$; in this case ρ_1 is the $(k-n)$-contact component of ρ, and ρ_2 is a form whose order of contactness is $\ge k-n+1$.

Our second result represents a criterion for a form to be π-decomposable.

Theorem 2. *Let $W \subset Y$ be an open set, $\rho \in \Omega^k_r W$ a form, and let (V, ψ), $\psi = (x^i, y^\sigma)$, be any fibered chart such that $V \subset W$.*

(a) *Let $k = 1$. Then ρ is π-decomposable if and only if*

$$\rho = \phi_i dx^i + \phi^J_\sigma \omega^\sigma_J \tag{3.17}$$

for some functions $\phi_i, \phi^J_\sigma : V_r \to R$, where $|J| \le r-1$.

(b) *Let $2 \le k \le n$. Then ρ is π-decomposable if and only if*

$$\rho = \phi_{i_1 \ldots i_k} dx^{i_1} \wedge \ldots \wedge dx^{i_k} + \omega^\sigma_J \wedge \phi^J_\sigma + d\omega^\sigma_I \wedge \Psi^I_\sigma, \tag{3.18}$$

where $\phi_{i_1 \ldots i_k}$ (resp. ϕ^J_σ, resp. Ψ^I_σ) are some functions (resp. $(k-1)$-forms, resp. $(k-2)$-forms) on V_r, and $|J| \le r-1$, $|I| = r-1$.

(c) *Let $n+1 \le k \le \dim J^r Y$. Then ρ is π-decomposable if and only if*

$$\rho = \phi_{i_1 \ldots i_k} \wedge dx^{i_1} \wedge \ldots \wedge dx^{i_n} + \omega^{\sigma_1}_{J_1} \wedge \ldots \wedge \omega^{\sigma_p}_{J_p} \wedge d\omega^{\nu_1}_{I_i} \wedge \ldots$$
$$\wedge\, d\omega^{\nu_q}_{I_q} \wedge \phi^{J_1 \quad J_p I_1 \quad I_q}_{\sigma_1 \ldots \sigma_p \nu_1 \ldots \nu_q}, \quad p + q = k - n + 1, \tag{3.19}$$

for some $(k-n)$-forms $\phi_{i_1 \ldots i_n}$ and $(k-p-2q)$-forms $\phi^{J_1 \quad J_p I_1 \quad I_q}_{\sigma_1 \ldots \sigma_p \nu_1 \ldots \nu_q}$ on V_r where $|J_1|, \ldots, |J_p| \le r-1$, $|I_1|, \ldots, |I_q| = r-1$.

The proof of (a) is straightforward.

To prove (b), suppose that $2 \le k \le n$. Then ρ has an expression

$$\rho = \rho_0 + A_{i_1 i_2 \ldots i_k} dx^{i_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k}$$
$$+ A^{I_1}_{\sigma_1 i_2 \ldots i_k} dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge \ldots \wedge dx^{i_k} + \ldots$$
$$+ A^{I_1 \quad I_{k-1}}_{\sigma_1 \ldots \sigma_{k-1} i_k} dy^{\sigma_1}_{I_1} \wedge \ldots \wedge dy^{\sigma_{k-1}}_{I_{k-1}} \wedge dx^{i_k} + A^{I_1 \quad I_k}_{\sigma_1 \ldots \sigma_k} dy^{\sigma_1}_{I_1} \wedge \ldots \wedge dy^{\sigma_k}_{I_k}, \tag{3.20}$$

where ρ_0 belongs to the ideal of forms generated by the 1-forms ω^σ_J, where $|J| \leq r-1$. Since

$$h\rho = \big(A_{i_1 i_2 \dots i_k} + A^{I_1}_{\sigma_1 \sigma_2 \dots i_k} y^{\sigma_1}_{I_1 i_1} + \dots + A^{I_1 \quad I_{k-1}}_{\sigma_1 \dots \sigma_{k-1} i_k} y^{\sigma_1}_{I_1 i_1} \dots y^{\sigma_{k-1}}_{I_{k-1} i_{k-1}} \\ + A^{I_1 \quad I_k}_{\sigma_1 \dots \sigma_k} y^{\sigma_1}_{I_1 i_1} \dots y^{\sigma_k}_{I_k i_k}\big) dx^{i_1} \wedge \dots \wedge dx^{i_k}, \tag{3.21}$$

the form $h\rho$ is $\pi_{r+1,r}$-projectable if and only if

$$A^{I_1 \quad I_j}_{\sigma_1 \dots \sigma_j i_{j+1} \dots i_k} \delta^{p_1}_{i_1} \dots \delta^{p_j}_{i_j} = 0 \quad \operatorname{antisym}(i_1 \dots i_j i_{j+1} \dots i_k) \\ \operatorname{sym}(I_1 p_1) \dots \operatorname{sym}(I_j p_j), \quad 1 \leq j \leq k. \tag{3.22}$$

If $j = k$, this implies

$$A^{I_1 \quad I_k}_{\sigma_1 \dots \sigma_k} = 0. \tag{3.23}$$

If $1 \leq j \leq k-1$, we apply Lemma 2, (3.10).

To prove (c), first consider the case $k = n+1$, and suppose that we have an $(n+1)$-form ρ such that $p_1\rho$ is $\pi_{r+1,r}$-projectable. ρ has a unique decomposition in the form

$$\rho = \rho_0 + \rho' + \rho'', \tag{3.24}$$

where ρ_0 is a multiple of ω_0, ρ' contains at most $n-1$ factors dx^i and is generated by the forms ω^ν_J, $|J| \leq r-1$, and ρ'' contains at most $n-1$ factors dx^i and none of the factors ω^ν_J, $|J| \leq r-1$. We wish to show that the $\pi_{r+1,r}$-projectability of $p_1\rho$ implies that both ρ' and ρ'' have the desired structure (3.19).

Notice that ρ' and ρ'' are of the form

$$\rho' = \omega^\nu_J \wedge \phi^J_\nu, \quad 0 \leq |J| \leq r-1, \tag{3.25}$$

$$\rho'' = A^{JI_1}_{\nu\sigma_1 i_2 \dots i_n} dy^\nu_J \wedge dy^{\sigma_1}_{I_1} \wedge dx^{i_2} \wedge \dots \wedge dx^{i_n} \\ + A^{JI_1 I_2}_{\nu\sigma_1\sigma_2 i_3 \dots i_n} dy^\nu_J \wedge dy^{\sigma_1}_{I_1} \wedge dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} + \dots \\ + A^{JI_1 \quad I_{n-1}}_{\nu\sigma_1 \dots \sigma_{n-1} i_n} dy^\nu_J \wedge dy^{\sigma_1}_{I_1} \wedge \dots \wedge dy^{\sigma_{n-1}}_{I_{n-1}} \wedge dx^{i_n} \\ + A^{JI_1 \quad I_n}_{\nu\sigma_1 \dots \sigma_n} dy^\nu_J \wedge dy^{\sigma_1}_{I_1} \wedge \dots \wedge dy^{\sigma_n}_{I_n}, \quad |J|, |I_1|, \dots, |I_n| = r. \tag{3.26}$$

We apply (b) to the form $h\phi^J_\nu$, and Lemma 2(a) to $p_1\rho''$, and get the required result after several steps.

If $k \geq n+1$, we proceed in the same way.

Remark 1. Taking $\phi_i = 0, \phi_{i_1 \dots i_k} = 0$, and $\phi_{i_1 \dots i_n} = 0$ in Theorem 2 we obtain necessary and sufficient conditions for a k-form ρ to be contact, resp. strongly contact.

4. Order of a lagrangian

Usually, a lagrangian $\lambda \in \Omega^n_{r,X} W$ is said to be of order r, if it is not $\pi_{r,r-1}$-projectable. In this paper we adopt a little different terminology. We shall say that $\lambda \in \Omega^n_{r,X} W$ is of *order* r, if there is no form $\rho \in \Omega^n_{r-1} W$ for which $h\rho = \lambda$ or, which is the same, if $h\rho \neq \lambda$ for every $\rho \in \Omega^n_{r-1} W$. If λ is not of order r we call the *order* of λ the smallest integer s such that there exists a form $\rho \in \Omega^n_s W$ for which $\lambda = h\rho$.

Clearly, if $\lambda \in \Omega^n_{r,X}W$ is $\pi_{r,s}$-projectable, and is not $\pi_{r,s-1}$-projectable, then the order of λ is s or $s-1$; the latter case takes place if and only if $\lambda = h\rho$ with $\rho \in \Omega^n_{s-1}W$ (up to $\pi^*_{r,s}$).

For example, the scalar curvature lagrangian (the Hilbert lagrangian for the Einstein equations in vacuo), is of the 1st order (see [9]).

Our problem to be considered in this section is to find the order of a lagrangian, that is, to solve the equation

$$\lambda = h\rho \tag{4.1}$$

with respect to ρ.

Let us consider the module of r-th order lagrangians $\Omega^n_r W$. Our aim will be to describe the image of $\Omega^n_r W$ by the horizontalization h, i.e., the subspace $h\Omega^n_r W \subset \Omega^n_{r+1,X}W$. We have the canonical decomposition of h,

$$\begin{array}{ccc} \Omega^n_r W & \xrightarrow{\quad h \quad} & \Omega^n_{r+1,X}W \\ \downarrow & & \uparrow \\ \Omega^n_r W/\Theta^n_r W & \xrightarrow{\ g\ } & h\Omega^n_r W \end{array} \tag{4.2}$$

in which the right vertical arrow represents the canonical inclusion, and g is a bijection. We wish to construct the inverse mapping

$$\Lambda = g^{-1} \tag{4.3}$$

in terms of coordinates.

To this purpose we specify the result of Theorem 1 to the case $k = n$.

Let us consider a form $\rho \in \Omega^n_r W$, and a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, where $V \subset W$. ρ has a unique decomposition

$$\rho = \rho_0 + \rho', \tag{4.4}$$

where ρ_0 is a contact form, i.e., is generated by $\omega^\sigma_J, |J| \leq r-1$ and $d\omega^\sigma_I, |I| = r-1$ (see Sec. 3, Remark 1), and

$$\rho' = \sum_{p=0}^{n} \frac{1}{p!(n-p)!} A^{I_1 \;\; I_p}_{\sigma_1 \dots \sigma_p i_{p+1} \dots i_n} dy^{\sigma_1}_{I_1} \wedge \dots \wedge dy^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_n}, \tag{4.5}$$

where $|I_1|, \dots, |I_p| = r$ and all the coefficients $A^{I_1}_{\sigma_1, i_2 \dots \, i_n}, \dots, A^{I_1 \;\; I_{n-1}}_{\sigma_1 \dots \sigma_{n-1} i_n}$ are completely traceless; in particular, ρ' does not contain any non-zero term of the form $d\omega^\nu_J \wedge \chi^J_\nu$ with $|J| = r-1$. This decomposition may be referred to as the *canonical decomposition* of ρ relative to the fibered chart (V, π).

Clearly, ρ' may be viewed as a representative of the class $[\rho] \in \Omega^n_r W/\Theta^n_r W$ in the chart (V, ψ). We have

$$h\rho' = \Big(\sum_{p=0}^{n} \frac{1}{p!(n-p)!} A^{I_1 \;\; I_p}_{\sigma_1 \dots \sigma_p i_{p+1} \dots i_n} y^{\sigma_1}_{I_1 i_1} \dots y^{\sigma_p}_{I_p i_p} \Big) dx^{i_1} \wedge \dots \wedge dx^{i_n}. \tag{4.6}$$

If we write $h\rho'$ in the form $h\rho' = Bdx^1 \wedge \ldots \wedge dx^n$, or

$$h\rho' = \frac{1}{n!} B_{i_1 \ldots i_n} dx^{r_1} \wedge \ldots \wedge dx^{i_n}, \tag{4.7}$$

it is clear that one can reconstruct ρ' from B, or $B_{i_1 \ldots i_n}$, by a differentiation process (the same method has been used in [8], [12], [11]).

Consider the form (4.5) and write

$$\pi^*_{r+1,r}\rho' = \sum_{p=0}^{n} \frac{1}{p!(n-p)!} B^{I_1 \quad I_p}_{\sigma_1 \ldots \sigma_p i_{p+1} \ldots i_n} \omega^{\sigma_1}_{I_1} \wedge \ldots \wedge \omega^{\sigma_p}_{I_p} \wedge dx^{i_p+1} \wedge \ldots \wedge dx^{i_n}. \tag{4.8}$$

This form has a unique decomposition

$$\pi^*_{r+1,r}\rho' = \rho'_0 + \rho'', \tag{4.9}$$

where ρ'_0 is of the form $d\omega^\nu_J \wedge \chi^J_\nu, |J| = r-1$,

$$\rho'' = \sum_{p=0}^{n} \frac{1}{p!(n-p)'} C^{I_1 \quad I_p}_{\sigma_1 \ldots \sigma_p i_{p+1} \ldots i_n} \omega^{\sigma_1}_{I_1} \wedge \ldots \wedge \omega^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge \ldots \wedge dx^{i_n}, \tag{4.10}$$

and the coefficients $C^{I_1}_{\sigma_1 i_2 \ldots i_n}, C^{I_1 \quad I_{n-1}}_{\sigma_1 \ldots \sigma_{n-1} i_n}$ are completely traceless. If $\iota : V_r \to V_{r+1}$ is the section of $J^{r+1}Y$ described by the equations

$$y^\sigma_I \circ \iota = 0, \quad |I| = r+1, \tag{4.11}$$

then Theorem 1, (3.15) implies, since the coefficients in (4.6) are supposed to be traceless,

$$\iota^* \rho'_0 = 0 \tag{4.12}$$

hence

$$\rho' = \iota^* \rho''. \tag{4.13}$$

On the other hand, we have the following result.

Lemma 4. *If ρ' is expressed by (4.5) and ρ'' by (4.10), we have*

$$C^{I_1}_{\sigma_1 i_2 \ldots i_n} = \frac{\partial C_{q i_2 \ldots i_n}}{\partial y^{\sigma_1}_{I_1 q}},$$

$$C^{I_1 I_2}_{\sigma_1 \sigma_2 i_3 \ldots i_n} = \frac{r+1}{r+2} \frac{\partial C^{I_1}_{\sigma_1 q i_3 \ldots i_n}}{\partial y^{\sigma_2}_{I_2 q}},$$

$$\ldots$$

$$C^{I_1 \quad I_{n-1}}_{\sigma_1 \ldots \sigma_{n-1} i_n} = \frac{r+1}{r+n-1} \frac{C^{I_1 \quad I_{n-2}}_{\sigma_1 \ldots \sigma_{n-2} q i_n}}{\partial y^{\sigma_{n-1}}_{I_{n-1} q}},$$

$$C^{I_1 \quad I_n}_{\sigma_1 \ldots \sigma_n} = \frac{r+1}{r+n} \frac{C^{I_1 \quad I_{n-1}}_{\sigma_1 \ldots \sigma_{n-1}} q_{i_n}}{\partial y^{\sigma_n}_{I_n q}}, \tag{4.14}$$

and

$$\frac{C^{I_1 \quad I_n}_{\sigma_1 \dots \sigma_n}}{\partial y^\sigma_L} = 0, \quad |L| = r+1,$$

$$\frac{C^{I_1 \quad I_{j-1}}_{\sigma_1 \dots \sigma_{j-1} i_j \dots i_n}}{\partial y^{\sigma_j}_{I_j p}} - \frac{(r+1)(n-j+1)}{r+1} \frac{C^{I_1 \quad I_{j-1}}_{\sigma_1 \dots \sigma_{j-1} q i_{j+1} \dots i_n}}{\partial y^{\sigma_j}_{I_j q}} \delta^{\sigma_j}_{I_j q} = 0,$$

$$1 \leq j \leq n, \text{ antisym}(i_1 i_2 \dots i_n) \text{ sym}(I_j p). \tag{4.15}$$

Formulas (4.14) and (4.15) follow from (3.15).

In particular, ρ'' is completely determined by $h\rho'' = h\rho' = h\rho$, where

$$h\rho = \frac{1}{n!} C_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n}. \tag{4.16}$$

Using (4.14) we get for every $p, 1 \leq p \leq n$,

$$C^{I_1 \quad I_p}_{\sigma_1 \dots \sigma_p i_{p+1} \dots i_n} = \frac{(r+1)^p}{(r+p)\dots(r+2)(r+1)} \frac{\partial^p C_{q_1 \dots q_p i_{p+1} \dots i_n}}{\partial y^{\sigma_1}_{I_1 \, q_1} \dots \partial y^{\sigma_p}_{I_p \, q_p}}. \tag{4.17}$$

Now we shall extend this formula to *arbitrary* lagrangians belonging to $\Omega^n_{r+1,X} W$.

Let $\lambda \in \Omega^n_{r+1,X} W$ be a lagrangian. Write in a fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$,

$$\lambda = L dx^1 \wedge \dots \wedge dx^n = \frac{1}{n!} L \varepsilon_{i_1 \dots i_n} dx^{i_1} \wedge \dots \wedge dx^{i_n},$$
$$L_{i_1 \dots i_n} = L \varepsilon_{i_1 \dots i_n}, \tag{4.18}$$

and put

$$\Lambda_{(V,\psi)}(\lambda) = \sum_{p=0}^{n} \frac{1}{p!(n-p)!} C^{I_1 \quad I_p}_{\sigma_1 \dots \sigma_p i_{p+1} \dots i_n} \omega^{\sigma_1}_{I_1} \wedge \dots \wedge \omega^{\sigma_p}_{I_p} \wedge dx^{i_{p+1}} \wedge \dots \wedge dx^{i_n}, \tag{4.19}$$

where

$$C^{I_1 \quad I_p}_{\sigma_1 \dots \sigma_p i_{p+1} \dots i_n} = \frac{(r+1)^p}{(r+p)\dots(r+2)(r+1)} \frac{\partial^p L_{q_1 \dots q_p i_{p+1} \dots i_n}}{\partial y^{\sigma_1}_{I_1 q_1} \dots \partial y^{\sigma_p}_{I_p q_p}},$$
$$|I_1|, \dots, |I_p| = r. \tag{4.20}$$

Theorem 3. *Let $\lambda \in \Omega^n_{r+1,X} W$. The following two conditions are equivalent:*

(1) *There exists a form $\rho \in \Omega^n_r W$ such that $\lambda = h\rho$.*

(2) *For every fibered chart $(V, \psi), \psi = (x^i, y^\sigma)$, the form $\Lambda_{(V,\psi)}(\lambda)$ is $\pi_{r+1,r}$-projectable.*

Suppose that (1) is satisfied. Then we have a decomposition

$$\rho = \rho_0 + \Lambda_{(V,\psi)}(h\rho) = \rho_0 + \Lambda_{(V,\psi)}(\lambda) \tag{4.21}$$

in which ρ_0 is a contact form, and both ρ_0, $\Lambda_{(V,\psi)}(\lambda)$ are defined on $V_r \subset W$.

Conversely, suppose that (2) holds. Is (V_ι, ψ_ι), $\iota \in I$, is a system of fibered charts such that $V_\iota \subset W$ for every $\iota \in I$ and $\cup V_\iota = W$, and if (ζ_ι) is a partition of unity subordinate to the covering (V_ι) of W, we put

$$\rho = \sum \zeta_\iota \Lambda_{(V_\iota, \psi_\iota)}(\lambda). \tag{4.22}$$

Then $h\rho = \lambda$.

Recall that a lagrangian λ on a fibered manifold $\pi : Y \to X$ is called *variationally trivial*, if its Euler-Lagrange form $E_\lambda = E_n(\lambda)$ vanishes, i.e., $E_\lambda = 0$. Our goal now is to discuss the local and global structure of variationally trivial lagrangians.

Theorem 4. *Let $\lambda \in \Omega^n_{r,X}Y$ be a variationally trivial lagrangian on a fibered manifold $\pi : Y \to X$, $n = \dim X$. Then to each point $J^r_x\gamma \in J^rY$ there exist a neighborhood V of $\gamma(x) \in Y$ and an n-form ρ_V defined on $V_{r-1} \subset J^{r-1}Y$ such that (1) $\lambda = h\rho_V$ on V_r, and (2) $d\rho_V = 0$.*

To prove Theorem 4 we may suppose that $V = U \times W$, where $U \subset R^n$ is an open set and $W \subset R^m$ is an open ball with centre 0. Let x^i, y^σ, $1 \le i \le n$, $1 \le \sigma \le m$, be the canonical coordinate on V. We define a mapping $\chi : R \times V_r \to V_r$ by the formula

$$\chi(s, (x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1 \dots j_r})) = (x^i, y^\sigma, y^\sigma_{j_1}, \dots, y^\sigma_{j_1 \dots j_{r-1}}, s y^\sigma_{j_1 \dots j_r}). \tag{4.23}$$

Let η be any $(n-1)$-form on V_r such that $hd\eta$ is $\pi_{r+1,r}$-projectable. Write

$$\eta = \eta_0 + \eta_1, \tag{4.24}$$

where η_0 is generated by the forms $\omega^\sigma, \omega_{j_1}, \dots, \omega^\sigma_{j_1 \dots j_{r-1}}$ and

$$\begin{aligned} \eta_1 = {} & B_{i_2 \dots i_n} dx^{i_2} \wedge \dots \wedge dx^{i_n} + B^{I_2}_{\sigma_2 i_3 \dots i_n} dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n} \\ & + B^{I_2 I_3}_{\sigma_2 \sigma_3 i_4 \dots i_n} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_n} + \dots \\ & + B^{I_2 \quad I_{n-1}}_{\sigma_2 \dots \sigma_{n-1} i_n} dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_{n-1}}_{I_{n-1}} \wedge dx^{i_n} + B^{I_2 \quad I_n}_{\sigma_2 \dots \sigma_n} dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_n}_{I_n}. \end{aligned} \tag{4.25}$$

Clearly,

$$hd\eta = hd\eta_1. \tag{4.26}$$

In particular, $hd\eta_1$ is $\pi_{r+1,r}$-projectable. By a direct computation

$$\begin{aligned} hd\eta_1 = \Big[& d'_{i_1} B_{i_2 \dots i_n} + \Big(\frac{\partial B_{i_2 \dots i_n}}{\partial y^{\sigma_1}_{I_1}} - d'_{i_1} B^{I_1}_{\sigma_1 i_3 \dots i_n} \Big) \\ & + \Big(\frac{\partial B^{I_2}_{\sigma_2 i_3 \dots i_n}}{\partial y^{\sigma_1}_{I_1}} + d'_{i_3} B^{I_1 I_2}_{\sigma_1 \sigma_2 i_4 \dots i_n} \Big) y^{\sigma_1}_{I_1 i_1} y^{\sigma_2}_{I_2 i_2} + \dots \\ & + \Big(\frac{\partial B^{I_2 \quad I_{n-1}}_{\sigma_2 \dots \sigma_{n-1} i_n}}{\partial y^{\sigma_1}_{I_1}} + (-1)^{n-1} d'_{i_n} B^{I_1 \quad I_{n-1}}_{\sigma_1 \dots \sigma_{n-1}} \Big) \cdot y^{\sigma_1}_{I_1 i_1} \cdots y^{\sigma_{n-1}}_{I_{n-1} i_{n-1}} \\ & + \frac{\partial B^{I_2 \quad I_n}_{\sigma_2 \dots \sigma_n}}{\partial y^{\sigma_1}_{I_1}} y^{\sigma_1}_{I_1 i_1} \cdots y^{\sigma_n}_{I_n i_n} \Big] dx^{i_1} \wedge \dots \wedge dx^{i_n}. \end{aligned} \tag{4.27}$$

This formula enables us to express the conditions of $\pi_{r+1,r}$-projectability of $hd\eta_1$ explicitly.

Put

$$\chi_1 = C^{I_3 \ldots I_n}_{\sigma_3 \ldots \sigma_n} dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_n}_{I_n}, \tag{4.28}$$

where

$$C^{I_3 \ldots I_n}_{\sigma_j \ldots \sigma_n} = y^\nu_J \int B^{JI_3 \ldots I_n}_{\nu\sigma_3 \ldots \sigma_n} \circ \chi \cdot s^{n-2} ds \tag{4.29}$$

(integration from 0 to 1). Write $d\chi_1$ in the form

$$\begin{aligned} d\chi_1 = {} & d' C^{I_3 \ldots I_n}_{\sigma_3 \ldots \sigma_n} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_n}_{I_n} \\ & + \frac{\partial C^{I_3 \ldots I_n}_{\sigma_3 \ldots \sigma_n}}{\partial y^{\sigma_2}_{I_2}} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_n}_{I_n}, \end{aligned} \tag{4.30}$$

where

$$\begin{aligned} \frac{\partial C^{I_3 \ldots I_n}_{\sigma_3 \ldots \sigma_n}}{\partial y^{\sigma_2}_{I_2}} dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n} = {} & \Big(\int B^{I_2 \ldots I_n}_{\sigma_2 \ldots \sigma_n} \circ \chi \cdot s^{n-2} ds \\ & + y^\nu_J \int \frac{\partial B^{JI_3 \ldots I_n}_{\nu\sigma_3 \ldots \sigma_n}}{\partial y^{\sigma_2}_{I_2}} \circ \chi \cdot s^{n-1} ds \Big) dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n} \\ = {} & \Big[\int B^{I_2 \ldots I_n}_{\sigma_2 \ldots \sigma_n} \circ \chi \cdot s^{n-2} ds + \frac{1}{n-1} \int \Big(\Big(\frac{\partial B^{JI_3 \ldots I_n}_{\nu\sigma_3 \ldots \sigma_n}}{\partial y^{\sigma_2}_{I_2}} - \frac{\partial B^{JI_2 I_4 \ldots I_n}_{\nu\sigma_2\sigma_4 \ldots \sigma_n}}{\partial y^{\sigma_3}_{I_3}} - \ldots \\ & - \frac{\partial B^{JI_3 \ldots I_{n-1} I_2}_{\nu\sigma_3 \ldots \sigma_{n-1}\sigma_2}}{\partial y^{\sigma_n}_{I_n}} \Big) \cdot y^\nu_J \Big) \circ \chi \cdot s^{n-2} ds \Big] dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n}. \end{aligned} \tag{4.31}$$

By (4.27)

$$\frac{\partial B^{I_2 I_3 \ldots I_n}_{\sigma_2\sigma_3 \ldots \sigma_n}}{\partial y^\nu_J} = \frac{\partial B^{JI_3 \ldots I_n}_{\nu\sigma_3 \ldots \sigma_n}}{\partial y^{\sigma_2}_{I_2}} - \frac{\partial B^{JI_2 I_4 \ldots I_n}_{\nu\sigma_2\sigma_4 \ldots \sigma_n}}{\partial y^{\sigma_3}_{I_3}} - \ldots - \frac{\partial B^{JI_3 \ldots I_{n-1} I_2}_{\nu\sigma_3 \ldots \sigma_{n-1}\sigma_2}}{\partial y^{\sigma_n}_{I_n}}, \tag{4.32}$$

from which it follows that

$$\begin{aligned} \frac{\partial C^{I_3 \ldots I_n}_{\sigma_3 \ldots \sigma_n}}{\partial y^{\sigma_2}_{I_2}} dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n} = {} & \Big(\int B^{I_2 \ldots I_n}_{\sigma_2 \ldots \sigma_n} \circ \chi \cdot s^{n-2} ds \\ & + \frac{1}{n-1} \int \Big(\frac{\partial B^{I_2 \ldots I_n}_{\sigma_2 \ldots \sigma_n}}{\partial y^{\sigma_2}_{I_2}} y^\nu_J \Big) \circ \chi \cdot s^{n-2} ds \Big) dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n} \\ = {} & \int \frac{d}{ds} \Big(\sigma_2 \ldots {}^{I_2 I_n}_{\sigma_n} \circ \chi \cdot \frac{1}{n-1} s^{n-1} \Big) ds \cdot dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n} \\ = {} & B^{I_2 \ldots I_n}_{\sigma_2 \ldots \sigma_n} dy^{\sigma_2}_{I_2} \wedge \ldots \wedge dy^{\sigma_n}_{I_n}. \end{aligned} \tag{4.33}$$

Consequently

$$\eta_1 = \eta_2 + d\chi_1 - p\big(d' C^{I_3 \ldots I_n}_{\sigma_3 \ldots \sigma_n}\big) \wedge dy^{\sigma_3}_{I_3} \wedge \ldots \wedge dy^{\sigma_n}_{I_n}, \tag{4.34}$$

where

$$\eta_2 = B_{i_2\dots i_n} dx^{i_2} \wedge \dots \wedge dx^{i_n} + B^{I_2}_{\sigma_2 i_3 \dots i_n} dy^{\sigma_2}_{I_2} \wedge dx^{i_3} \wedge \dots \wedge dx^{i_n}$$
$$+ B^{I_2 I_3}_{\sigma_2 \sigma_3 i_4 \dots i_n} dy^{\sigma_2}_{I_2} \wedge dy^{\sigma_3}_{I_3} \wedge dx^{i_4} \wedge \dots \wedge dx^{i_n} + \dots$$
$$+ \left(B^{I_2 \quad I_{n-1}}_{\sigma_2 \dots \sigma_{n-1} i_n} - (-1)^{n-2} d'_{i_n} C^{I_2 \quad I_{n-1}}_{\sigma_2 \dots \sigma_{n-1} i_n}\right) dy^{\sigma_2}_{I_2} \wedge \dots \wedge dy^{\sigma_{n-1}}_{I_{n-1}} \wedge dx^{i_n}. \quad (4.35)$$

Clearly

$$hd\eta_1 = hd\eta_2, \quad (4.36)$$

Now we can use (4.27) again replacing η_1 by η_2. After several steps we conclude that η is expressible in the form

$$\eta = \nu + d\varphi + \psi, \quad (4.37)$$

where ν is π_r-horizontal,

$$\nu = \bar{B}_{i_2 \dots i_n} dx^{i_2} \wedge \dots \wedge dx^{i_n}, \quad (4.38)$$

and ψ is contact; the form $hd\eta = hd\nu$ is $\pi_{r+1,r}$-projectable which means that

$$\frac{\partial \bar{B}_{i_2 \dots i_n}}{\partial y^{\sigma_1}_{I_1}} = \frac{r(n-1)}{r+1} \delta^{j_1}_{i_2} \frac{\partial \bar{B}_{q i_3 \dots i_n}}{\partial y^{\sigma}_{q j_2 \dots j_r}}$$
$$\operatorname{antisym}(i_2 i_3 \dots i_n) \ \operatorname{sym}(j_1 j_2 \dots j_r); \quad I_1 = (j_1 j_2 \dots j_r). \quad (4.39)$$

Let $U \subset R^n$ be an open set, $W \subset R^m$ an open ball with centre 0, and denote by (x^i, y^σ) the canonical coordinates on $U \times W$. Put for every $(s, (x^i, y^\sigma)) \in R \times (U \times W)$

$$\chi(s, (x^i, y^\sigma)) = (x^i, s y^\sigma). \quad (4.40)$$

χ is a mapping from $R \times (U \times W)$ into $U \times W$. Let $k \geq 1$, and let ρ be a k-form on $U \times W$. There exists a unique decomposition.

$$\chi^* \rho = ds \wedge \rho_0(s) + \rho'(s), \quad (4.41)$$

where the form $\rho_0(s), \rho'(s)$ do not contain ds. We set

$$I\rho = \int \rho_0(s), \quad (4.42)$$

where the right-hand side means that we integrate the coefficients in the form $\rho_0(s)$ over s from 0 to 1. Let $\pi : U \times W \to U$ be the canonical projection, and let $\iota : U \to U \times W$ be the zero section. Then it is standart to show that

$$\rho = Id\rho + dI\rho + \pi^* \iota^* \rho. \quad (4.43)$$

In what follows we apply this formula to the case of the jet projection $\pi_{r,r-1} = V_r \to V_{r-1}$, where $V = U \times W$.

Let λ be a variationally trivial lagrangian on V_r. Using the variational sequence (2.25) we get $\pi^*_{r+1,r} \lambda = E_{n-1} h\nu = hd\nu$, where ν is an $(n-1)$-form on V_r. Since $hd\nu$ is

$\pi_{r+1,r}$-projectable we may suppose, by the first part of the proof, that ν is π_r-horizontal. Put

$$\rho = d\nu, \quad \eta = Id\nu, \tag{4.44}$$

where I is defined by (4.42) (with respect by the projection $\pi_{r,r-1} : V_r \to V_{r-1}$. Is $\iota : V_{r-1} \to V_r$). By (4.43)

$$\rho = d\eta + \pi^*_{r,r-1}\iota^*\rho. \tag{4.45}$$

Clearly,

$$\iota^* d\eta = d\iota^*\eta = 0, \tag{4.46}$$

and it is easily verified with the help of (4.41) and (4.42) that η is π_r-horizontal.

Let $\Omega \subset U$ be any compact, n-dimensional submanifold with boundary $\partial\Omega$. Let us consider a section $w : V_{r-1} \to V_r$ satisfying

$$supp\ w \subset V_{r-1} \cap \pi^{-1}_{r-1}(\Omega). \tag{4.47}$$

If $\gamma : U \to V$ is a section and $x \in \partial\Omega$, then

$$(w \circ J^{r-1}\gamma)(x) = (w \circ J^{r-1}\gamma)(x). \tag{4.48}$$

Now we integrate both sides of the equality

$$w^*\rho = dw^*\eta + \iota^*\rho \tag{4.49}$$

arising from (4.45). We obtain by (4.48) and (4.46)

$$\begin{aligned} \int_\Omega J^{r-1}\gamma^* dw^*\eta &= \int_{\partial\Omega} J^{r-1}\gamma^* w^*\eta = \int_{\partial\Omega} (w \circ J^{r_1}\gamma)^*\eta \\ &= \int_{\partial\Omega} (\iota \circ J^{r_1}\gamma)^*\eta = \int_{\partial\Omega} J^{r-1}\gamma^* i^*\eta = \int_\Omega J^{r-1}\gamma^* di^*\eta = 0 \end{aligned} \tag{4.50}$$

so that

$$\int_\Omega J^{r-1}\gamma^*(w^*\rho - \iota^*\rho) = 0. \tag{4.51}$$

Since this condition holds for every γ we get on Ω

$$hw^*\rho = hi^*\rho \tag{4.52}$$

and (4.49) gives

$$hdw^*\eta = 0. \tag{4.53}$$

Writing

$$\eta = B_{i_2 \ldots i_n} dx^{i_2} \wedge \ldots \wedge dx^{i_n} \tag{4.54}$$

we get

$$hdw^*\eta = d_{i_1}(B_{i_2 \ldots i_n} \circ w) dx^{i_1} \wedge \ldots \wedge dx^{i_n} = 0, \tag{4.55}$$

i.e.,

$$\begin{aligned}
&\frac{\partial B_{i_2\dots i_n}\circ w}{\partial x^{i_1}} + \frac{\partial B_{i_2\dots i_n}\circ w}{\partial y^\sigma} y^\sigma_{i_1} + \dots + \frac{\partial B_{i_2\dots i_n}\circ w}{\partial y^\sigma_{q_1\dots q_{r-1}i_1}} y^\sigma_{q_1\dots q_{r-1}i_1}\\
&\quad = \frac{\partial B_{i_2\dots i_n}}{\partial x^{i_1}}\circ w + \frac{\partial B_{i_2\dots i_n}}{\partial y^\nu_{j_1\dots j_r}}\circ w\cdot \frac{\partial y^\nu_{j_1\dots j_r}\circ w}{\partial x^{i_1}}\\
&\qquad + \Big(\frac{\partial B_{i_2\dots i_n}}{\partial y^\sigma}\circ w + \frac{\partial B_{i_2\dots i_n}}{\partial y^\nu_{j_1\dots j_r}}\circ w\cdot \frac{\partial y^\nu_{j_1\dots j_r}\circ w}{\partial y^\sigma}\Big) y^\sigma_{i_1} + \dots\\
&\qquad + \Big(\frac{\partial B_{i_2\dots i_n}}{\partial y^\sigma_{q_1\dots q_{r-1}}}\circ w + \frac{\partial B_{i_2\dots i_n}}{\partial y^\nu_{j_1\dots j_r}}\circ w\cdot \frac{\partial y^\nu_{j_1\dots j_r}\circ w}{\partial y^\sigma_{q_1\dots q_{r_1}}}\Big) y^\sigma_{q_1\dots q_{r-1}i_1}\\
&\quad = \frac{\partial B_{i_2\dots i_n}}{\partial x^{i_1}}\circ w + \frac{\partial B_{i_2\dots i_n}}{\partial y^\sigma}\circ w\cdot y^\nu_{i_1} + \dots\\
&\qquad + \frac{\partial B_{i_2\dots i_n}}{\partial y^\sigma_{q_1\dots q_{r-1}}}\circ w\cdot y^\sigma_{q_1\dots q_{r-1}i_1} + \frac{\partial B_{i_2\dots i_n}}{\partial y^\sigma_{j_1\dots j_r}}\circ w\cdot d_{i_1}(y^\nu_{j_1\dots j_r}\circ w)\\
&\quad = 0.
\end{aligned}\tag{4.56}$$

Since the functions $y^\nu_{j_1\dots j_r}\circ w$ are arbitrary, we have

$$\frac{\partial B_{i_2\dots i_n}}{\partial y^\nu_{j_1\dots j_r}}\circ w = 0,\tag{4.57}$$

and

$$\frac{\partial B_{i_2\dots i_n}}{\partial y^\nu_{j_1\dots j_r}} = 0.\tag{4.58}$$

In particular, η is defined on V_{r-1}, and so must be ρ (4.43) as required by Theorem 4.

Using the variational sequence it is easy to obtain the following slightly different version of Theorem 4 refining the results known so far on the kernel of the higher order Euler-Lagrange mapping (see [1]).

Theorem 5. *Let $\lambda \in \Omega^n_{r,X}Y$ be a lagrangian on a fibered manifold $\pi : Y \to X$. λ is variationally trivial if and only if there exists an n-form ρ on $J^{r-1}Y$ such that (1) $\lambda = h\rho$, (2) $d\rho \in \Theta^{n+1}_{r-1}Y$.*

Only necessity of conditions (1), (2) needs proof. We have an exact sequence

$$0 \to \Theta^n_{r-1} \to \Omega^n_{r-1} \to \Omega^n_{r-1}/\Theta^n_{r-1} \to 0\tag{4.59}$$

in which $\Theta^n_{r-1}, \Omega^n_{r-1}$ are soft sheaves. Hence $\Omega^n_{r-1}/\Theta^n_{r-1}$ is also soft, and the induced sequence of global sections must be exact. But by Theorem 4 a variationally trivial lagrangian $\lambda \in \Omega^n_{r,X}Y$ belongs to $\Omega^n_{r-1}/\Theta^n_{r-1}$. This ensures the existence of a form ρ on $J^{r-1}Y$ such that $h\rho = \lambda$. Then obviously $d\rho \in \Theta^{n+1}_{r-1}$.

Corollary 1. *A lagrangian $\lambda \in \Omega^n_{1,X}Y$ is variationally trivial if and only if there exists an n-form ρ on Y such that (1) $\lambda = h\rho$, (2) $d\rho = 0$.*

By Theorem 4, if $E_n(\lambda) = 0$, then $\lambda = h\rho$ for an n-form on $J^0Y = Y$ such that $d\rho \in \Theta^{n+1}_0$. But $\Theta^{n+1}_0 = \{0\}$.

Remark 2. Condition (1) of Theorem 5 completely describes the *polynomial structure* of variationally trivial lagrangians in the variables $y^\sigma_{j_1\dots j_r}$; the polynomial structure is therefore defined by the horizontalization h. Condition (2) of Theorem 4 then determines relations among the coefficients of the polynomials.

In the following theorem we summarize our result on the structure of variationally trivial lagrangians; in its second part we use a global hypothesis and derive global results, based on the variational sequence (2.25). We denote by H^nY the n-th cohomology group of Y.

Theorem 6. *Let $\lambda \in \Omega^n_{r,X}Y$ be a lagrangian, $n = \dim X$. λ is variationally trivial if and only if to each point $J^r_x\gamma \in J^rY$ there exists a neighborhood V of $\gamma(x) \in Y$ and an $(n-1)$-form η_V defined on $V_{r-1} \subset J^{r-1}Y$ such that on V_r,*

$$\lambda = hd\eta_V. \tag{4.60}$$

(b) *Suppose that λ is variationally trivial and moreover, $H^nY = \{0\}$. Then there exists an $(n-1)$-form η defined on $J^{r-1}Y$ such that*

$$\lambda = hd\eta. \tag{4.61}$$

To prove (a) we use the identity $E_{n-1}h\eta_V = hd\eta_V$, where $\lambda = E_{n-1}h\eta_V$.

Suppose that λ is variationally trivial and $H^nY = \{0\}$. Then $\lambda = E_{n-1}\tau$, where τ is a global section of $\Omega^{n-1}_{r-1}/\Theta^{n-1}_{r-1}$. Since the induced sequence of global section associated to the exact sequence

$$0 \to \Theta^n_{r-1} \to \Omega^n_{r-1} \to \Omega^n_{r-1}/\Theta^n_{r-1} \to 0 \tag{4.62}$$

is exact, $\tau = h\eta$ for some global section η of Ω^n_{r-1}. Then $\lambda = E_{n-1}h\eta = hd\eta$.

If a lagrangian λ is variationally trivial and there exists a form η such that $\lambda = hd\eta$, we say that λ is *globally variationally trivial.* The condition $H^nY = \{0\}$ guarantees that every variationally trivial lagrangian $\lambda \in \Omega^n_{r,X}Y$ is globally variational, and is of the form $\lambda = hd\eta$, where $\eta \in \Omega^n_{r-1}Y$.

As mentioned in Introduction, a complete characterization of the so called generalized invariant transformations of a lagrangian λ requires the knowledge of the kernel of the Euler-Lagrange mapping. The definition of a generalized invariant transformation of λ, saying that the transformed lagrangian should belong to the kernel of the Euler-Lagrange mapping, can then effectively be applied.

Having now a proper theorem on the structure of the variationally trivial lagrangians (Theorem 6), we are in a position to give a precise description of generators of the generalized invariants transformations, quite in the fashion of the first order case [10]. Recall that a π-projectable vector field ξ on Y is said to generate *generalized invariant transformations* of a lagrangian $\lambda \in \Omega^n_{r,X}Y$, if its one-parameter group preserves the Euler-Lagrange form $E_\lambda = E_n(\lambda)$ i.e.,

$$\partial_{J^{2r}\xi}E_\lambda = 0, \tag{4.63}$$

where $J^{2r}\xi$ is the $2r$-jet prolongation of ξ. The following result gives a complete description of the generators of generalized invariants transformations.

Theorem 7. *Let $\lambda \in \Omega^n_{r,X}Y$ be a lagrangian, and let ξ be a π-projectable vector field. The following three conditions are equivalent:*

(1) ξ generates generalized invariants transformations of λ.

(2) There exists an n-form ρ on $J^{r-1}Y$ such that

$$\partial_{J^r\xi}\lambda = h\rho, \quad d\rho \in \Theta^{n+1}_{r-1}Y. \tag{4.64}$$

(3) For every fibered chart (V,ψ) on $Y, d\Lambda_{(V,\psi)}\partial_{J^r\xi}\lambda$ is $\pi_{r,r-1}$-projectable, and

$$d\Lambda_{(V,\psi)}\partial_{J^r\xi}\lambda \in \Theta^{n+1}_{r-1}V. \tag{4.65}$$

Suppose that ξ generates generalized invariants transformations of λ. Then

$$\partial_{J^{2r}\xi}E_\lambda = E_{\partial_{J^{2r}\xi}\lambda} = 0, \tag{4.66}$$

and we apply Theorem 6.

Suppose that (2) is satisfied, and choose a fibered chart (V,ψ) on Y. Write

$$\rho = \rho_0 + \Lambda_{(V,\psi)}h\rho = \rho_0 + \Lambda_{(V,\psi)}\partial_{J^r\xi}\lambda, \tag{4.67}$$

where ρ_0 is contact. Since by hypothesis, $d\rho \in \Theta^{n+1}_{r-1}Y$, and obviously $d\rho_0 \in \Theta^{n+1}_{r-1}V$, we have

$$d\Lambda_{(V,\psi)}\partial_{J^r\xi}\lambda \in \Theta^{n+1}_{r-1}V \tag{4.68}$$

as required.

Suppose that (3) holds, and consider the lagrangian

$$h\Lambda_{(V,\psi)}\partial_{J^r\xi}\lambda = \partial_{J^r\xi}\lambda. \tag{4.69}$$

The Euler-Lagrange form $E_n(\partial_{J^r\xi}\lambda)$ is the class of the form $d\Lambda_{(V,\psi)}\partial_{J^r\xi}\lambda$. By hypothesis this class is equal to 0.

Equation (4.65) replaces the *generalized Noether-Bessel-Hagen equation* (1.6) on a higher order jet space J^rY. If $r = 1$ it reduces to (1.6).

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D. Krupka
Department of Mathematics
Silesian University Opava
Bezručovo nám. 17
746 01 Opava
Czech Republic

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