Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 461–472

NATURAL OPERATORS RELATED WITH THE VARIATIONAL CALCULUS

Ivan Kolář

Abstract. The basic operators of the variational calculus in fibered manifolds are discussed from the point of view of the theory of general natural bundles and operators. This approach yields a clarification of the non-uniqueness phenomena related with the higher order Poincaré-Cartan morphisms in the case of several independent variables.

Keywords. Natural operator, Euler equation, Poincaré-Cartan morphisms.
MS Classification. 58A20, 58E30

Let λ be an r-th order Lagrangian on a fibered manifold with an m-dimensional base. The main purpose of the paper is to clarify some non-uniqueness phenomena related with the Poincaré-Cartan forms of λ in the case $m \geqslant 2$ and $r \geqslant 3$ by using the theory of general natural bundles and operators, [4]. In Section 1 we introduce the Poincaré-Cartan morphisms of λ by means of a variational decomposition formula. Such a morphism always exists globally, but it is unique if and only if $r = 1$ (with arbitrary m) or $m = 1$ (with arbitrary r). In the case $m \geqslant 2$ and $r = 2$ we define a distinguished Poincaré-Cartan morphism called quasisymmetric in Section 6. Then we determine all natural operators of the type of Poincaré-Cartan morphisms. Proposition 4 reads that for $m \geqslant 2$ and $r \geqslant 3$ the only natural operator of such a type is the zero operator. This implies that one has to use an additional structure if one wants to distinguish one Poincaré-Cartan morphism in the case $m \geqslant 2$ and $r \geqslant 3$. (In [3] we have selected a linear connection on the base manifold.) In Sections 2–5 we determine all natural operators of the type of the Euler morphism (except the case $m = 1$ and $r \geqslant 2$). This problem is technically easier and we develop some procedures useful in the main part of the paper. Moreover, the results of Section 2 testify once again that the Euler morphism is deeply related with the Lagrangian in a purely geometric way. - All manifolds and maps are assumed to be infinitely differentiable and all morphisms of fibered manifolds are base-preserving.

1. The decomposition formula. Consider a fibered manifold $\pi : Y \to M$, $\dim M = m$, $\dim Y = m + n$, and denote by J^rY its r-th jet prolongation. An r-th order Lagrangian on Y is a morphism

$$\lambda : J^rY \to \Lambda^m T^*M.$$

If $x^i, y^p, i, j, \ldots = 1, \ldots, m,\ p, q, \ldots = 1, \ldots n$, are some local fiber coordinates on Y, the induced coordinates on J^rY are x^i, y^p_α for all multiindices $|\alpha| \leqslant r$. Since we have

This paper is in final form and no version of it will be submitted for publication elsewhere.

to discuss several problems of tensorial character, we shall also write $y^p_\alpha = y^p_{i_1\dots i_k}$ with the right-hand side symmetric in all subscrits. The coordinate expression of λ is

$$\lambda \equiv L_{i_1\dots i_m} dx^{i_1} \wedge \dots \wedge dx^{i_m} =: L_\mu dx^\mu \tag{1}$$

where μ will be a shorthand for the polyindex $i_1, \dots, i_m$. The only essential component of λ is $L_{12\dots m}$, so that (1) is usually written as $Ldx^1 \wedge \dots \wedge dx^m$. However, for investigating the naturality problems it will be useful to have indicated explicitely the tensorial character of λ.

For every function $F : J^rY \to \mathbb{R}$, we set

$$F_i = \partial F/\partial x^i, \qquad F_p^{i_1\dots i_k} = \frac{\alpha\,!}{k\,!}\,\partial F/\partial y^p_\alpha$$

where $\alpha = (\alpha_1, \dots, \alpha_m)$ is the multiindex corresponding to $i_1 \dots i_k$ and $\alpha! = \alpha_1! \dots \alpha_m!$. Then we can write

$$dF = F_i dx^i + F_p dy^p + \dots + F_p^{i_1\dots i_r} dy^p_{i_1\dots i_r}.$$

For every morphism $\varphi : J^rY \to \Lambda^k T^*M$ one defines its formal exterior differential $D\varphi : J^{r+1}Y \to \Lambda^{k+1}T^*M$ by $(j^{r+1}s)^* D\varphi = d((j^r s)^*\varphi)$ for every local section s of Y. In particular, the coordinate expression of $DF : J^{r+1}Y \to T^*M$ is $(D_iF)\, dx^i$, where

$$D_iF = F_i + F_p y^p_i + \dots + F_p^{i_1\dots i_r} y^p_{i_1\dots i_r i} \tag{2}$$

is the formal derivative of F with respect to x^i.

Let $\eta : Y \to VY$ be a vertical vector field on Y. Its flow prolongation $\mathcal{J}^r\eta : J^rY \to VJ^rY = V(J^rY \to M)$ is defined by

$$\mathcal{J}^r\eta = \frac{\partial}{\partial t}\Big|_0 (J^r \exp t\eta).$$

If $\eta^p \partial/\partial y^p$ is the coordinate expression of η, then $\mathcal{J}^r\eta$ is of the form

$$\eta^p \frac{\partial}{\partial y^p} + D_i\eta^p \frac{\partial}{\partial y^p_i} + \dots + D_{i_1\dots i_r}\eta^p \frac{\partial}{\partial y^p_{i_1\dots i_r}}.$$

Let $\delta\lambda : VJ^rY \to \Lambda^m T^*M$ be the vertical differential of λ, the coordinate expression of which is

$$(L_{\mu p} dy^p + \dots + L_{\mu p}^{\;i_1\dots i_r} dy^p_{i_1\dots i_r}) \otimes dx^\mu.$$

Then the contraction $\langle \delta\lambda, \mathcal{J}^r\mu\rangle$ corresponds to the variation of λ with respect to η.

In [3] we have deduced that there exists a unique morphism $E : J^{2r}Y \to V^*Y \otimes \Lambda^m T^*M$ and a morphism $H : J^{2r-1}Y \to V^*J^{r-1}Y \otimes \Lambda^{m-1}T^*M$ such that

$$\langle \delta\lambda, \mathcal{J}^r\eta\rangle = D\langle H, \mathcal{J}^{r-1}\eta\rangle + \langle E, \eta\rangle \tag{3}$$

holds for every vertical vector field η on Y. If $r = 1$ (with arbitrary m) or $m = 1$ (with arbitrary r), then H is unique. We remark that even some other approaches yield that H is defined globally even though it is not unique in general, see e.g. [1], [8].

Definition 1. The morphism $E = E(\lambda)$ is called the Euler morphism of λ.

In coordinates, we have $E \equiv E_{p\mu} dy^p \otimes dx^\mu$ with

$$E_{p\mu} = L_{\mu p} - D_i L_{\mu p}^{\ i} + \cdots + (-1)^r D_{i_1 \dots i_r} L_{\mu p}^{\ i_1 \dots i_r}.$$

Definition 2. Every morphism H from (3) is called a Poincaré-Cartan morphism of λ.

In the case $r = 1$ we also write $H(\lambda)$ for the unique morphism H satisfying (3). Its coordinate expression is

$$H(\lambda) \equiv (L_{i\lambda p}^{\ i})\, dy^p \otimes dx^\lambda$$

where the subscript λ is a shorthand for $i_1, \dots, i_{m-1}$.

We remark that in [2] it is clarified that the Poincaré-Cartan morphisms correspond to the simpliest class of the Lepagean equivalents studied by D. Krupka, [5], [6].

2. Natural operators of the Euler type. Given two fibered manifolds Y and Z over the same base M, we denote by $C^\infty_M(Y, Z) \subset C^\infty(Y, Z)$ the set of all base-preserving morphisms of Y into Z. The construction of the Euler morphism of an r-th order Lagrangian on Y is an operator

$$E_Y : C^\infty_M(J^r Y, \Lambda^m T^* M) \to C^\infty_M(J^{2r} Y, V^* Y \otimes \Lambda^m T^* M).$$

By the uniqueness property from (3), E is a natural operator, i.e. it commutes with the action of local isomorphisms of fibered manifolds.

We are going to determine all operators

$$C^\infty_M(J^r Y, \Lambda^m T^* M) \to C^\infty_M(J^{2r} Y, V^* Y \otimes \Lambda^m T^* M) \tag{4}$$

natural on the category $\mathcal{FM}_{m,n}$ of all fibered manifolds with m-dimensional bases and n-dimensional fibers and their local isomorphisms. In Sections 4 and 5 we shall prove the following two propositions.

Proposition 1. For $m \geqslant 2$, all natural operators

$$C^\infty_M(J^1 Y, \Lambda^m T^* M) \to C^\infty_M(J^2 Y, V^* Y \otimes \Lambda^m T^* M)$$

are of the form kE, $k \in \mathbb{R}$. For $m = 1$, these operators form the 2-parameter family

$$k_1 E + k_2 W, \qquad k_1, k_2 \in \mathbb{R}$$

where the operator W is constructed below.

In general, we have an exact sequence of vector bundles over $J^1 Y$

$$O \to VY \otimes T^* M \to V J^1 Y \to VY \to O. \tag{5}$$

Restricting the vertical differential $\delta\lambda : V J^1 Y \to \Lambda^m T^* M$ to the kernel and dualizing, we obtain a map $\Lambda : J^1 Y \to V^* Y \otimes TM \otimes \Lambda^m T^* M$. In the case $m = 1$, we construct

$$\lambda \otimes \Lambda : J^1 Y \to V^* Y \otimes TM \otimes T^* M \otimes T^* M.$$

Applying the tensor contraction on the right-hand side (which is unique because of $\dim M = 1$), we obtain a natural operator $W : J^1 Y \to V^* Y \otimes \Lambda^1 T^* M$ with the coordinate expression

$$L_1 \frac{\partial L_1}{\partial y^p_1}\, dy^p \otimes dx^1.$$

Proposition 2. For $m \geqslant 2$, all natural operators

$$C^\infty_M(J^rY, \Lambda^m T^*M) \to C^\infty_M(J^{2r}Y, V^*Y \otimes \Lambda^m T^*M)$$

are of the form kE, $k \in \mathbb{R}$.

In the case $m = 1$ and $r \geqslant 2$, our methods suggest that there exists some additional natural operators, which are somewhat similar to the operator W. But a complete discussion of this case is beyond the scope of the present paper.

3. The reduction lemma. An s-th order natural operator of type (4) is determined by a morphism

$$J^s C^\infty_M(J^rY, \Lambda^m T^*M) \times_M J^{2r}Y \to V^*Y \otimes \Lambda^m T^*M.$$

Dealing with natural operators, it suffices to discuss the standard fibers of the bundles in question, [4].

Let $G^r_{m,n}$ be the group of all r-jets at $0 \in \mathbb{R}^{m+n}$ of the origin-preserving isomorphisms

$$\bar{x}^i = f^i(x), \qquad \bar{y}^p = f^p(x, y).$$

The inverse isomorphism will be denoted by

$$x^i = \tilde{f}^i(\bar{x}), \qquad y^p = \tilde{f}^p(\bar{x}, \bar{y}).$$

The projection to $\mathbb{R}^m$ and the restriction to $\{0\} \times \mathbb{R}^n$ determine a group homomorphism

$$G^r_{m,n} \to G^r_m \times G^r_n \tag{6}$$

where G^r_m or G^r_n is the r-th jet group in dimension m or n, respectively, [4].

Denote by $Y^r = (y^p_\alpha)$, $0 < |\alpha| \leqslant r$ the fiber of $J^r(\mathbb{R}^{m+n} \to \mathbb{R}^m)$ over $0 \in \mathbb{R}^{m+n}$. This is a $G^r_{m,n}$-space. Let L^s_r be the fiber of $J^s C^\infty_{\mathbb{R}^m}(J^r(\mathbb{R}^{m+n} \to \mathbb{R}^m), \Lambda^m T^*\mathbb{R}^m)$ over $0 \in \mathbb{R}^{m+n}$. From the transformation law of L_μ

$$\bar{L}_{i_1 \dots i_m} = L_{j_1 \dots j_m} \frac{\partial \tilde{f}^{j_1}}{\partial x^{i_1}} \dots \frac{\partial \tilde{f}^{j_m}}{\partial x^{i_m}} \tag{7}$$

we deduce that L^s_r is a $G^{r+s}_{m,n}$-space. The standard fibre of $V^*Y \otimes \Lambda^m T^*M$ is

$$Z = \mathbb{R}^{n*} \otimes \Lambda^m \mathbb{R}^{m*} \qquad\qquad Z = (z_{p\mu})$$

with the tensor action of $G^1_m \times G^1_n$. By the general theory, [4], the s-th order natural operators of type (4) are in bijection with the $G^q_{m,n}$-maps $L^s_r \times Y^{2r} \to Z$, $q = \max(r+s, 2r)$.

From the technical point of view, the most important step consists in replacing the partial derivatives (even iterated) $\partial/\partial x^i$ on L^s_r by the formal derivatives D_i. Analogously to the first order case, the coordinates on L^s_r are of the form

$$L_{\mu p_1}^{\;\;\;\; I_1} \dots {}^{I_k}_{p_k j_1 \dots j_l} \tag{8}$$

where each I is a sequence of length at most r (even empty) of the base indices and j indicates the partial derivative with respect to x^j. On the product $L^s_r \times Y^{r+s}$ we replace (8) by

$$D_{j_1} \dots D_{j_l} L_{\mu p_1}^{I_1} \dots {}_{p_k}^{I_k} =: \tilde{L}_{\mu p_1}^{I_1} \dots {}_{p_k j_1 \dots j_l}^{I_k}.$$

This introduces new coordinates on $L^s_r \times Y^{r+s}$ and together with the original coordinates y^p_α on Y^{r+s} defines a decomposition $L^s_r \times Y^{r+s} = \tilde{L}^s_r \times Y^{r+s}$ and a projection

$$\rho : L^s_r \times Y^{r+s} \to \tilde{L}^s_r.$$

Consider the subgroup $H^r_{m,n} \subset G^r_{m,n}$ of the r-jets of the following isomorphisms

$$\bar{x}^i = f^i(x), \quad \bar{y}^p = a^p_q y^q + f^p(x), \quad a^p_q \in \mathbb{R} \tag{9}$$

and write $\tilde{a}^p_q$ for the inverse matrix to a^p_q. Since the vertical differentials $d\bar{y}^p_\alpha$ do not depend on $f^p(x)$, the action of $H^{r+s}_{m,n}$ on $\tilde{L}^s_r$ factorizes through the canonical projection

$$p : H^{r+s}_{m,n} \to G^{r+s}_m \times G^1_n.$$

Hence $\tilde{L}^s_r$ is a $G^{r+s}_m \times G^1_n$-space and ρ is p-equivariant.

Lemma 1. *Let S be a $G^{r+s}_m \times G^1_n$-space. Then for every $H^{r+s}_{m,n}$-map $f : L^s_r \times Y^{r+s} \to S$ there exists a unique $G^{r+s}_m \times G^1_n$-map $g : \tilde{L}^s_r \to S$ satisfying $f = g \circ \rho$.*

Proof. The kernel K of p is formed by the jets of the isomorphisms

$$\bar{x}^i = x^i, \quad \bar{y}^p = y^p + f^p(x). \tag{10}$$

By Proposition 28.1 from [4] or by [7] it suffices to prove that each fiber $\rho^{-1}(x)$, $x \in \tilde{L}^r_s$ is a K-orbit. But $\rho^{-1}(x) = Y^{r+s}$ and the group (10) acts simply transitively on the sections of $\mathbb{R}^{m+n} \to \mathbb{R}^m$, QED.

4. Proof of Proposition 1. Consider first the second order natural operators $C^\infty_M(J^1Y, \Lambda^m T^*M) \to C^\infty_M(J^2Y, V^*Y \otimes \Lambda^m T^*M)$. They correspond to the equivariant maps $L^2_1 \times Y^2 \to Z$. Taking into account the jet projection $Y^3 \to Y^2$, we are going to determine all $G^3_{m,n}$-maps $L^2_1 \times Y^3 \to Z$. By Lemma 1, we can start with finding all $G^3_m \times G^1_n$-maps $f : \tilde{L}^2_1 \to Z$. In general, there is a canonical injection $i : G^1_m \to G^r_m$, [4]. On the subgroup $i(G^1_m) \times G^1_n$, all quantities from $\tilde{L}^2_1$ are tensorial of the types indicated by their indices. Consider the equivariance of f with respect to the fiber homotheties $\bar{y}^p = ky^p$. By the homogeneous function theorem, [4], f is independent of the quantities with two fiber subscripts, linear in the quantities $L_{\mu p}$, $L^{\ i}_{\mu p}$, $\tilde{L}_{\mu p i}$, $\tilde{L}^{\ \ j}_{\mu p i}$ with one fiber subscript and the coefficients are smooth functions in L_μ, $\tilde{L}_{\mu i}$, $\tilde{L}_{\mu ij}$. Considering the equivariance with respect to the base homotheties $\bar{x}^i = kx^i$, we have the following four cases.

I. The coefficients in the first part $z_{p\mu} = f^{\nu q}_{p\mu}(L_\mu, \tilde{L}_{\mu i}, \tilde{L}_{\mu ij})L_{\nu q}$ satisfy the homogeneity condition

$$0 = ma + (m+1)b + (m+2)c. \tag{11}$$

By the homogeneous function theorem, $f^{\nu q}_{p\mu}$ are constant.

II. The coefficients by $L_{\mu p}^{\ i}$ satisfy the homogeneity condition

$$1 = ma + (m+1)b + (m+2)c. \tag{12}$$

For $m \geqslant 2$, there are no non-negative integers satisfying (12), so that the coefficients vanish. For $m = 1$ we have an additional case $a = 1$, $b = 0$, $c = 0$, in which the coefficients are linear in L_μ and independent of $\tilde{L}_{\mu i}$ and $\tilde{L}_{\mu ij}$.

III. The coefficients by $\tilde{L}_{\mu p i}$ satisfy the homogeneity condition

$$-1 = ma + (m+1)b + (m+2)c$$

so that they are zero functions.

IV. The coefficients by $\tilde{L}_{\mu p i}^{\ \ \ j}$ are constant for they satisfy the homogeneity condition (11).

Assume $m \geqslant 2$ and apply the generalized invariant tensor theorem, [4]. In the case I we obtain $z_{p\mu} = k_1 L_{\mu p}$. In the case IV we have both $k_2 \tilde{L}_{\mu p i}^{\ \ \ i}$ and the antisymmetrization of another contraction

$$L_{j[\lambda|p|i]}^{\ \ \ \ \ \ \ j}. \tag{13}$$

The only essential component $\lambda = 12\dots m-1$, $i = m$ of (13) is

$$\tilde{L}_{j1\dots m-1pm}^{\ \ \ \ \ \ \ \ \ \ \ \ j} - \tilde{L}_{jm2\dots m-1p1}^{\ \ \ \ \ \ \ \ \ \ \ \ \ j} - \dots - \tilde{L}_{j1\dots mpm-1}^{\ \ \ \ \ \ \ \ \ \ \ \ j}.$$

The first term is non-vanishing for $j = m$, the second one for $j = 1$ and the last one for $j = m-1$. Hence (13) is a constant multiple of $\tilde{L}_{\mu p i}^{\ \ \ i}$. So we have found the following 2-parameter family

$$k_1 L_{\mu p} + k_2 \tilde{L}_{\mu p i}^{\ \ \ i}. \tag{14}$$

Consider now the equivariance of (14) with respect to the jets of the following isomorphisms

$$\bar{x}^i = x^i, \quad \bar{y}^p = f^p_q(x) y^q. \tag{15}$$

This implies easily $k_1 + k_2 = 0$. Then (14) corresponds to the constant multiples of the Euler operator.

The associated map of an s-th order natural operator A is $L^s_1 \times Y^2 \to Z$. Taking into account the jet projection $Y^{s+1} \to Y^2$, we determine all $G^{s+1}_{m,n}$-maps $L^s_1 \times Y^{s+1} \to Z$. By Lemma 1 we can look for all $G^{s+1}_m \times G^1_n$-maps $f : \tilde{L}^s_1 \to Z$. Using the homogeneity technique from the first part of the proof we find that A is a second order operator.

It remains to deduce that every natural operator of the type in question has finite order. One sees easily that every element of $\tilde{L}^s_1$ can be transformed into any neighbourhood of zero by a suitable combination of the fiber and base homotheties. Then the nonlinear Peetre theorem implies the finiteness of the order, [4], [12]. This completes the proof of Proposition 1.

5. Proof of Proposition 2. Consider first a natural operator of the order $r+1$. If we compose its associated map $L_r^{r+1}\times Y^{2r}\to Z$ with the jet projection $Y^{2r+1}\to Y^{2r}$, we obtain a $G_{m,n}^{2r+1}$-map $L_r^{r+1}\times Y^{2r+1}\to Z$. By Lemma 1 we can look for all $G_m^{2r+1}\times G_n^1$-maps $f:\tilde L_r^{r+1}\to Z$. Using the fiber homotheties, we find that f is independent of the quantities with at least two fiber subscripts, linear in the quantities with one fiber subscript and the coefficients are smooth functions in the quantities with no fiber subscript. Hence f is a linear combination of the terms

$$f^{\nu q j_1\dots j_l}_{p\mu i_1\dots i_k}(L_\mu,\tilde L_{\mu i},\dots,\tilde L_{\mu i_1\dots i_{r+1}})\tilde L_{\nu q j_1\dots j_l}^{\;\;i_1\dots i_k}.$$

Write $k=l+h$. The base homotheties yield the homogeneity condition

$$h=ma_0+(m+1)a_1+\dots+(m+r+1)a_{r+1}. \tag{16}$$

We have the following possibilities.

I. For $h<0$, the f's are zero functions.

II. For $h=0$, the f's are constant.

III. For $h=1$ and $m=1$, (16) is satisfied for $a_0=1$, $a_1=\dots=a_{r+1}=0$. But the case $m=1$ is excluded. If $m\geqslant 2$, the f's are zero functions.

IV. For $h=2$ and $m=2$, the only possibility is $a_0=1$, $a_1=\dots=a_{r+1}=0$. Hence we have a natural linear combination of

$$L_{k_1k_2}\tilde L_{k_3k_4pj_1\dots j_l}^{\;\;i_1\dots i_l i_{l+1}i_{l+2}}.$$

This expression is symmetric in i's and j's and antisymmetric in k_1, k_2 and k_3, k_4. So the only non-zero term can be obtained by contracting $i_1\dots i_l$ with $j_1\dots j_l$, i_{l+1} with k_1, i_{l+2} with k_4 and antisymmetrizing in k_2,k_3. The only essential component $k_2=1$, $k_3=2$ of the result is

$$L_{12}\tilde L_{12pi_1\dots i_l}^{\;\;i_1\dots i_l 12}-L_{21}\tilde L_{21pi_1\dots i_l}^{\;\;i_1\dots i_l 21}=0.$$

If $m\geq 3$, (16) implies directly that the f's are zero functions.

V. For $h\geqslant 3$ and $m\geqslant 2$ we find analogously to IV that the f's are zero functions.

Next we use the generalized invariant tensor theorem, [4]. In the same way as in Section 4 we find that $\tilde L_{i_1[\lambda|p|i]i_2\dots i_k}^{\;\;i_1\dots i_k}$ is a multiple of $\tilde L_{\mu p i_1\dots i_k}^{\;\;i_1\dots i_k}$. Hence f must be of the form

$$k_0L_{\mu p}+k_1\tilde L_{\mu p i}^{\;\;i}+\dots+k_r\tilde L_{\mu p i_1\dots i_r}^{\;\;i_1\dots i_r}. \tag{17}$$

Now we proceed by induction with respect to r. Formula (17) shows that our operator is projectable to a natural operator $C^\infty_M(J^{r-1}Y,\Lambda^mT^*M)\to C^\infty_M(J^{2r-2}Y,V^*Y\otimes\Lambda^mT^*M)$. The induction hypothesis implies $k_0+k_1=0,\dots,k_{r-2}+k_{r-1}=0$. On the subgroup (15) we find easily the following transformation laws

$$\bar L_{\mu p}^{\;\;i_1\dots i_r}=\tilde f_p^q L_{\mu q}^{\;\;i_1\dots i_r}\qquad f_q^p\tilde f_r^q=\delta_r^p\,,$$

$$\vdots$$

$$\bar L_{\mu p}^{\;\;i_1\dots i_k}=\tilde f_p^q L_{\mu q}^{\;\;i_1\dots i_k}+\cdots$$

where the dots denote a linear combination of

$$\frac{\partial^l \tilde f^q_p}{\partial \bar x^{j_1}\cdots\partial \bar x^{j_l}} L_{\mu r}^{\;i_1\dots i_{k+l}}, \qquad l = 1,\dots,r-k.$$

Comparing the coefficients by $\partial^r \tilde f^p_q/\partial x^{i_1}\dots\partial x^{i_r}$ in the equivariance condition for (17), we deduce that k_r is a linear combination of $k_0,\dots,k_{r-1}$, i.e. a multiple of k_0 by the previous relations. But the Euler morphism is natural, so that we must obtain the coordinate form of kE.

Having the associated map of an s-th order natural operator A, $s \geqslant r+1$, we find quite similarly to Section 4 that the order of A is $r+1$, so that we have the previous situation. In the same way as in Section 4 we deduce that every natural operator in question has finite order. This completes the proof of Proposition 2.

6. Natural operators of the Poincaré-Cartan type. There is a simple geometric way how to determine a unique Poincaré-Cartan morphism of a second order Lagrangian λ. In general, consider a map $F : N \to V^*J^1Y \otimes \Lambda^{m-1}T^*M$. The dual exact sequence to (5) yields a projection $q : V^*J^1Y \to V^*Y \otimes TM$. Then $(q \otimes id)\circ F : N \to V^*Y \otimes TM \otimes \Lambda^{m-1}T^*M$ and the tensor contraction defines a map $\tilde F : N \to V^*Y \otimes \Lambda^{m-2}T^*M$. If $\tilde F$ vanishes, then we say that F is quasisymmetric. In [3] we deduced that there exists a unique quasisymmetric morphism $Q(\lambda) : J^3Y \to V^*J^1Y \otimes \Lambda^{m-1}T^*M$ satisfying (3). Its coordinate expression is

$$\Big[(L_{i\lambda p}^{\;i} - D_j L_{i\lambda p}^{\;ij})\,dy^p + L_{i\lambda p}^{\;ij} dy^p_j\Big] \otimes dx^\lambda.$$

Hence Q is a natural operator $C^\infty_M(J^2Y, \Lambda^mT^*M) \to C^\infty_M(J^3Y, V^*J^1Y \otimes \Lambda^{m-1}T^*M)$. However, this construction does not work in any order $r \geqslant 3$, see also [10], [11]. We remark that another approach to Q can be found in [9].

We are going to determine all natural operators

$$C^\infty_M(J^rY, \Lambda^mT^*M) \to C^\infty_M(J^{2r-1}Y, V^*J^{r-1}Y \otimes \Lambda^{m-1}T^*M)$$

except $m = 1$ and $r \geqslant 2$. We underline that even though the motivation goes from the decomposition formula (3), there is no direct relation of our problem to (3).

Proposition 3. All natural operators $C^\infty_M(J^1Y, \Lambda^mT^*M) \to C^\infty_M(J^1Y, V^*Y \otimes \Lambda^{m-1} T^*M)$ are of the form kH, $k \in \mathbb{R}$. For $m \geqslant 2$, all natural operators $C^\infty_M(J^2Y, \Lambda^mT^*M) \to C^\infty_M(J^3Y, V^*J^1Y \otimes \Lambda^{m-1}T^*M)$ are of the form kQ, $k \in \mathbb{R}$.

Proposition 4. For $m \geqslant 2$ and $r \geqslant 3$, the only natural operator $C^\infty_M(J^rY, \Lambda^mT^*M) \to C^\infty_M(J^{2r-1}Y, V^*J^{r-1}Y \otimes \Lambda^{m-1}T^*M)$ is the zero operator.

Every element of the standard fiber Z^{r-1} of $V^*J^{r-1}Y \otimes \Lambda^{m-1}T^*M$ can be written as

$$(z_{p\lambda}dy^p + z^i_{p\lambda}dy^p_i + \dots + z^{i_1\dots i_{r-1}}_{p\lambda} dy^p_{i_1\dots i_{r-1}}) \otimes dx^\lambda.$$

In this sense we have

$$Z^{r-1} = Z_0 \times Z_1 \times \dots \times Z_{r-1}$$

with $Z_i = \mathbb{R}^{n*} \otimes S^i\mathbb{R}^{m*} \otimes \Lambda^{m-1}R^{m*}$, provided S^i indicates the i-th symmetric tensor power.

7. Proof of Proposition 3. Consider first the case $r = 1$ and start with the first order operators. The space $Z_0 = \mathbb{R}^{n*} \otimes \Lambda^{m-1}\mathbb{R}^{m*}$ is endowed with the tensorial action of $G^1_m \times G^1_n$. According to the general theory, [4], the operators in question are in bijection with the $G^2_{m,n}$-maps $L^1_1 \times Y^1 \to Z_0$. If we add the jet projection $Y^2 \to Y^1$, we can look for a $G^2_m \times G^1_n$-map $f : \tilde{L}^1_1 \to Z_0$ according to Lemma 1.

Using the fiber homotheties, we obtain f in the form

$$z_{p\lambda} = f^{\nu q}_{p\lambda}(L_\mu, \tilde{L}_{\mu i})L_{\nu q} + f^{\nu q}_{p\lambda i}(L_\mu, \tilde{L}_{\mu i})L_{\nu q}{}^{i}.$$

The base homotheties yield the homogeneity condition $-1 = ma + (m+1)b$ for the first term, so that it vanishes. For the second term we obtain $0 = ma + (m+1)b$, so that the coefficients are constant. Then the generalized invariant tensor theorem yields $z_{p\lambda} = kL_{i\lambda p}{}^{i}$. This corresponds to a constant multiple of H.

For an s-th order natural operator A we deduce quite similarly to Section 4 that its associated map factorizes through $\tilde{L}^s_1 \to Z_0$. Then the homogeneity technique implies A is a first order operator. Finally we find in the same way as in Section 4 that every natural operator in question has finite order.

In the case $r = 2$ we start with the second order operators. The standard fiber Z^1 is a $G^2_m \times G^1_n$-space with respect to the transformations (8). By the reduction lemma, we can look for the equivariant maps $f : \tilde{L}^2_2 \to Z^1 = Z_0 \times Z_1$.

The equivariance with respect to $G^1_m \times G^1_n$ implies in the standard way

$$\begin{aligned} z^i_{p\lambda} &= k_1 L_{j\lambda p}{}^{ij} \\ z_{p\lambda} &= k_2 L_{i\lambda p}{}^{i} + k_3 D_j L_{i\lambda p}{}^{ij} \end{aligned}$$

where we write $D_j L_{i\lambda p}{}^{ij} = \tilde{L}_{i\lambda p j}{}^{ij}$ because of (19) below. Then we discuss the equivariance with respect to the isomorphisms

$$\bar{x}^i = f^i(x),\ \bar{y}^p = f^p_q(x) y^q. \tag{18}$$

In the next section we deduce in a more general situation the following transformation laws, provided $a^i_j = \partial f^i(0)/\partial x^j = \delta^i_j$, $a^p_q = f^p_q(0) = \delta^p_q$, $\partial^2 f^i(0)/\partial x^i \partial x^k = a^i_{jk}$, $\partial f^p_q(0)/\partial x^i = a^p_{qi}$,

$$\begin{aligned} \bar{z}^i_{p\lambda} &= z^i_{p\lambda}, & \bar{z}_{p\lambda} &= z_{p\lambda} - a^q_{pi} z^i_{q\lambda} \\ \bar{L}_{\mu p}{}^{ij} &= L_{\mu p}{}^{ij}, & \bar{L}_{\mu p}{}^{i} &= L_{\mu p}{}^{i} - 2a^q_{pj} L_{\mu q}{}^{ij} + a^i_{jk} L_{\mu p}{}^{jk} \\ \bar{D}_j L_{\mu p}{}^{ij} &= D_j L_{\mu p}{}^{ij} - a^a_{pj} L_{\mu q}{}^{ij} + a^i_{jk} L_{\mu p}{}^{jk} \end{aligned} \tag{19}$$

The equivariance of f yields the relations

$$k_1 = 2k_2 + k_3, \qquad k_2 + k_3 = 0.$$

This corresponds to a constant multiple of Q.

In the same way as in Section 4 we deduce even in this case that every finite order natural operator has order two and that every natural operator has finite order. This completes the proof of Proposition 3.

Let us deduce here a formula we need in the next section. Let K_0 be the kernel of the jet projection $G^2_m \to G^1_m$.

Lemma 2. *The action of K_0 on $D_i L_{i_1\dots i_m}$ has the form*

$$\bar{D}_i \bar{L}_{i_1\dots i_m} = D_i L_{i_1\dots i_m} - a^j_{ji} L_{i_1\dots i_m}. \tag{20}$$

Proof. Differentiating formally (7), we find

$$\begin{aligned}\bar{D}_i \bar{L}_{i_1\dots i_m} &= D_i L_{i_1\dots i_m}\\ &\quad - a^j_{i_1 i} L_{j i_2\dots i_m} - \dots - a^i_{i_m i} L_{i_1\dots i_{m-1} j}.\end{aligned}$$

The essential component $i_1 = 1, \dots, i_m = m$ of the second row is

$$-a^1_{1i} L_{12\dots m} - \dots - a^m_{mi} L_{12\dots m} = -a^j_{ji} L_{12\dots m}.$$

This implies (20).

8. Proof of Proposition 4. We start with the r-th order operators. The standard fiber Z^{r-1} is a $G^r_m \times G^1_n$-space with respect to (8). By Lemma 1 we can discuss the equivariant maps $f : \tilde{L}^r_r \to Z^{r-1}$. The equivariance with respect to $G^1_m \times G^1_n$ yields in the standard way

$$\begin{aligned} z^{i_1\dots i_{r-1}}_{p\lambda} &= k_1 L_{j\lambda p}{}^{i_1\dots i_{r-1} j}\\ z^{i_1\dots i_{r-2}}_{p\lambda} &= k_2 L_{i\lambda p}{}^{i i_1\dots i_{r-2}} + k_3 D_j L_{i\lambda p}{}^{i_1\dots i_{r-2} i j}\\ &\vdots\\ z^{i_1\dots i_l}_{p\lambda} &= k_{l,1} L_{i\lambda p}{}^{i i_1\dots i_l} + \dots + k_{l,r-l} D_{j_1\dots j_{r-l-1}} L_{i\lambda p}{}^{i j_1\dots j_{r-l-1} i_1\dots i_l}\\ &\vdots \end{aligned} \tag{21}$$

We are going to study the equivariance of the first two lines with respect to (18), provided $a^i_j = \delta^i_j$, $a^p_q = \delta^p_q$. By induction we find

$$\begin{aligned} d\bar{y}^p_{i_1\dots i_r} &= dy^p_{i_1\dots i_r} + a^p_{q i_1} dy^q_{i_2\dots i_r} + \dots + a^p_{q i_r} dy^q_{i_1\dots i_{r-1}} -\\ &\qquad - a^j_{i_1 i_2} dy^p_{j i_3\dots i_r} - \dots - a^j_{i_{r-1} i_r} dy^p_{j i_1\dots i_{r-2}} + \cdots\\ d\bar{y}^p_{i_1\dots i_{r-1}} &= dy^p_{i_1\dots i_{r-1}} + \cdots \end{aligned} \tag{22}$$

where the dots at the end means an expression not containing $dy^p_{i_1\dots i_r}$ and $dy^p_{i_1\dots i_{r-1}}$. This implies

$$\begin{aligned} \bar{z}^{i_1\dots i_{r-1}}_{p\lambda} &= z^{i_1\dots i_{r-1}}_{p\lambda}\\ \bar{z}^{i_1\dots i_{r-2}}_{p\lambda} &= z^{i_1\dots i_{r-2}}_{p\lambda} - (r-1) a^q_{pi} z^{i i_1\dots i_{r-2}}_{q\lambda} + \frac{r-1}{2}\big(a^{i_1}_{jk} z^{i_2\dots i_{r-2} jk}_{p\lambda} + \dots +\\ &\qquad a^{i_{r-2}}_{jk} z^{i_1\dots i_{r-3} jk}_{p\lambda}\big). \end{aligned} \tag{23}$$

By the general properties of jet spaces, [4], the transformation law of $L_{\mu p}^{i_1 \dots i_r}$ is tensorial

$$\bar{L}_{\mu p}^{i_1 \dots i_r} = \tilde{f}_p^q L_{\nu q}^{j_1 \dots j_r} \frac{\partial f^{i_1}}{\partial x^{j_1}} \cdots \frac{\partial f^{i_r}}{\partial x^{j_r}} \frac{\partial \tilde{f}^\nu}{\partial \bar{x}^\mu}. \tag{24}$$

Differentiating formally (24) and using Lemma 2, we obtain

$$\begin{aligned} \bar{D}_j \bar{L}_{\mu p}^{i_1 \dots i_r} = & D_j L_{\mu p}^{i_1 \dots i_r} - a_{pj}^q L_{\mu q}^{i_1 \dots i_r} + a_{jk}^{i_1} L_{\mu p}^{k i_2 \dots i_r} + \dots + \\ & a_{jk}^{i_r} L_{\mu p}^{i_1 \dots i_{r-1} k} - a_{kj}^k L_{\mu p}^{i_1 \dots i_r}. \end{aligned} \tag{25}$$

From (22) we also deduce

$$\begin{aligned} \bar{L}_{\mu p}^{i_1 \dots i_{r-1}} = L_{\mu p}^{i_1 \dots i_{r-1}} - r\, a_{pj}^q L_{\mu q}^{j i_1 \dots i_{r-1}} + \frac{r}{2} \Big[& a_{jk}^{i_1} L_{\mu p}^{i_2 \dots i_{r-1} jk} + \dots + \\ & a_{jk}^{i_{r-1}} L_{\mu p}^{i_1 \dots i_{r-2} jk} \Big]. \end{aligned} \tag{26}$$

In the case $r = 2$, (23), (25) and (26) reduce to (19).

For $r \geqslant 3$, the equivariance of the first two lines of (21) is equivalent to the equations

$$\begin{aligned} (r-1)k_1 &= r k_2 + k_3, \\ \frac{r-1}{2} k_1 &= \frac{r}{2} k_2 + k_3, \\ 0 &= \frac{r}{2} k_2 + k_3. \end{aligned}$$

(In the case $r = 2$ the middle equation is missing.) This yields $k_1 = k_2 = k_3 = 0$.

(21) implies that every natural operator in question is projectable, i. e. its restriction to the $(r-1)$-st order Lagrangians is a natural operator as well. Thus, for $r = 3$ we know $k_2 = k_{0,1} = -k_{0,2}$ in the third line of (21) from the second order case. But $k_2 = 0$, so that

$$z_{p\lambda} = k_{0,3} D_{ij} L_{k\lambda p}^{ijk} \tag{27}$$

and $z_{p\lambda}^i = 0$, $z_{p\lambda}^{ij} = 0$. The last two equations imply that the left-hand side of (27) is a tensorial quantity, but the right-hand side is not. Hence $k_{0,3} = 0$.

For $r \geqslant 4$ we know by induction that the underlying operator on the $(r-1)$-st order Lagrangians vanishes as well as the first two lines in (21). Hence we have

$$z_{p\lambda}^{i_1 \dots i_{r-3}} = k_{r-3,3} D_{ij} L_{k\lambda p}^{ijk i_1 \dots i_{r-3}}$$

with a tensorial quantity on the left-hand side and a non-tensorial one on the right-hand side, so that $k_{r-3,3} = 0$. Iterating this recurrence procedure we find that all coefficients in (21) vanish.

Having an s-th order natural operator A, $s \geqslant r$, we deduce by Lemma 1 and the homogeneity technique that A is an r-th order operator. In the same way as in Section 4 we find even in this case that every natural operator has finite order. This completes the proof of Proposition 4.

References

[1] M. Ferraris, M. Francaviglia, *Global formalism in higher order calculus of variations*, in: Geometrical Methods in Physics, Conference Proceedings (J. E. Purkyně University, Brno, 1984) 93-117.

[2] I. Kolář, *Some geometric aspects of the higher order variational calculus*, in: Geometrical Methods in Physics, Conference Proceedings (J. E. Purkyně University, Brno, 1984) 155-166.

[3] I. Kolář, *A geometrical version of the higher order Hamilton formalism in fibered manifolds*, J. Geom. Phys. 1 (1984) 127-137.

[4] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry*, to appear in Springer-Verlag.

[5] D. Krupka, *A Geometric Theory of Ordinary First Order Variational Problems in Fibered Manifolds*, I. Critical Sections, J. Math. Anal. Appl. 49 (1975) 180-206.

[6] D. Krupka, *Lepagean Forms in Higher Order Variational Theory*, in: Proceedings of the IUTAM–ISIMM Symposium on Modern Developments in Analytical Mechanics (Torino, 1982) 197-238.

[7] D. Krupka, J. Janyška, *Lectures on Differential Invariants (J. E. Purkyně University Brno, 1990).*

[8] M. Marvan, *On global Lepagean equivalents*, in: Geometrical Methods in Physics, Conference Proceedings (J. E. Purkyně University, Brno, 1984) 185-190.

[9] J. Muñoz Masqué, *An axiomatic characterization of the Poincaré-Cartan form for second order variational problems*, in: Differential Geometric Methods in Mathematical Physics, Lecture Notes in Mathematics 1139 (Springer-Verlag, 1983) 74-84.

[10] D. J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series 142 (Cambridge, 1989).

[11] D. J. Saunders, M. Crampin, *On the Legendre map in the higher order field theories*, J. Phys. A: Math. Gen. 23 (1990) 3169-3182.

[12] J. Slovák, *Peetre theorem for nonlinear operators*, Annals of Global Anal. Geom. 6 1988 273-283.

Ivan Kolář
Department of Algebra and Geometry MU
Janáčkovo nám. 2a
662 95 Brno, Czechoslovakia

Received 22 October 1992