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A GEOMETRIC LAGRANGIAN FORMALISM FOR EXTENDED OBJECTS

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Abstract. A geometric generalization of the first-order Lagrangian formalism is used to analyse extended objects (i.e. p-branes).

We show that the condition that reparametrizations and Galilei transformations are Noetherian symmetries, lead uniquely to a Lagrangian which is a sum of two contributions: a "Nambu-Goto" term (which appears only for particles i.e. $p = 1$) and a "Chern-Simons"-like contribution.
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0. INTRODUCTION

The study of time-dependent Lagrangian systems needs a more general formalism than the usual one based on the tangent bundle approach. Such a formalism was developed by Poincaré, Cartan and Lichnerowicz (see for instance [1]). In its most convenient formulation this framework is based on a 2-form having as associated system exactly the Euler-Lagrange equations. This 2-form is defined on the projective tangent bundle over the space-time manifold of the system, called by Souriau [2] the evolution space of the system.

This formulation allows a very elegant treatment of the Noetherian symmetries and of the connection with the symplectic action of groups appearing in the Hamiltonian formalism [2].

It is interesting to note that these geometrical ideas can be generalized to classical field theory, namely one can try to take as the basic object a generalization of the Lagrange-Souriau form [3-6]. This object is globally defined on the bundle of 1-jets of n-dimensional manifolds over the corresponding "space-time" manifold of the system; also, it behaves naturally with respect to Noetherian symmetries. We emphasize that this formalism covers only first-order Lagrangian theories.

In this paper we use this formalism to derive the most general Lagrangian theory for an extended object (a p-brane) imposing the condition that the reparametrization transformations and the Galilei transformations are Noetherian symmetries.

Let us outline briefly the structure of the paper. In Section 1 we consider the general formalism, following essentially [6]. In particular we will clear up the connection with the usual Lagrangian formalism and the usual definition of Noetherian symmetries. In Section 2 we give the definition of an extended object in our formalism. In Section 3

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we derive that the most general Lagrangian compatible with the symmetry conditions considered above is a sum between a "Nambu-Goto" Lagrangian (appearing only for $p = 1$) and a "Chern-Simons"-type Lagrangian. The computations are similar with the ones for the case of Poincaré invariance in the target space studied in [7], [8].

1. GENERAL THEORY

1.1 Let S be a differentiable manifold of dimension $n + N$. The first order Lagrangian formalism is based on an auxiliary object, namely the bundle of 1-jets of n-dimensional submanifolds of S, denoted by $J^1_n(S)$. This differentiable manifold is, by definition:

$$J^1_n(S) \equiv \bigcup_{p \in S} J^1_n(S)_p$$

where $J^1_n(S)_p$ is the manifold of n-dimensional linear subspaces of the tangent space $T_p(S)$ at S in the point $p \in S$. This manifold is naturally fibered over S and we denote by π the canonical projection. Let us construct charts on $J^1_n(S)$ adapted to this fibered structure. We first choose a local coordinate system (x^μ, ψ^A) on the open set $U \subseteq S$; here $\mu = 1, ..., n$ and $A = 1, ..., N$. Then on the open set $V \subseteq \pi^{-1}(U)$, we shall choose the local coordinate system $(x^\mu, \psi^A, \chi^A{}_\mu)$, defined as follows: if (x^μ, ψ^A) are the coordinates of $p \in U$, then the n-dimensional plane in $T_{p_0}(S)$ corresponding to $(x^\mu, \psi^A, \chi^A{}_\mu)$ is spanned by the tangent vectors:

$$\frac{\delta}{\delta x^\mu} \equiv \frac{\partial}{\partial x^\mu} + \chi^A{}_\mu \frac{\partial}{\partial \psi^A}. \tag{1.1}$$

We will systematically use the summation convention over the dummy indices.

By an *evolution space* we mean any (open) subbundle E of $J^1_n(S)$.

1.2 Let us define for a given evolution space E:

$$\Lambda_{LS} \equiv \{\sigma \in \wedge^{n+1}(J^1_n(S)) | i_{Z_1} i_{Z_2} \sigma = 0, \forall Z_i,\ s.\ t.\ \pi_* Z_i = 0, i = 1, 2\}. \tag{1.2}$$

It is clear that any $\sigma \in \Lambda_{LS}$ can be written in the local coordinates from above as follows:

$$\sigma = \varepsilon_{\mu_1,...,\mu_n} \sum_{k=0}^{n} \frac{1}{k!} C^k_n \sigma^{\mu_0,...,\mu_k}_{A_0,...,A_k} d\chi^{A_0}{}_{\mu_0} \wedge \delta\psi^{A_1} \wedge ... \wedge \delta\psi^{A_k} \wedge dx^{\mu_{k+1}} \wedge ... \wedge dx^{\mu_n}$$

$$+ \varepsilon_{\mu_1,...,\mu_n} \sum_{k=0}^{n} \frac{1}{(k+1)!} C^k_n \tau^{\mu_1,...,\mu_k}_{A_0,...,A_k} \delta\psi^{A_0} \wedge ... \wedge \delta\psi^{A_k} \wedge dx^{\mu_{k+1}} \wedge ... \wedge dx^{\mu_n}. \tag{1.3}$$

Here $\varepsilon_{\mu_1,...,\mu_n}$ is the signature of the permutation $(1, ..., n) \mapsto (\mu_1, ..., \mu_n)$, $\delta\psi^A$ is by definition:

$$\delta\psi^A \equiv d\psi^A - \chi^A{}_\mu dx^\mu. \tag{1.4}$$

We can suppose that the functions $\sigma^{\mu_0,...,\mu_k}_{A_0,...,A_k}$ are completely antisymmetric in the indices $\mu_1, ..., \mu_k$ and also in the indices $A_1, ..., A_k$, and the functions $\tau^{\mu_1,...,\mu_k}_{A_0,...,A_k}$ are completely antisymmetric in the indices $\mu_1, ..., \mu_k$ and also in the indices $A_0, ..., A_k$.

Next, one defines first the local operator K on Λ_{LS} by:

$$K\sigma \equiv i_{\frac{\delta}{\delta x^\mu}} i_{\frac{\partial}{\partial \chi^A{}_\mu}} (\delta\psi^A \wedge \sigma). \tag{1.5}$$

and proves that K is in fact globally defined.

We say that $\sigma \in \Lambda_{LS}$ is a *Lagrange-Souriau form on E* if it verifies

$$K\sigma = 0. \tag{1.6}$$

and is also closed:

$$d\sigma = 0. \tag{1.7}$$

A *Lagrangian system over S* is a couple (E, σ) where $E \subseteq J^1_n(S)$ is some evolution space over S and σ is a Lagrange-Souriau form on E.

1.3 The purpose of the Lagrangian formalism is to describe *evolutions* i.e. immersions $\Psi : M \to S$, where M is some n-dimensional manifold, usually interpreted as the space-time manifold of the system.

Let us note that frequently, one supposes that S is fibered over M, but we do not need this additional restriction in developing the general formalism. Let us denote by $\dot\Psi : M \to J^1_n(S)$ the natural lift of Ψ. If (E, σ) is a Lagrangian system over S, we say that $\Psi : M \to S$ verifies the *Euler-Lagrange equations* if:

$$\dot\Psi^* i_Z \sigma = 0. \tag{1.8}$$

for any vector field Z on E.

1.4 By a *symmetry* of the Euler-Lagrange equations we understand a map $\phi \in \mathrm{Diff}(S)$ such that if $\Psi : M \to S$ is a solution of these equations, then $\phi \circ \Psi$ is a solution of these equations also.

It is easy to see that if $\phi \in \mathrm{Diff}(S)$ is such that $\dot\phi$ leaves E invariant and:

$$\dot\phi^* \sigma = \sigma. \tag{1.9}$$

then it is a symmetry of the Euler-Lagrange equations (1.8). We call the symmetries of this type *Noetherian symmetries* for (E, σ).

If a group G act on S: $G \ni g \mapsto \phi_g \in \mathrm{Diff}(S)$ then we say that G is a *group of Noetherian symmetries* for (E, σ) if for any $g \in G$, ϕ_g is a Noetherian symmetry. In particular we have:

$$(\dot\phi_g)^* \sigma = \sigma. \tag{1.10}$$

It is considered of physical interest to solve the following classification problem: given the manifold S with an action of some group G on S, find all Lagrangian systems (E, σ) where $E \subseteq J^1_n(S)$ is on open subset and G is a group of Noetherian symmetries for (E, σ). This goal will be achieved by solving simultaneously (1.6), (1.7) and (1.10) in local coordinates and then investigating the possibility of globalizing the result.

1.5 Now we make the connection with the usual Lagrangian formalism. We can consider that the open set $V \subseteq \pi^{-1}(U)$ is simply connected by choosing it small enough.

The first task is to exhibit somehow a Lagrangian. One shows rather easily from (1.7) that $\sigma^{\cdots}_{\cdots}$ and $\tau^{\cdots}_{\cdots}$ can be written in the following form:

$$\sigma^{\mu_0,\dots,\mu_k}_{A_0,\dots,A_k} = \frac{\partial L^{\mu_1,\dots,\mu_k}_{A_1,\dots,A_k}}{\partial \chi^{A_0}{}_{\mu_0}} - L^{\mu_0,\dots,\mu_k}_{A_0,\dots,A_k}. \tag{1.11}$$

$$\tau^{\mu_1,\dots,\mu_k}_{A_0,\dots,A_k} = \sum_{i=0}^{k} (-1)^i \frac{\partial L^{\mu_1,\dots\dots\dots,\mu_k}_{A_0,\dots,\hat{A}_i,\dots,A_k}}{\partial \psi^{A_i}} - \frac{\delta L^{\mu_0,\dots,\mu_k}_{A_0,\dots,A_k}}{\delta x^{\mu_0}}. \tag{1.12}$$

where $L^{\mu_1,\dots,\mu_k}_{A_1,\dots,A_k}$ are (real) functions defined on V and completely antisymmetric in the indices $\mu_1, \dots, \mu_n$ and also in the indices $A_1, \dots, A_n$.

Finally, using the structure equation (1.6) one shows that there exists a smooth function $L : V \rightarrow R$ such that:

$$L^{\mu_1,\dots,\mu_k}_{A_1,\dots,A_k} \equiv \frac{1}{k!} \sum_{\sigma \in P_k} (-1)^{|\sigma|} \frac{\partial^k L}{\partial \chi^{A_1}{}_{\mu_{\sigma(1)}} \dots \partial \chi^{A_k}{}_{\mu_{\sigma(k)}}}. \tag{1.13}$$

(P_k is the permutation group of the numbers $1, \dots, k$) and $|\sigma|$ is the signature of σ) [6]. L is called a *local Lagrangian*. If σ is of the form (1.3) with the coefficients $\sigma^{\cdots}_{\cdots}$ and $\tau^{\cdots}_{\cdots}$ given by (1.11), (1.12) and (1.13) then we denote it by σ_L.

Now one can easily show that the local form of the Euler-Lagrange equations (1.8) coincides with the usual one. Namely, one chooses convenient local coordinates (x^μ, ψ^A) on an open set $U \subseteq S$, such that the evolution $\Psi : M \rightarrow S$ will be locally given by $x^\mu \mapsto (x^\mu, \Psi^A(x))$. Then $\dot{\Psi} : M \rightarrow J^1_n(S)$ is given by $\dot{\Psi}$ by $x^\mu \mapsto (x^\mu, \Psi^A(x), \frac{\partial \Psi^A}{\partial x^\mu}(x))$ and the equations (1.8) have the local expression:

$$\frac{\partial L}{\partial \psi^A} \circ \dot{\Psi} - \frac{\partial}{\partial x^\mu} \left(\frac{\partial L}{\partial \chi^A{}_\mu} \circ \dot{\Psi} \right) = 0. \tag{1.14}$$

i.e. the usual Euler-Lagrange equations.

It is clear that even in the case when one can cover the whole E with a single chart and L is defined everywhere, it is not correct to consider L as a globally defined function on E.

An interesting result is the following one [6]: The Euler-Lagrange equations (1.8) are trivial *iff* $\sigma_L = 0$.

Let us suppose now for the moment that σ is exact i.e.

$$\sigma = d\theta. \tag{1.15}$$

on the whole E. Then one can define the action functional by the formula:

$$A(\Psi) \equiv \int \dot{\Psi}^* \theta. \tag{1.16}$$

One can show that Ψ is a solution of the Euler-Lagrange equations *iff* it is an extremal of the action functional.

We say that the functional A is a *trivial action* if it gives trivial Euler-Lagrange equations. Then the usual definition of Noetherian symmetries is encoded in the relation:

$$A(\phi \circ \Psi) = A(\Psi) + a\ trivial\ action. \tag{1.17}$$

Now one can establish that in this case (1.17) is equivalent to the definition (1.9) given in the general case. This shows clearly why (1.9) is the most suitable definition of Noetherian symmetries: it makes sense also in the case when σ is not exact.

We close this Section with a definition. We say that a Lagrangian system (E, σ) is of the *Chern-Simons type* if:

$$\sigma \in \Lambda_{CS} \equiv \{\sigma \in \Lambda^{n+1}(E) | i_Z \sigma = 0, \forall Z,\ s.\ t.\ \pi_* Z = 0\}. \tag{1.18}$$

(compare with (1.2)).

2. EXTENDED OBJECTS AS A FIRST-ORDER LAGRANGIAN SYSTEMS

2.1 We take in the general scheme from Section 1, $S \equiv P \times M$ where P is a p-dimensional parameter manifold and $M = R \times Q$ with Q the n-dimensional Euclidean space. It is natural to suppose that $n \geq p$. The coordinates on S are (τ^a, X^0, X^i); $a = 1, ..., p$ and $i = 1, ..., n$. It will be convenient to use also the compact notation: $X^\mu \equiv (X^0, X^i)$; $\mu = 0, ..., n$. The global coordinates on $J^1_n(S)$ are: $(\tau^a, X^\mu, U^\mu{}_a)$.

The expression (1.3) of the Lagrange-Souriau form σ is:

$$\sigma = \varepsilon_{a_1,...,a_p} \sum_{k=0}^{p} \frac{1}{k!} C^k_p \sigma^{a_0,...,a_k}_{\mu_0,...,\mu_k} dU^{\mu_0}{}_{a_0} \wedge \delta X^{\mu_1} \wedge ... \wedge \delta X^{\mu_k} \wedge d\tau^{a_{k+1}} \wedge ... \wedge d\tau^{a_p}$$

$$+\varepsilon_{a_1,...,a_p} \sum_{k=0}^{p} \frac{1}{(k+1)!} C^k_p \tau^{a_1,...,a_k}_{\mu_0,...,\mu_k} \delta X^{\mu_0} \wedge ... \wedge \delta X^{\mu_k} \wedge d\tau^{a_{k+1}} \wedge ... \wedge d\tau^{a_p}. \tag{2.1}$$

where:

$$\delta X^\mu \equiv dX^\mu - U^\mu{}_a d\tau^a. \tag{2.2}$$

The structure equations (1.6) and (1.7) are in this case:

$$\sum_{i,j=1}^{k} (-1)^{i+j} \sigma^{a_j, a_1, ..., \hat{a}_j, ..., a_k}_{\mu_i, \mu_1, ..., \hat{\mu}_i, ..., \mu_k} = 0. \tag{2.3}$$

for $k = 1, ..., p$ and respectively:

$$\frac{\partial \sigma^{a_0,...,a_k}_{\mu_0,...,\mu_k}}{\partial U^{\mu_{k+1}}{}_{a_{k+1}}} - \sigma^{a_0,...,a_{k+1}}_{\mu_0,...,\mu_{k+1}} - (\mu_0 \leftrightarrow \mu_{k+1}, a_0 \leftrightarrow a_{k+1}) = 0. \tag{2.4}$$

$$\frac{\delta \sigma^{b,c,a_1,...,a_k}_{\nu,\mu_0,...,\mu_k}}{\delta \tau^c} - \sum_{i=0}^{k} (-1)^i \frac{\partial \sigma^{b,a_1,..........,a_k}_{\nu,\mu_0,...,\hat{\mu}_i,...,\mu_k}}{\partial X^{\mu_i}} + \frac{\partial \tau^{a_1,...,a_k}_{\mu_0,...,\mu_k}}{\partial U^\nu{}_b} - \tau^{a_1,...,a_k,b}_{\mu_0,...,\mu_k,\nu} = 0. \tag{2.5}$$

$$\frac{\delta \tau^{a_0,...,a_k}_{\mu_0,...,\mu_{k+1}}}{\delta \tau^{a_0}} + \sum_{i=0}^{k+1} (-1)^i \frac{\partial \tau^{a_1,............,a_k}_{\mu_0,...,\hat{\mu}_i,...\mu_{k+1}}}{\partial X^{\mu_i}} = 0. \tag{2.6}$$

for $k = 0, ..., p$. Here:

$$\frac{\delta}{\delta\tau^a} \equiv \frac{\partial}{\partial\tau^a} + U^\mu{}_a \frac{\partial}{\partial X^\mu}. \tag{2.7}$$

and a hat over a symbol means omission.

2.2 We impose now the reparametrization invariance. If $\xi \in \mathrm{Diff}(P)$, then the action of this reparametrization on S is simply:

$$\phi_\xi(\tau, X) = (\xi(\tau), X). \tag{2.8}$$

We will impose the condition that ϕ_ξ is a Noetherian symmetry for any $\xi \in \mathrm{Diff}(P)_+$ i.e. for diffeomorphisms of positive Jacobian:

$$(\dot{\phi}_\xi)^* \sigma = \sigma. \tag{2.9}$$

for any $\xi \in \mathrm{Diff}(P)_+$.

It is not hard to translate (2.9) into equations on $\sigma^{\cdots}_{\cdots}$ and $\tau^{\cdots}_{\cdots}$. Let $A_a{}^b(\tau)$ be the inverse matrix of $\frac{\partial\xi^a}{\partial\tau^a}$:

$$\frac{\partial\xi^b}{\partial\tau^a}(\tau) A_b{}^c(\tau) = \delta^c_a. \tag{2.10}$$

Then (2.9) is equivalent to:

$$\prod_{i=0}^k A_{a_i}{}^{b_i} \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \circ \dot{\phi}_\xi = (det(A)) \sigma^{b_0,\ldots,b_k}_{\mu_0,\ldots,\mu_k}. \tag{2.11}$$

and:

$$\prod_{i=1}^k A_{a_i}{}^{b_i} \left(\tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} \circ \dot{\phi}_\xi - \frac{\partial A_b{}^d}{\partial\tau^c} U^\nu{}_b \sigma^{b,c,a_1,\ldots,a_k}_{\nu,\mu_0,\ldots,\mu_k} \circ \dot{\phi}_\xi \right) = (det(A)) \tau^{b_1,\ldots,b_k}_{\mu_0,\ldots,\mu_k}. \tag{2.12}$$

for $k = 0, ..., p$

Because $\xi \in \mathrm{Diff}(P)_+$ we can consider ξ an infinitesimal diffeomorphism i.e.:

$$\xi^a(\tau) = \tau^a + \theta^a(\tau). \tag{2.13}$$

with θ^a infinitesimally small functions. Then (2.11) is equivalent to:

$$\frac{\partial\sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k}}{\partial\tau^b} = 0. \tag{2.14}$$

and:

$$U^\nu{}_b \frac{\partial\sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k}}{\partial U^\nu{}_c} = \delta^c_b \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k} - \sum_{i=0}^k \delta^{a_i}_b \sigma^{a_0,\ldots,a_{i-1},c,a_{i+1},\ldots,a_k}_{\mu_0,\ldots\ldots\ldots\ldots,\mu_k}. \tag{2.15}$$

In the same way (2.12) is equivalent to:

$$\frac{\partial \tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k}}{\partial \tau^b} = 0. \tag{2.16}$$

$$U^\nu{}_b \frac{\partial \tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k}}{\partial U^\nu{}_c} = \delta^c_b \tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} - \sum_{i=1}^{k} \delta^{a_i}_b \tau^{a_1,\ldots,a_{i-1},c,a_{i+1},\ldots,a_k}_{\mu_0,\ldots\ldots\ldots\ldots,\mu_k}. \tag{2.17}$$

and:

$$U^\nu{}_d \sigma^{b,c,a_1,\ldots,a_k}_{\nu,\mu_0,\ldots\ldots,\mu_k} + (b \leftrightarrow c) = 0. \tag{2.18}$$

In addition we impose Galilei invariance in the "target" space M. Namely:

$$\phi_{R,\boldsymbol{v},\eta,\boldsymbol{a}}(\tau, X^0, \boldsymbol{X}) = (\tau, X^0 + \eta, R\boldsymbol{X} + \boldsymbol{a}). \tag{2.19}$$

is a Noetherian symmetry for any $R \in SO(n)$, any $\boldsymbol{v}, \boldsymbol{a} \in Q$ and any $\eta \in R$.

It is not hard to translate (2.19) into conditions on $\sigma^{\cdots}_{\cdots}$ and $\tau^{\cdots}_{\cdots}$. For $R = id$ and $\boldsymbol{v} = \boldsymbol{0}$ this condition translates into the X-independence of those functions. For $\eta = 0$ and $\boldsymbol{a} = \boldsymbol{0}$ we obtain the invariance of the tensor functions $\sigma^{\cdots}_{\cdots}$ and $\tau^{\cdots}_{\cdots}$ with respect to the natural action of the Euclidean group $SE(n)$ induced by the action of this group on the target space M. Namely:

$$\prod_{i=0}^{k} \Lambda^{\nu_i}{}_{\mu_i} \sigma^{a_0,\ldots,a_k}_{\nu_0,\ldots,\nu_k}(\Lambda^\mu{}_\nu U^\nu{}_a) = \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k}(U^\mu{}_a). \tag{2.20}$$

and:

$$\prod_{i=0}^{k} \Lambda^{\nu_i}{}_{\mu_i} \tau^{a_1,\ldots,a_k}_{\nu_0,\ldots,\nu_k}(\Lambda^\mu{}_\nu U^\nu{}_a) = \tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k}(U^\mu{}_a). \tag{2.21}$$

for any $k = 0, \ldots, p$ and any $\Lambda \in Aut(M)$ of the form:

$$\Lambda_{R,\boldsymbol{v}} \equiv \begin{pmatrix} 1 & \boldsymbol{0} \\ \boldsymbol{v} & R \end{pmatrix}. \tag{2.22}$$

Using the fact that in this case the tensors $\sigma^{\cdots}_{\cdots}$ and $\tau^{\cdots}_{\cdots}$ are only U-dependent one can easily show that (2.15) and (2.17) are the infinitesimal form of:

$$\prod_{i=0}^{k} A_{a_i}{}^{b_i} \sigma^{a_0,\ldots,a_k}_{\mu_0,\ldots,\mu_k}(A_a{}^b U^\mu{}_b) = (det(A))\sigma^{b_0,\ldots,b_k}_{\mu_0,\ldots,\mu_k}(U^\mu{}_a). \tag{2.23}$$

and:

$$\prod_{i=1}^{k} A_{a_i}{}^{b_i} \tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k}(A_a{}^b U^\mu{}_b) = (det(A))\tau^{b_1,\ldots,b_k}_{\mu_0,\ldots,\mu_k}(U^\mu{}_a). \tag{2.24}$$

for any $A \in GL(p,R)_+$.

We close this Section noting that the structure equations simplify considerably: equation (2.6) is trivially satisfied and (2.5) becomes:

$$\frac{\partial \tau^{a_1,\dots,a_k}_{\mu_0,\dots,\mu_k}}{\partial U^\nu{}_b} - \tau^{a_1,\dots,a_k,b}_{\mu_0,\dots,\mu_k,\nu} = 0. \tag{2.25}$$

3. THE DETERMINATION OF THE LAGRANGIAN

3.1 We first determine the the functions $\sigma^{\dots}_{\dots}$. From (2.4) and (2.5) one shows by induction that $\sigma^{\dots}_{\dots}$ are of the form:

$$\sigma^{a_0,\dots,a_k}_{\mu_0,\dots,\mu_k} = \frac{\partial (L_0)^{a_1,\dots,a_k}_{\mu_1,\dots,\mu_k}}{\partial U^{\mu_0}{}_{a_0}} - (L_0)^{a_0,\dots,a_k}_{\mu_0,\dots,\mu_k}. \tag{3.1}$$

where:

$$(L_0)^{a_0,\dots,a_k}_{\mu_0,\dots,\mu_k} = \frac{1}{k!}\sum_{\sigma\in P_k}(-1)^{|\sigma|}\frac{\partial^k L_0}{\partial U^{\mu_1}{}_{a_{\sigma(1)}}\dots\partial U^{\mu_k}{}_{a_{\sigma(k)}}}. \tag{3.2}$$

Here L_0 is a U-dependent smooth function.

Using a cohomological argument (see for instance [8]) one shows that (2.23) implies that L_0 can be chosen such that it is $GL(p,R)_+$ invariant i.e.

$$L_0(A_a{}^b U^\mu{}_b) = L_0(U^\mu{}_a). \tag{3.3}$$

for any $A \in GL(p,R)_+$. (It is better to prove (3.3) first for the case $A \in SL(p,R)$ and next the case $A = \lambda I$ for $\lambda \in R_+$).

We exploit now the Euclidean covariance of the functions $\sigma^{\dots}_{\dots}$ using again a cohomological analysis. First we can redefine L_0 such it is rotation covariant without modifying σ and without modifying (3.3). It remains to use the invariance with respect to pure Galilei transforms. One can show that there are two cases:

(a) $p = 1$; in this case L_0 can be taken in the familiar form:

$$L_0(U^0, \boldsymbol{U}) = \frac{m}{2}\frac{\boldsymbol{U}^2}{U^0}. \tag{3.4}$$

(b) $p > 1$; in this case L_0 can be taken strictly Galilei invariant. This fixes:

$$L_0 = 0. \tag{3.5}$$

This completely elucidates the structure of the functions $\sigma^{\dots}_{\dots}$.

3.2 The determination of the functions $\tau^{\dots}_{\dots}$ is very simple. First, from (2.25) one determines by induction that $\tau^{\dots}_{\dots}$ are of the form:

$$\tau^{a_1,\dots,a_k}_{\mu_0,\dots,\mu_k} = \sum_{l=k}^{p}\frac{1}{(l-k)!}C^{a_1,\dots,a_l}_{\mu_0,\dots,\mu_l}\prod_{i=k+1}^{l}U^{\mu_i}{}_{a_i}. \tag{3.6}$$

where $C^{\dots}_{\dots}$ are some constants, completely antisymmetric in the upper and also in the lower indices.

The Euclidean covariance of the tensor $\tau^{\cdots}_{\cdots}$ is the equivalent to the Euclidean invariance of the tensors $C^{\cdots}_{\cdots}$ (in the lower indices) and (2.24) is equivalent to the $GL(p, R)_+$-invariance of the same tensors (in the upper indices).

In particular we must have:

$$\tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k}(\lambda U) = \lambda^{n-k}\tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k}(U). \tag{3.7}$$

for any $\lambda \in R_+$. If we insert (3.6) here we get that:

$$C^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} = 0. \tag{3.8}$$

for $k < p$ so only the coefficient $C^{a_1,\ldots,a_p}_{\mu_0,\ldots,\mu_p}$ survives. Because the property of complete antisymmetry we must have in fact:

$$C^{a_1,\ldots,a_p}_{\mu_0,\ldots,\mu_p} = \varepsilon^{a_1,\ldots,a_p} C_{\mu_0,\ldots,\mu_p}. \tag{3.9}$$

where $C_{\mu_0,\ldots,\mu_p}$ is a completely antisymmetric tensor. It remains to use the Euclidean invariance of this tensor. One can easily show that there are two possibilities:

(A) $n \neq p$; in this case:

$$C_{\mu_0,\ldots,\mu_p} = 0. \tag{3.10}$$

so in fact:

$$\tau^{a_1,\ldots,a_k}_{\mu_0,\ldots,\mu_k} = 0. \tag{3.11}$$

for any $k = 0, \ldots, p$. So, in this case we get:

$$\sigma = \sigma_{L_0}. \tag{3.12}$$

with L_0 determined by the situations (a) and (b) from Section 3.1.

(B) $n = p$; in this case only the components of the tensor $C_{0,i_1,\ldots,i_p}$ of the tensor $C_{\ldots}$ can be non-zero:

$$C_{0,i_1,\ldots,i_p} = \lambda\varepsilon_{i_1,\ldots,i_p}. \tag{3.13}$$

for some $\lambda \in R$.

It easily follows that the corresponding $\tau^{\cdots}_{\cdots}$'s obtained from (3.6) and (3.13) are produced by the Chern-Simons Lagrangian:

$$L_{CS}(X, U) \equiv \lambda\frac{1}{(p+1)!}\varepsilon^{a_1,\ldots,a_p}\varepsilon_{i_1,\ldots,i_p}\left[X^0\prod_{j=1}^{p} U^{i_j}{}_{a_j} + \sum_{j=1}^{p} X^{i_j} U^0{}_{a_j}\prod_{k\neq j} U^{i_k}{}_{a_k}\right]. \tag{3.14}$$

So, in this case:

$$\sigma = \sigma_{L_0+L_{CS}}. \tag{3.15}$$

where L_0 is described at the end of Subsection 4.1.

We note that for $p = 1$ (particles) we get the some well-known expressions derived for the first time in [9] by a different method. For $n = p$ we obtain Chern-Simons terms of the same type as the one produced by the Poincaré invariance in the target space [7].

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