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Index Theory and Quaternionic Kähler Manifolds

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Abstract. Index theorems are discussed and applied to coupled Dirac operators in a quaternionic setting.

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Introduction

The Dirac operator plays a fundamental role in the geometry and topology of Riemannian manifolds, and special classes of manifolds have properties that reflect features of their Dirac operators. A case in point is the description of Dolbeault cohomology on a Kähler manifold in terms of the Dirac operator with coefficients in an appropriate bundle. The aim of this note is to highlight some aspects of the analogous theory concerning quaternionic Kähler manifolds, which form a large class of Riemannian manifolds with reduced holonomy.

In the last section, we announce some new linear relations amongst the Betti numbers of compact quaternionic Kähler manifolds with positive scalar curvature. These are analogues of Kähler-Einstein manifolds whose cohomology is exclusively of type (p,p). Our results give further evidence for supposing that the only such manifolds are the quaternionic Kähler symmetric spaces, especially when combined with knowledge that the complex Grassmannians $\mathrm{Gr}_2(\mathbb{C}^{n+2})$ are the only examples with $b_2 > 0$ [15]. Exterior power operations and the γ-filtration in K-theory are used in the formulation of the main computational lemma **5.3**. The results were obtained as part of a joint project with C.R. LeBrun on the topology of quaternionic Kähler manifolds, and full details will appear in a forthcoming paper [16].

The rest of the note is organized with the last section in sight. Two preliminary sections set up the rudiments of index theory from the point of view of Dirac operators. The situation for Kähler manifolds, described by Hitchin [10], is summarized so as to reveal exact parallels with the quaternionic case, and a concise survey of quaternionic Kähler theory is inserted as an interlude. The material was presented at the conference on Differential Geometry and its Applications in Opava, Czechoslovakia in August 1992, and thanks are due to the organizers of that event.

1. Dirac operators

Throughout the paper M denotes a compact $4n$-dimensional oriented Riemannian manifold. In this first section, we shall suppose that M is a spin manifold, so that there exists a principal $Spin(4n)$-bundle P over M such that $P/\mathbb{Z}_2$ is isomorphic to the bundle of oriented orthonormal frames. The local isomorphism between P and $P/\mathbb{Z}_2$ allows the Levi Civita connection to be lifted to a connection on the principal bundle P invariant by $Spin(4n)$. It is this connection that is especially well suited to the definition of differential operators linked to the geometry of M.

A representation ρ of a compact Lie group is determined by its restriction to a maximal torus, and the simultaneous eigenvalues of this restricted action are the weights of ρ. Of relevance to us is the existence of a faithful representation of $Spin(4n)$ on a complex vector space V of dimension 2^{2n} with weights

$$\tfrac{1}{2}(\pm x_1 \pm x_2 \cdots \pm x_{2n}) \tag{1}$$

with respect to standard coordinates on the Lie algebra of a maximal torus. This representation arises from the realization of $Spin(4n)$ as a subgroup of the Clifford algebra C_{4n}, and the group action preserves an antilinear mapping

$$\sigma\colon V \to V \quad \text{with} \quad \sigma^2 = (-1)^n; \tag{2}$$

this allows us to view V as the complexification of a real space (the fixed points of σ) when n is even, or as underlying a quaternionic space (with $\sigma = j$) when n is odd. Combining the real or quaternionic structure σ with an invariant Hermitian metric gives an equivariant isomorphism $V \cong V^*$.

The representation of $Spin(4n)$ on $\operatorname{End} V \cong V \otimes V$ has kernel $\mathbb{Z}_2$ and therefore factors through $SO(4n)$. Indeed, there is a $SO(4n)$-equivariant isomorphism

$$V \otimes V \cong \bigoplus_{k=0}^{4n} \Lambda^k, \tag{3}$$

where Λ^k denotes the exterior power $\bigwedge^k T$ of the basic representation given by

$$Spin(4n) \longrightarrow SO(4n) \hookrightarrow \operatorname{Aut} \mathbb{C}^{4n}. \tag{4}$$

The representation T has a real structure, and we shall generally replace it with the equivalent representation T^* to conform to certain conventions. The fundamental formula (3) may be deduced from a standard method of decomposing tensor products involving dominant weights.

Given a complex representation of $Spin(4n)$, one may associate to P a complex vector bundle. In particular, the associated vector bundle $P \times_{Spin(4n)} V$ is called the total spin bundle, and we shall denote it by $\underline{V}$. The connection on P induces one on $\underline{V}$ which provides a covariant differentiation operator

$$\nabla\colon \Gamma(\underline{V}) \longrightarrow \Gamma(\underline{T}^* \otimes \underline{V}), \tag{5}$$

where $\underline{T}^* = P \times_{Spin(4n)} T^*$ denotes the complexified cotangent bundle of M.

Example. The well-known isomorphism $Spin(4) \cong SU(2) \times SU(2)$ allows one to describe the representation V of $Spin(4)$ in terms of the action of the two $SU(2)$'s; indeed $V = V_+ \oplus V_-$, where V_+, V_- are the standard complex 2-dimensional representations of the two $SU(2)$'s. Similarly, the cotangent representation is defined by $T^* \cong V_+ \otimes V_-$, and the invariant element of the symmetric product $\mathsf{S}^2 T^*$ is the product of invariant skew forms that trivialize $\bigwedge^2 V_\pm$. In the isomorphism

$$V \otimes V \cong \mathbb{C} \oplus (V_+ \otimes V_-) \oplus (\mathsf{S}^2 V_+ \oplus \mathsf{S}^2 V_-) \oplus (V_+ \otimes V_-) \oplus \mathbb{C}$$

illustrating (3), $\mathsf{S}^2 V_\pm$ can be identified with the eigenspaces $\Lambda^2_\pm$ of the $*$ operator on Λ^2. These facts are crucial to an understanding of 4-dimensional Riemannian geometry.

Although $Spin(4n)$ is a simple group for $n > 1$, the representation V of $Spin(4n)$ always decomposes as

$$V = V_+ \oplus V_-, \tag{6}$$

where V_+, V_- are irreducible representations of equal dimension. Indeed, both restrict to the unique 2^{2n-1}-dimensional irreducible representation of $Spin(4n-1)$, under the natural inclusion $Spin(4n-1) \subset Spin(4n)$. The isomorphisms

$$\begin{aligned} V_+ \otimes V_+ &\cong \Lambda^0 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^{2n-2} \oplus \Lambda^{2n}_+, \\ V_+ \otimes V_- &\cong \Lambda^1 \oplus \Lambda^3 \oplus \cdots \oplus \Lambda^{2n-3} \oplus \Lambda^{2n-1}, \\ V_- \otimes V_- &\cong \Lambda^0 \oplus \Lambda^2 \oplus \cdots \oplus \Lambda^{2n-2} \oplus \Lambda^{2n}_-, \end{aligned} \tag{7}$$

refine (3), and involve the eigenspaces $\Lambda^{2n}_\pm$ of $*$, which defines an involution of Λ^{2n}.

The inclusion $T^* \cong \Lambda^1 \subset \operatorname{End} V$ determined by (3) gives rise to a $Spin(4n)$-equivariant homomorphism $\mu: T^* \otimes V \to V$, called Clifford multiplication, with the property

$$\mu(T^* \otimes V_\pm) = V_\mp.$$

The total Dirac operator is the composition

$$\mu \circ \nabla : \Gamma(\underline{V}) \longrightarrow \Gamma(\underline{V}),$$

and is both elliptic and self-adjoint. It is convenient to decompose the total Dirac operator into the two operators

$$D: \Gamma(\underline{V}_+) \longrightarrow \Gamma(\underline{V}_-), \qquad D^*: \Gamma(\underline{V}_-) \longrightarrow \Gamma(\underline{V}_+).$$

If $\{e_i\}$, $\{e^i\}$ are dual orthonormal bases of local sections of $\underline{T}$, $\underline{T}^*$ respectively then we may write

$$Dv = \sum_{i=1}^{4n} \mu(e^i \otimes \nabla_{e_i} v), \qquad v \in \Gamma(\underline{V}_+).$$

Associated to the operator D in the context of K-theory is the virtual vector bundle

$$\underline{\widetilde{V}} = \underline{V}_+ - \underline{V}_-, \tag{8}$$

which is central to much of the discussion below, and we shall refer to the corresponding element $\widetilde{V} = V_+ - V_-$ in the representation ring of $Spin(4n)$ as the "signed" spin representation. In practice, we shall be more concerned with Dirac operators with coefficients in some auxiliary complex vector bundle F itself equipped with a covariant derivative $\nabla(F)\colon \Gamma(F) \longrightarrow \Gamma(\underline{T}^* \otimes F)$. The coupled Dirac operator

$$D(F)\colon \Gamma(\underline{V}_+ \otimes F) \longrightarrow \Gamma(\underline{V}_- \otimes F) \tag{9}$$

is then the linear operator determined by

$$D(v \otimes f) = Dv \otimes f + \sum_{i=1}^{4n} \mu(e^i \otimes v)\nabla(F)_{e_i} f, \qquad v \in \Gamma(\underline{V}),\ f \in \Gamma(F).$$

Example. Take F to be $\underline{V}_+$, and $\nabla(F)$ to be the natural connection. The first two formulae from (7) and an examination of the underlying algebra show that

$$D(F)\colon \Gamma(\underline{\Lambda}^0 \oplus \underline{\Lambda}^2 \oplus \cdots \oplus \underline{\Lambda}^{2n}_+) \longrightarrow \Gamma(\underline{\Lambda}^1 \oplus \underline{\Lambda}^3 \oplus \cdots \oplus \underline{\Lambda}^{2n-1}) \tag{10}$$

coincides with the operator built up in a natural way from exterior differentiation d and its adjoint d^*. The space $\ker D(F)$ may be computed summand by summand, and is the direct sum of the spaces

$$H^{2k} = \{\alpha \in \Gamma(\underline{\Lambda}^{2k}) : d\alpha = 0 = d^*\alpha\}$$

of harmonic forms for $0 \le k < 2n$ and the space $H^+ = \{\alpha \in \Gamma(\underline{\Lambda}^{2n}_+) : d^*\alpha = 0\}$ of self-dual harmonic forms.

The restriction of $D^*(F)$ to $\underline{\Lambda}^{2n-1}$ equals $d^+ + d^*$, where d^+ is the composition of d with the linear projection $\Lambda^{2n} \to \Lambda^{2n}_+$. Integrating the formula

$$d(\alpha \wedge d\alpha) = (d\alpha)^2 = (\|d^+\alpha\|^2 - \|d^-\alpha\|^2)(\text{vol form})$$

shows that $d^+\alpha = 0$ implies $d\alpha = 0$, and it follows that $\ker D^*(F)$ is a direct sum of H^{2k-1} for $1 \le k \le 2n$. The Dirac operator coupled to $\underline{V}_+$ has special significance when $n = 1$ in the treatment of Yang-Mills theory over 4-manifolds.

2. Index classes

On a compact manifold M, the kernel of any elliptic operator is finite-dimensional. In particular, the kernel and cokernel of any coupled Dirac operator (9) (which may be regarded as the cohomology spaces H^0 and H^1 of the corresponding 2-step complex) are finite-dimensional. The index of $D(F)$ is the integer defined by

$$\begin{aligned} \text{ind } D(F) &= \dim\ker D(F) - \dim \operatorname{coker} D(F) \\ &= \dim\ker D(F) - \dim\ker D^*(F). \end{aligned}$$

This quantity is invariant under deformation, and can in fact be computed entirely in terms of the Pontrjagin classes of M and the Chern classes of F by means of the Atiyah-Singer Theorem which is stated below.

Characteristic classes assign cohomology classes to bundles over a manifold M. One of the simplest is the first Chern class of a complex line bundle, or equivalently the Euler class of an oriented real rank 2 vector bundle. This is an element $x = c_1(L)$ in $H^2(M,\mathbb{Z})$ that is easy to define directly in terms of transition functions using Čech cohomology. A complex vector bundle which decomposes as a direct sum

$$F = L_1 \oplus \cdots \oplus L_r \tag{11}$$

of line bundles gives rise to classes $x_1, \ldots, x_r$, and the elementary symmetric polynomials $c_1, \ldots, c_r$ in these classes depend only on F, and not on the choice of splitting. These are the Chern classes of F, and can be defined for an arbitrary complex rank r vector bundle by pulling back to a suitable space over which a splitting of the form (11) exists.

Given a complex line bundle L on a manifold M, there exists an integer k and a mapping $f\colon M \to \mathbb{C}P^k$ such that L is isomorphic to the pullback $f^{-1}\mathcal{O}(1)$ of the standard line bundle over $\mathbb{C}P^k$ and $c_1(L) = f^*x$, where x generates the cohomology of $\mathbb{C}P^k$. The first Chern class of a line bundle is then seen to arise from the cohomology of the classifying space $B_{U(1)} = \mathbb{C}P^\infty$ associated to the Lie group $U(1) = SO(2)$. From this point of view, $x_1, \ldots, x_r$ correspond to the weights of the standard representation of $U(r)$ and are really cohomology classes on the classifying space of a maximal torus T. An induced mapping

$$H^*(B_{U(r)}) \longrightarrow H^*(B_T)$$

identifies the Chern classes $c_1, \ldots, c_r$ with these weights that are invariant by the Weyl group of $U(r)$. We adopt this approach of treating characteristic classes as formal power series with all terms present, without regard to the dimension of any underlying manifold. This will be important in the statement of **2.1** below.

Let F be a complex vector bundle of rank r over M. Its Chern character is given by

$$\mathrm{ch}(F) = r + \sum_{k=1}^{\infty} \frac{1}{k!} s_k,$$

where s_k is defined inductively in terms of the Chern classes of F by Newton's formula

$$s_k - c_1 s_{k-1} + \cdots + (-1)^{k-1} c_{k-1} s_1 + (-1)^k k c_k = 0.$$

In terms of the formal factorization that expresses c_k as the kth elementary symmetric function in $x_1, x_2, \ldots, x_r$, we have $s_k = \sum_{i=1}^r {x_i}^k$, and

$$\mathrm{ch}(F) = \sum_{i=1}^{r} e^{x_i}.$$

This definition is designed so that

$$\begin{aligned} \mathrm{ch}(F_1 \oplus F_2) &= \mathrm{ch}(F_1) \oplus \mathrm{ch}(F_2), \\ \mathrm{ch}(F_1 \otimes F_2) &= \mathrm{ch}(F_1)\,\mathrm{ch}(F_2). \end{aligned}$$

Consider now a familiar index in differential geometry, namely the Euler characteristic

$$\chi(M) = \sum_{k=1}^{4n} (-1)^k b_k,$$

of a manifold M (of dimension $4n$) with Betti numbers $b_k = \dim H^k(M, \mathbb{R})$. By Hodge theory, $\chi(M)$ coincides with the index of the 2-step de Rham complex

$$d+d^* : \Gamma\Big(\bigoplus_{k=0}^{2n} \underline{\Lambda}^{2k}\Big) \longrightarrow \Gamma\Big(\bigoplus_{k=1}^{2n} \underline{\Lambda}^{2k-1}\Big) \tag{12}$$

Recalling the discussion of (10), it is easy to see that this index equals the difference of the indices of $D(\underline{V}_+)$ and $D(\underline{V}_-)$. It follows that $\chi(M)$ is formally the index of the Dirac operator coupled to the virtual vector bundle (8).

The Gauss-Bonnet Theorem states that $\chi(M)$ is obtained by evaluating a certain class $e \in H^{4n}(M, \mathbb{R})$ on the fundamental cycle $[M]$; e is the Euler class of the tangent bundle of M. For computational purposes, we shall suppose that M has an almost complex structure, so that its complexified cotangent bundle has the form

$$\underline{T} = \underline{T}^{1,0} \oplus \underline{T}^{0,1}. \tag{13}$$

Then e coincides with the top Chern class c_{2n} of the holomorphic tangent bundle $\underline{T}^{1,0}$, which we know is determined by the polynomial $\prod_{i=1}^{2n} x_i$, where the x_i are weights on a common maximal torus of $U(2n)$ and $SO(4n)$. Since e is the restriction of a well-defined invariant polynomial on the Lie algebra $\mathfrak{so}(4n)$, it characterizes the Euler class in general.

By comparison, observe that

$$\begin{aligned} \mathrm{ch}(\widetilde{\underline{V}}) &= \mathrm{ch}(\underline{V}_+) - \mathrm{ch}(\underline{V}_-), \\ &= \prod_{i=1}^{2n} (e^{x_i/2} - e^{-x_i/2}) \\ &= \Big(\prod_{i=1}^{2n} x_i\Big) P, \end{aligned} \tag{14}$$

where P is a power series in ${x_1}^2, \dots, {x_{2n}}^2$ with constant term 1. The Chern classes of the complexified tangent bundle $\underline{T}$ are given by

$$1 + c_1 + \cdots + c_{4n} = \prod_{i=1}^{2n} (1 + x_i)(1 - x_i),$$

in view of the decomposition (13). The kth elementary symmetric polynomial

$$p_k = (-1)^k c_{2k}$$

in ${x_1}^2, \dots, {x_{2n}}^2$ is the kth Pontrjagin class of (the tangent bundle of) M. To sum up,

2.1 Lemma *There exists an invertible formal power series P in the Pontrjagin classes such that $e = ch(\widetilde{\underline{V}})P^{-1}$.*

The fact that $\mathrm{ch}(\widetilde{\underline{V}})$ vanishes to high order will be of vital importance in Section 5. If α is a cohomology class of mixed degree, we use $\int_M \alpha$ to denote the number obtained by evaluating the component of α of top degree ($4n$ in our case) on the fundamental cycle $[M]$. The Atiyah-Singer Theorem [1] may now be quoted in the following form.

2.2 Theorem *The index of the Dirac operator coupled to F is given by*

$$ind\ D(F) = \int\limits_M ch(F)\, P^{-1}.$$

The characteristic class

$$P^{-1} = \prod_{i=1}^{2n} \frac{x_i/2}{\sinh(x_i/2)}$$

is called the $\hat{A}$-class of (the tangent bundle of) M, and its first terms are readily computed:

$$P^{-1} = \hat{A} = 1 - \tfrac{1}{24}p_1 + \tfrac{1}{45.2^7}(7{p_1}^2 - 4p_2) + \cdots. \tag{15}$$

It can itself be regarded as a type of Chern character determined by the symbol of D. The Gauss-Bonnet theorem is the special case of **2.2** in which the terms of P^{-1} of non-zero degree do not enter, and amounts to identifying e and $\mathrm{ch}(\widetilde{\underline{V}})$ as elements of $H^*(M,\mathbb{R})$.

Each coefficient bundle F that we shall need to consider will have a real or quaternionic structure in the sense of (2), so its Chern classes will, in common with $\hat{A}$, be present only in degrees that are multiples of 4. With this assumption,

$$\begin{aligned}\mathrm{ch}(F) &= r + \tfrac{1}{2}s_2 + \tfrac{1}{24}s_4 + \cdots\\ &= r - c_2 + \tfrac{1}{12}({c_2}^2 - 2c_4) + \cdots.\end{aligned}$$

Example. Let F be the total spin bundle $\underline{V} = \underline{V}_+ \oplus \underline{V}_-$ with its natural connection. The resulting operator

$$D(F)\colon \Gamma(\underline{\Lambda}^0 \oplus \underline{\Lambda}^1 \oplus \cdots \oplus \underline{\Lambda}^{2n-1} \oplus \underline{\Lambda}^{2n}_+) \longrightarrow \Gamma(\underline{\Lambda}^0 \oplus \underline{\Lambda}^1 \oplus \cdots \oplus \underline{\Lambda}^{2n-1} \oplus \underline{\Lambda}^{2n}_-) \tag{16}$$

is, in contrast to (12), formally the *sum* of $D(\underline{V}_+)$ and $D(\underline{V}_-)$. Its index equals $b^+ - b^-$, where $b^\pm = \dim H^\pm$ (see (10)), and by Hodge theory coincides with the signature of M, i.e. of the bilinear form $\mathsf{S}^2(H^2(M,\mathbb{R})) \to H^4(M,\mathbb{R})$ given by cup product. The class appearing in the corresponding integral formula

$$b^+ - b^- = \int\limits_M \mathrm{ch}(\underline{V})\hat{A} \tag{17}$$

is the *Hirzebruch L-class*, and arises from the formal factorization

$$\prod_{i=1}^{2n}(e^{x_i/2}+e^{-x_i/2})\hat{A}=\prod_{i=1}^{2n}x_i\coth(x_i/2).$$

Its first terms are given by

$$L=1-\tfrac{1}{3}p_1+\tfrac{1}{45}(7p_2-{p_1}^2)+\cdots.$$

3. Kähler spinors and cohomology

In this section, M continues to denote a compact Riemannian manifold of dimension $4n$. We shall also suppose that the Riemannian structure is Kähler, which means that there exists an orthogonal almost complex structure J which is parallel with respect to the Levi Civita connection. The last condition guarantees (via the Newlander-Nirenberg Theorem) that J is integrable so that M is a complex manifold possessing local holomorphic coordinates $z^1,\dots,dz^{2n}$ for which $J\,dz^k=i\,dz^k$.

The choice of an orthogonal almost complex structure on the Riemannian manifold M may be interpreted by the existence of a principal $U(2n)$-subbundle of the $SO(4n)$-bundle of oriented orthonormal frames. The Kähler condition is that the Levi Civita connection reduce to this subbundle, which is equivalent to the rather imprecise statement that "the holonomy group of M is contained in $U(2n)$". A temporary assumption will allow us to go further. Suppose that the structure on M lifts to a principal $SU(2n)\times U(1)$-subbundle Q compatible with the commutative diagram

$$\begin{array}{ccc} SU(2n)\times U(1) & \longrightarrow & Spin(4n)\\ \downarrow & & \downarrow\\ U(2n) & \hookrightarrow & SO(4n), \end{array}$$

in which the first vertical homomorphism has kernel $\mathbb{Z}_n$. Natural vector bundles over M can now be associated, via Q, to representations of $SU(2n)\times U(1)$. Consider, in particular, the summand $\underline{\Lambda}^{1,0}$ of the complexified cotangent bundle

$$\underline{T}^*=\underline{\Lambda}^{1,0}\oplus\underline{\Lambda}^{0,1},$$

spanned by forms of type $(1,0)$ relative to J (i.e., those on which J acts as the complex number i). This bundle is isomorphic to $Q\times_{SU(2n)\times U(1)}\Lambda^{1,0}$, where $\Lambda^{1,0}$ denotes the dual of the standard representation of $U(2n)$ which we choose to write in the form

$$\Lambda^{1,0}\cong E\otimes L^*, \tag{18}$$

where E ($\cong\mathbb{C}^{2n}$) is the dual of the standard representation of $SU(2n)$, and L ($\cong\mathbb{C}$) is the standard representation of $U(1)$. It follows that the canonical line bundle $\underline{\kappa}=\underline{\Lambda}^{2n,0}$ of the complex manifold M is isomorphic to $(\underline{L}^*)^{\otimes 2n}=\underline{L}^{-2n}$. In particular, the spin assumption ensures the global existence of a square root $\underline{\kappa}^{\frac{1}{2}}\cong\underline{L}^{-n}$.

The first step to investigate the Dirac operator on a Kähler manifold is to examine how the total spin representation V of $Spin(4n)$ breaks up relative to $SU(2n)\times U(1)$.

There is a standard procedure for doing this in terms of weights, and by general principles the answer will be a sum of terms, each of which is a product of a representation of $SU(2n)$ with a power of L. We omit the details, because the result can be quickly derived from the well-known isomorphism

$$V\cong\Big(\bigoplus_{q=0}^{2n}\Lambda^{0,q}\Big)\otimes\kappa^{\frac{1}{2}},\tag{19}$$

[10, 13, 22]. We adopt the standard notation $\Lambda^{p,q}$ for the tensor product of $\bigwedge^p(\Lambda^{1,0})$ and $\bigwedge^q(\Lambda^{0,1})=\bigwedge^q(\overline{\Lambda^{1,0}})$, and the presence of $\kappa^{\frac{1}{2}}$ ensures that $V\cong V^*$ since $\Lambda^{0,q}\otimes\kappa=\Lambda^{2n,q}\cong(\Lambda^{0,2n-q})^*$. Then using (18) and an appropriate orientation convention,

$$\widetilde{V}\cong\sum_{q=0}^{2n}(-1)^q\textstyle\bigwedge^{2n-q}E\otimes L^{q-n}=\displaystyle\sum_{p=0}^{2n}(-1)^p\textstyle\bigwedge^pE\otimes L^{n-p}.$$

At this point, we take the opportunity to introduce the exterior power operations of K-theory [7]. If A,B are elements of a representation ring, one defines

$$\textstyle\bigwedge^m(A-B)=\displaystyle\sum_{q=0}^{m}(-1)^q\textstyle\bigwedge^{m-q}A\otimes\mathsf{S}^qB;$$

this formula generalizes the following elementary isomorphism of vector spaces: if $A=B\oplus C$ then $(A\otimes B)\oplus\bigwedge^2C\cong\bigwedge^2A\oplus\mathsf{S}^2B$. Then

3.1 Proposition *The signed spin representation satisfies $L^n\otimes\widetilde{V}\cong\bigwedge^{2n}(E-L)$ with respect to the action of $SU(2n)\times U(1)$.*

The isomorphisms (7) tell us what to expect when we compute $\widetilde{V}\otimes V$. The result is confirmed by tensoring $\widetilde{V}$ with just one of the summands of V:

$$\widetilde{V}\otimes(\textstyle\bigwedge^pE\otimes L^{n-p})\cong\displaystyle\sum_{q=0}^{2n}(-1)^q\Lambda^{p,q}.$$

The associated virtual vector bundle is that which arises in the definition of the coupled Dirac operator $D(\bigwedge^p\underline{E}\otimes\underline{L}^{n-p})$. The resulting 2-step complex may be strung out into the Dolbeault complex

$$0\to\Gamma(\underline{\Lambda}^{p,0})\xrightarrow{\overline{\partial}}\Gamma(\underline{\Lambda}^{p,1})\xrightarrow{\overline{\partial}}\Gamma(\underline{\Lambda}^{p,2})\to\cdots\to\Gamma(\underline{\Lambda}^{p,2n})\to 0,\tag{20}$$

whose cohomology at the qth step is denoted by $H^{p,q}(M,\mathcal{O})$.

The "basic" Dolbeault complex obtained by taking $p=0$ can itself be coupled to any holomorphic vector bundle F, or for that matter $\underline{\Lambda}^{p,0}\otimes F$. The spaces $H^{p,q}(M,\mathcal{O}(F))$ are defined accordingly, and the holomorphic Euler characteristic of $\underline{\Lambda}^{p,0}\otimes F$ is the alternating sum

$$\chi(M,\mathcal{O}(\underline{\Lambda}^{p,0}\otimes F)=\sum_{q=0}^{2n}(-1)^q\dim H^{p,q}(M,\mathcal{O}(F)).\tag{21}$$

It follows that this equals the index of the coupled Dirac operator $D(F\otimes\bigwedge^p\underline{E}\otimes\underline{L}^{n-p})$, which can be computed with **2.2**. Indeed, if c_1 denotes the first Chern class of (the holomorphic tangent bundle of) M, then

$$\mathrm{ch}(\underline{\kappa}^{-\frac{1}{2}})\hat{A} = e^{\frac{1}{2}c_1}\hat{A}$$

is the Todd class td of M, and we obtain the following Riemann-Roch theorem.

3.2 Corollary. $\chi(M,\mathcal{O}(\underline{\Lambda}^{p,0}\otimes F)) = \int_M \mathrm{ch}(\underline{\Lambda}^{p,0})\mathrm{ch}(F)\,\mathrm{td}.$

The emphasis on the Dirac operator and the $\hat{A}$ class in the above derivation pays enormous computational dividends, since it enables one to control the cohomology classes of degrees not divisible by 4. We shall need an application of **3.2** in which M is replaced by the twistor space of a quaternionic Kähler manifold. Observe that in many of the preceding statements we have implicitly dropped the assumption that the structure of M lift to $SU(2n)\times U(1)$ or $Spin(4n)$; the complexes (20) and the computation of their indices are valid in all cases.

Example. On a compact Kähler manifold, the compatibility between the operators d, d^*, $\overline{\partial}$, $\overline{\partial}^*$ implies that the Laplacians dd^*+d^*d and $\overline{\partial}\overline{\partial}^*+\overline{\partial}^*\overline{\partial}$ are proportional and it is well known that there is a natural isomorphism

$$H^k(M,\mathbb{R}) \cong \bigoplus_{p+q=k} H^{p,q}(M,\mathcal{O}), \qquad \text{with} \qquad H^{q,p}(M,\mathcal{O}) = \overline{H^{p,q}(M,\mathcal{O})}.$$

As an illustration, suppose that M has the very special property that its Dolbeault cohomology spaces $H^{p,q}(M,\mathcal{O})$ vanish whenever $p\neq q$. Then the odd Betti numbers of M vanish, and

$$\mathrm{ind}\ D(\textstyle\bigwedge^p\underline{E}\otimes\underline{L}^{n-p}) = (-1)^p b_{2p}, \tag{22}$$

a result that is a model for **5.2** below. The integral formulae that result from applying **3.2** to (22) refine both the Gauss-Bonnet theorem for the Euler characteristic $\sum_{p=0}^{2n} b_{2p}$ and the Hirzebruch formula (17) for the signature $\sum_{p=0}^{2n}(-1)^p b_{2p}$.

4. Quaternionic Kähler manifolds

The quaternionic Kähler condition may be defined in terms of the subgroup

$$Sp(n)Sp(1) = Sp(n)\times_{\mathbb{Z}_2}Sp(1), \tag{23}$$

of $SO(4n)$ that normalizes right multiplication by quaternions i,j,k on $\mathbb{H}^n\cong\mathbb{R}^{4n}$. The description (23) is analogous to that of the group $U(2n)\cong SU(2n)\times_{\mathbb{Z}_{2n}}U(1)$ used to define the Kähler condition. Accordingly, a quaternionic Kähler manifold is a Riemannian manifold M for which the Levi Civita connection reduces to a principal subbundle of the bundle of orthonormal frames with fibre (23), or equivalently "the holonomy group is contained in $Sp(n)Sp(1)$". Despite the terminology, quaternionic Kähler manifolds are not in general Kähler manifolds for the simple reason that $Sp(n)Sp(1) \not\subseteq U(2n)$.

There is a commutative diagram

$$\begin{array}{ccc} Sp(n)\times Sp(1) & \longrightarrow & Spin(4n) \\ \downarrow & & \downarrow \\ Sp(n)Sp(1) & \hookrightarrow & SO(4n), \end{array} \tag{24}$$

in which the top horizontal homomorphism is injective when n is odd, and factors through an inclusion $Sp(n)Sp(1) \subset Spin(4n)$ when n is even [17]. This is a direct consequence of **5.1** below, and means that a quaternionic Kähler manifold of even quaternionic dimension n is always spin and its structure may or may not be covered by an $Sp(n)\times Sp(1)$ bundle. The existence of such a $Sp(n)\times Sp(1)$-bundle would give rise to an isomorphism $\underline{T}^* \cong \underline{E}\otimes\underline{H}$, analogous to (18), where $\underline{E}$, $\underline{H}$ are vector bundles associated to the standard complex representations E, H of $Sp(n)$, $Sp(1)$ respectively.

The representation E is compatible with that of the previous section with respect to the inclusion $Sp(n) \subset SU(2n)$, but E acquires a quaternionic structure under the action of $Sp(n)$ (cf. (2)) and can be identified with $\mathbb{H}^n$. Similarly H can be identified with $\mathbb{H}$. When $n = 1$ the spaces E, H coincide with the spin representations V_+, V_- (see the first example in Section 1). The choice of a 1-dimensional subspace L^* (notation compatible with (18)) in H determines a maximal isotropic subspace

$$E\otimes L^* \subset E\otimes H \cong T; \tag{25}$$

in this way the complex projective line $P(H)$ parametrizes a family of complex structures on T.

The following established facts [6] have helped to motivate further advances in the subject.

1. Any quaternionic Kähler manifold (of real dimension at least 8) is Einstein. If the scalar curvature is zero, then M is (locally) hyperkähler, and admits parallel complex structures I, J, K arising from (25) and satisfying the usual quaternion identities.

2. There is a natural family of quaternionic Kähler symmetric spaces. In fact there is one of the form G/K for each compact simple Lie group G, where K is the normalizer of a three-dimensional subalgebra of $\mathfrak{g}$ corresponding to a highest root [24]. These spaces consist of the Grassmannians

$$\mathbb{H}P^n = \frac{Sp(n+1)}{Sp(n)\times Sp(1)},\ \mathrm{Gr}_2(\mathbb{C}^{n+2}) = \frac{U(n+2)}{U(n)\times U(2)},\ \mathrm{Gr}_4(\mathbb{R}^{n+4}) = \frac{SO(n+4)}{SO(n)\times SO(4)}, \tag{26}$$

and five exceptional symmetric spaces.

3. There exist homogeneous non-symmetric quaternionic Kähler manifolds, but none with positive scalar curvature [2, 3].

From now on, we shall consider exclusively quaternionic Kähler manifolds that are "positive", that is for which the scalar curvature is a positive constant. This ensures, amongst other things, that the quaternionic structure is not subordinate to a Kähler metric. So far no complete positive quaternionic Kähler metrics are known other than the symmetric ones, and various results cited below suggest that there may be no others. First though, we need to introduce a basic technique in the theory.

The twistor space Z of a quaternionic Kähler manifold is the total space of the bundle $\pi: P(\underline{H}) \to M$ parametrizing almost complex structures on each tangent space by means of (25). It has the structure of a complex $(2n+1)$-dimensional manifold foliated by rational curves, namely the fibres of π. The Levi Civita connection of M determines a horizontal holomorphic distribution which is locally isomorphic to the tensor product $\pi^*\underline{E}(1)$ of $\pi^*\underline{E}$ with a holomorphic line bundle $\mathcal{O}(1)$ (cf. (18)), whose square $\mathcal{O}(2)$ is globally defined on Z. Moreover, there is a short exact sequence

$$0 \longrightarrow \pi^*\underline{E}(1) \longrightarrow T^{1,0}Z \longrightarrow \mathcal{O}(2) \longrightarrow 0. \tag{27}$$

These facts are equally valid when M is a self-dual Einstein manifold, the 4-dimensional incarnation of a quaternionic Kähler manifold.

The sequence (27) determines a complex contact structure on Z if the scalar curvature of M is non-zero, in which case one deduces that the anticanonical bundle κ^{-1} of Z is isomorphic to $\mathcal{O}(2n+2)$. The positivity assumption on M then implies that some power of κ^{-1} embeds M into a complex projective space, i.e. that Z is a Fano manifold. Moreover, long exact cohomology sequences associated to (27) show that $H^{p,q}(Z) = 0$ unless $p = q$ [20]. Since $H^*(Z,\mathbb{R})$ is a free $H^*(M,\mathbb{R})$-module with one generator of degree 2, it follows that all the odd Betti numbers b_{2p+1} of M are zero.

Full proofs of the following statements are beyond the scope of this note.

4.1 Theorem *Let M be a complete positive quaternionic Kähler $4n$-manifold.*
(i) *If $H^2(M,\mathbb{Z}_2) = 0$ then M is isometric to $\mathbb{H}P^n$;*
(ii) *If $b_2(M) > 0$ then M is isometric to $Gr_2(\mathbb{C}^{n+2})$.*
(iii) *If $n = 2$ then M is a symmetric space.*

Explanation. Part (i) is a corollary of a somewhat stronger result in [20]; namely M is isometric to $\mathbb{H}P^n$ whenever its structure lifts to $Sp(n)\times Sp(1)$ (cf. (24)). The obstruction to this lifting is measured by a cohomology class $\varepsilon \in H^2(M,\mathbb{Z}_2)$ analogous to the second Stiefel-Whitney class [17]. When $\varepsilon = 0$, one may extract a $(n+1)$st root of κ^{-1} and a standard argument implies that Z is biholomorphically equivalent to $\mathbb{C}P^{2n+1}$, the twistor space of $\mathbb{H}P^n$.

Part (ii) is a spin-off of Mori programme, first observed in [15]. The crucial property of the twistor space Z of a positive quaternionic Kähler manifold with $b_2(M) > 0$ is the existence of a rational curve C of “twistor degree” equal to 1 and whose homology class is not proportional to that of a fibre of $Z \to M$. Through each point the family of such rational curves actually spans out a projective space $\mathbb{C}P^n$ and a Fano contraction identifies Z with the total space of the projectivization of a holomorphic vector bundle over a variety X, as described in [23]. The fibres of the contraction are tangent to the contact distribution, essentially because the pullback of the contact form to C vanishes for cohomological reasons. A further argument is needed in order to deduce that X is isomorphic to $\mathbb{C}P^{n+1}$, and the result follows from this.

Part (iii) was proved in [19], but can be deduced easily from the newer relatively deep result (ii). The key point is that $b_4 = 1 + b_2$ on any positive quaternionic Kähler 8-manifold (a relation that will be generalized to arbitrary dimensions in section 5). The assumption that $b_2 = 0$ then implies that all 4-dimensional characteristic classes are proportional, and the result follows from index formulae [16]. There are three quaternionic symmetric spaces of dimension 8, namely $\mathbb{H}P^2$, $\mathrm{Gr}_2(\mathbb{C}^4) \cong \mathrm{Gr}_4(\mathbb{R}^6)$ and $G_2/SO(4)$.

The group $Sp(n)Sp(1)$ leaves fixed an element Ω of $\bigwedge^4 T$, and one way of characterizing the holonomy reduction of a quaternionic Kähler manifold is by the existence of a corresponding parallel (and so closed) 4-form. For some purposes, Ω is analogous to the Kähler 2-form ω defined by a reduction to the unitary group, but there are important differences. One advantage of the quaternionic case is that the metric is completely determined by Ω since its stabilizer in $GL(4n,\mathbb{R})$ is exactly $Sp(n)Sp(1)$; but this means that, unlike ω, the form Ω is far from being generic as the orbit $GL(4n,\mathbb{R})/Sp(n)Sp(1)$ has high codimension in $\bigwedge^4 T$. This explains the difficulty in finding metrics with holonomy $Sp(n)Sp(1)$, though we conclude this section with some techniques that have been successful in the construction of incomplete quaternionic Kähler metrics.

Suppose that G is a connected Lie group acting on a positive quaternionic Kähler manifold M as a group of isometries. Each element A in the Lie algebra $\mathfrak{g}$ of G determines a Killing vector field on M. The interior product of this vector field with the closed 4-form determined by Ω can be shown to equal the exterior derivative of a certain 2-form μ_A, itself a real section of the rank 3 vector bundle $\mathsf{S}^2\underline{H}$. This is exactly analogous to the procedure for defining a moment mapping in symplectic geometry, except that the symplectic 2-form as been replaced by Ω. Moreover, the section μ of $\mathsf{S}^2\underline{H}\otimes\mathfrak{g}^*$ defined by the assignment $A\mapsto\mu_A$ is G-equivariant. Corresponding to the Marsden-Weinstein reduction in symplectic geometry is the following result of Galicki and Lawson [9].

4.2 Theorem *If G acts freely on the zero set $M_0=\{m\in M:\mu(m)=0\}$, then M_0/G is a quaternionic Kähler manifold.*

This quotient is often denoted by $M/\!\!/G$. For example

$$\mathrm{Gr}_2(\mathbb{C}^{n+2})=\mathbb{H}P^{n+1}/\!\!/U(1),\qquad \mathrm{Gr}_4(\mathbb{R}^{n+4})=\mathbb{H}P^{n+3}/\!\!/Sp(1),$$

where the actions are given by natural inclusions $U(1)\subset Sp(n+2)$ and $Sp(1)\subset Sp(n+4)$. Less standard $U(1)$'s give rise to a host of quaternionic Kähler orbifolds, and the procedure also produces self-dual Einstein metrics in 4 dimensions, for example on weighted complex projective planes [9]. Although complete metrics with negative scalar curvature exist [8, 14], singularities seem inevitable in the positive case.

Example. An exceptional situation occurs when one considers the standard action of $U(1)$ on $\mathbb{R}^7$ by means of inclusions $U(1)\subset U(3)\subset SO(7)$, and has recently been described in [11]. An open set of the quotient $Q=\mathrm{Gr}_4(\mathbb{R}^7)/\!\!/U(1)$ can be identified with an $\mathbb{H}^*$ quotient of the adjoint $SL(3,C)$-orbit $\{A\in\mathfrak{sl}(3,\mathbb{C}):A^3=0,\,A^2\neq 0\}$ of principal nilpotent elements, which for quite separate reasons admits a quaternionic Kähler metric. Indeed, any complex nilpotent adjoint orbit gives rise to a quaternionic Kähler space, although this is singular unless the orbit is minimal in which case it is symmetric [21]. The compact group $SU(3)$, the centralizer of $U(1)$ in $SO(7)$, acts isometrically on Q, which is a locally symmetric orbifold formed as a $\mathbb{Z}_3$ quotient of $G_2/SO(4)$.

5. Quaternionic spinors and Betti numbers

In section 3 we saw that the building blocks for spin representation were the $SU(2n)$ exterior powers $\bigwedge^p E$ and the $U(1)$ tensor powers L^q. We shall first define the representations needed to describe spinors in the quaternionic case. The main differences are that the exterior powers of E are no longer irreducible under the action of $Sp(n)$, and the 1-dimensional representation L has to be replaced by the 2-dimensional one H.

The group $Sp(n)$ leaves invariant a skew form $\varepsilon \in \bigwedge^2 E$. The primitive subspace $\bigwedge_0^p E$ of $\bigwedge^p E$ is defined to be the Hermitian complement of $\varepsilon \wedge \bigwedge^{p-2} E$; it is an irreducible $Sp(n)$-module of dimension $\binom{2n}{p} - \binom{2n}{p-2}$ if $p \leq n$. The resulting decomposition

$$\bigwedge^p E \cong \bigwedge_0^p E \oplus \bigwedge_0^{p-2} E \oplus \bigwedge_0^{p-4} E \oplus \cdots \tag{28}$$

will also be used immediately below when E is replaced by a virtual representation or bundle. The role of the representation L^q in the Kähler case is taken by the symmetric tensor product $\mathsf{S}^q H$ which has dimension $q+1$ (cf. (25)). The formula

$$\widetilde{V} \cong \sum_{q=0}^{n} (-1)^q \bigwedge_0^{n-q} E \otimes \mathsf{S}^q H. \tag{29}$$

is proved in [4, 22] modulo a choice of orientation, and provides a direct analogue of **3.1**.

5.1 Proposition *The signed spin representation satisfies $\widetilde{V} = \bigwedge_0^n (E - H)$ with respect to the action of $Sp(n) \times Sp(1)$.*

It will be convenient to define

$$R^{p,q} = \bigwedge_0^p E \otimes \mathsf{S}^q H.$$

Since the summands of $\widetilde{V} \otimes V$ exhaust those appearing in a decomposition of (3) under the action of $Sp(n)Sp(1)$, the representations giving rise to the 2-step de Rham complex (12) of M can be grouped into pieces

$$\widetilde{V} \otimes R^{n-q,q}, \qquad q = 0, \ldots, n. \tag{30}$$

In particular, taking $q = n$,

$$\widetilde{V} \otimes \mathsf{S}^n H \cong \sum_{q=0}^{2n} (-1)^q \bigwedge^q E \otimes \mathsf{S}^q H, \tag{31}$$

where we may view $\bigwedge^q E \otimes \mathsf{S}^q H$ as a distinguished subspace of $\bigwedge^q T^*$. Exactly in analogy to (20), the right-hand side of (31) gives rise to a subcomplex

$$0 \to \Gamma(\underline{\Lambda}^0) \to \Gamma(\underline{\Lambda}^1) \to \Gamma(\bigwedge^2 \underline{E} \otimes \mathsf{S}^2 \underline{H}) \to \cdots \to \Gamma(\bigwedge^{2n} \underline{E} \otimes \mathsf{S}^{2n} \underline{H}) \to 0 \tag{32}$$

of the full length de Rham complex of M. Analogues of this complex are studied in [5], and develop the theory of Fueter's quaternionic Cauchy-Riemann equation.

The summands arising from (30) can be used to decompose the exterior forms of a quaternionic Kähler manifold M under $Sp(n)Sp(1)$; it would be interesting to know if they give rise to a filtration of the de Rham complex of M, as in the Kähler case. In any case, all the spaces concerned can be directly related to the algebra of (p,q)-forms on the twistor space Z, and an understanding of this relationship leads to the next theorem. First we define

$$i^{p,q} = \operatorname{ind} D(\underline{R}^{p,q}), \qquad p \geq 0, \quad q \geq 0, \quad n + p + q \text{ even};$$

the parity restriction ensures that the corresponding coupled Dirac operator is globally defined given that (by **4.1**(i)) the individual vector bundles $\underline{E}$, $\underline{H}$ will not usually be.

5.2 Theorem *On a positive quaternionic Kähler manifold of dimension* $4n$,

$$i^{p,q} = \begin{cases} 0 & \text{if } n = p+q+2r,\ r>0 \\ (-1)^p(b_{2p-2}+b_{2p}) & \text{if } n = p+q. \end{cases}$$

Explanation. We have discussed coupled Dirac operators on both Kähler and quaternionic Kähler manifolds, and the two theories can be united by recalling that the twistor space Z of a positive quaternionic Kähler manifold M is itself a Kähler manifold. The key result is that the cohomology of the Dirac operator on M coupled to $\mathsf{S}^q\underline{H}$ is isomorphic to that of the basic Dolbeault complex on Z coupled to the line bundle $\mathcal{O}(q-n)$, and

$$i^{p,q} = \chi(Z, \mathcal{O}(\pi^*\textstyle\bigwedge_0^p\underline{E}\,(q-n)).$$

Long exact cohomology sequences associated to exterior powers of (27) relate the Dolbeault cohomology spaces $H^{0,q}(Z, \mathcal{O}(\pi^*\bigwedge^p\underline{E}\,(-p)))$ and $H^{p,q}(Z,\mathcal{O})$, and yield the holomorphic Euler characteristic formula

$$\chi(Z, \mathcal{O}(\pi^*\textstyle\bigwedge^p\underline{E}\,(-2r-p)) = \begin{cases} 0 & \text{if } 1\le r\le n-p, \\ (-1)^p \dim H^{p,p}(Z,\mathcal{O}) & \text{if } r=0 \end{cases}$$

[20], Corollary 6.7. The theorem follows from the decomposition (28) and the fact that $\dim H^{p,p}(Z,\mathcal{O}) = b_{2p-2}+b_{2p}$ with the convention that $b_{-2}=0$.

Theorem **5.2** is of course consistent with the equation $\sum_{p=0}^n(-1)^p i^{p,n-p} = \chi(M)$ arising from (12). On the other hand, the signature of M is given by

$$\sum_{p=0}^n i^{p,n-p} = (-1)^n b_{2n},$$

and therefore a positive quaternionic Kähler manifold of dimension $4n$ has $b^+=0$ or $b^-=0$ according as n is odd or even. This result was discovered by Nagano and Takeuchi [18] (though they made a different choice of orientation).

We shall now explain that there are non-trivial identities amongst the expressions for $i^{p,q}$ given by the Atiyah-Singer Theorem **2.2**. For each quaternionic dimension n, we shall indicate the existence of a virtual representation

$$W_n = \sum_{\substack{n\,=\,p\,+\,q\,+\,2r\\ r\ge 0}} c_{p,q}R^{p,q}, \qquad c_{p,q}\in\mathbb{Z},$$

with the property that the index of the associated Dirac operator $D(\underline{W}_n)$ vanishes. The complexity of the $\hat{A}$ class (15) leads one to seek W_n to satisfy the stronger property that

$$\mathrm{ch}(\underline{W}_n) = 0+\cdots+0+y+\tfrac{1}{24}p_1y+\cdots, \tag{33}$$

where y has degree $4n-4$. This ensures that $\mathrm{ch}(\underline{W}_n)\hat{A}$ vanishes as a cohomology class on a $4n$-manifold, so that the required index is zero.

The problem has now been reduced to an algebraic one, and the first step is to find elements of the representation ring of $Sp(n)\times Sp(1)$ with the property that their associated Chern characters vanish to high order. Let $\mathcal{F}_n$ denote the space spanned by virtual representations R such that $\mathrm{ch}(\underline{R})$ has no terms of degree less than $4n$. The resulting sequence

$$\mathcal{F}_1 \supset \cdots \supset \mathcal{F}_n \supset \mathcal{F}_{n+1} \supset \cdots$$

is an example of the γ-filtration that can also be described directly in terms of exterior power operations [7]. Note that $R\in\mathcal{F}_1$ if and only if the virtual dimension of R is zero, and the filtration is compatible with tensor products in the sense that $\mathcal{F}_m\otimes\mathcal{F}_n\subset\mathcal{F}_{m+n}$.

If R is a genuine representation of dimension $2n-2$ then clearly $\bigwedge^n R\cong\bigwedge^{n-2}R$, so by definition $\bigwedge_0^n R=0$. This is not true if R is a virtual representation, but observe that **2.1** and **5.1** imply that $\bigwedge_0^n(E-H)\in\mathcal{F}_n$. Moreover,

$$\textstyle\bigwedge_0^{n-k}(E-(k+1)H)\in\mathcal{F}_{n-k},\qquad 0\le k\le n-1;$$

this follows from the computation of a generalization of the Euler class for the virtual vector bundle $\underline{E}-k\underline{H}$. We shall denote the virtual representation $\bigwedge_0^p(E-qH)$ by $\Lambda(p,q)$.

In trying to solve (33), it is futile to introduce representations of $Sp(n)$ other than the exterior powers of E. This leaves tensor products of the form

$$\Lambda(n-k+1,k)\otimes S^m H,\qquad k-1-m=2r,\ r\ge 0,$$

to play with to obtain linear combinations of the $R^{p,q}$ for $p+q\le n$, and at the same time satisfy the parity condition that $n-p-q$ be even. However, to obtain something in $\mathcal{F}_{n-1}$, we are forced to take a linear combination of

$$W_n'=\Lambda(n,1),\qquad W_n''=\Lambda(n-1,2)\otimes H,\qquad W_n'''=\Lambda(n-2,3)\otimes(\mathrm{S}^2H-3),$$

bearing in mind that $S^2H-3\in\mathcal{F}_1$, and remarkably,

5.3 Lemma *A solution to* (33) *is* $W_n=\frac{1}{2}(n-1)n\,W_n'+3(n-1)W_n''+6\,W_n'''$.

This result was initially established by computer using MATHEMATICA, but proofs of this and subsequent statements will appear in [16]. It is a straightforward matter to find the coefficients of $R^{p,q}$ with $p+q=n$ in W_n; indeed there exist $a_p\in\mathbb{Z}$ such that

$$W_n\cong\Big(\sum_{p=0}^n a_pR^{p,n-p}\Big)-6(n+2)\Lambda(n-2,2),$$

where

$$a_p=(-1)^n a_{n-p}\quad\text{ and }\sum_{p=0}^n(-1)^pa_p=0. \tag{34}$$

Because $\Lambda(n-2,n)$ is a linear combination of $R^{p,q}$ with $p+q<n$, **5.2** implies that

$$\sum_{p=0}^n(-1)^p(a_p-a_{p+1})b_{2p}=0,$$

with $a_{n+1}=0$. These relations turn out to be quadratic in both n and p:

5.4 Theorem *The Betti numbers of any positive quaternionic Kähler 4n-manifold satisfy the relation*

$$\sum_{p=0}^{n-1}\big[6p(n-1-p)-(n-1)(n-3)\big]b_{2p} = \tfrac{1}{2}n(n-1)b_{2n}.$$

We recall (Section 4) that the odd Betti numbers b_{2p+1} of M are already known to vanish. The new equations for $\dim M \le 32$ are listed in the table.

n	
2	$1+b_2=b_4$
3	$2b_2=b_6$
4	$-1+3b_2+3b_4-b_6=2b_8$
5	$-4+5b_2+8b_4+5b_6-4b_8=5b_{10}$
6	$-5+3b_2+7b_4+7b_6+3b_8-5b_{10}=5b_{12}$
7	$-8+2b_2+8b_4+10b_6+8b_8+2b_{10}-8b_{12}=7b_{14}$
8	$-35+b_2+25b_4+37b_6+37b_8+25b_{10}+b_{12}-35b_{14}=28b_{16}$

Each equation stops at the middle Betti number so Poincaré duality does not intervene; nevertheless there is a curious symmetry that arises from (34). More insight can be obtained by replacing b_{2p} by the sum $\sum_{i=0}^{[p/2]}\beta_{2p-4i}$, where β_{2p} are "primitive Betti numbers" arising from the injection

$$H^{2p-4}(M,\mathbb{R}) \hookrightarrow H^{2p}(M,\mathbb{R}), \qquad p \le n+1,$$

established in [12] by wedging with the closed 4-form Ω. Computations of Poincaré polynomials confirm that the equations are satisfied by the Grassmannians (26), and the line $n=7$ is also applicable to the 28-dimensional exceptional space $F_4/Sp(3)Sp(1)$.

Assuming that M is not the Grassmannian $\mathrm{Gr}_2(\mathbb{C}^{n+2})$, we know from **4.1**(ii) that $b_2=0$. The table then shows that a positive quaternionic Kähler 12-manifold, other than $\mathrm{Gr}_2(\mathbb{C}^5)$, has all Betti numbers zero except for $b_4=b_8$. In dimension 16 the single assumption that $b_4=1$ implies that M has the same real cohomology ring as $\mathbb{H}P^4$. The same assumption implies that the classes p_1 and $[\Omega]$ in $H^4(M,\mathbb{R})$ are proportional, a situation that brings into play other methods, such as an equality between the index $i^{n+2,0}$ and the dimension of the group of isometries of M [20]. As a consequence, we foresee that further progress on the classification of positive quaternionic Kähler manifolds will result from an analysis of the fourth Betti number b_4.

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