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DECOMPOSABLE ELEMENTS IN EIGENSPACES OF THE CURVATURE OPERATOR

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Abstract. We find an algebraic obstruction preventing the flatness of the normal bundle of any isometric immersion of the Riemannian manifold M into any space of constant sectional curvature $\kappa \in \mathbb{R}$ of an arbitrary dimension $N > \dim M$. The obstruction is expressed in terms of the curvature operator of the Riemannian manifold M.
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1. Result

Fix a Riemannian manifold (M, g) of dimension $n \geq 4$ and a point $x \in M$.

Theorem. *If, for the curvature operator*

$$R_x : \Lambda^2 T_x M \to \Lambda^2 T_x M,$$

there exists a one-dimensional eigenspace E ($E = \ker(R_x - \lambda I)$ for some $\lambda \in R$) spanned by an indecomposable element $\omega_0 \in \Lambda^2 T_x M$, then there exists an open neighbourhood W of x such that, for each open neighbourhood $U \subset W$ of x, any $N > n$, $\kappa \in \mathbb{R}$ and for any isometric immersion

$$f : (U, g|U) \to K^N(\kappa)$$

of the Riemannian manifold U into the space $K^N(\kappa)$ of constant sectional curvature of dimension N and curvature κ, the normal bundle $\nu f \to U$ is non-flat everywhere (i.e. at any point $y \in U$).

Proof. The eigenvalue λ_0 corresponding to the eigenvector ω_0 is a single root, so from the implicit function theorem we get a smooth funct ion $\lambda : W_0 \to \mathbb{R}$ such that, for any point $y \in W_0$, $\lambda(y)$ is an eigenvalue of multiplicity one of the operator R_y, and $\lambda(x) = \lambda_0$, where W_0 is some open neighbourhood of x. It is well known that the set of decomposable elements in $\Lambda^2 T M$ is closed. The implicit function theorem brings us again to the conclusion that there exist an open neighbourhood $W \subset W_0$ of x and a smooth section $\omega : W \to \Lambda^2 T M$ such that, for an arbitrary $y \in W$, $\omega(y)$ is the indecomposable eigenvector of the operator R_y, corresponding to the single eigenvalue $\lambda(y)$, and $\omega(x) = \omega_0$.

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Suppose that the normal bundle νf is flat at some point $y \in U$ and choose an orthonormal frame $\xi_1, \dots, \xi_{N-n}$ of $\nu_y f$.

The flatness of the normal bundle at the point y via the Ricci equation implies that all the shape operators commute with each other i.e. $[A_\xi, A_\eta] = 0$ where $\xi, \eta \in \nu_y f$. The set $\{A_{\xi_1}, \dots, A_{\xi_{N-n}}\}$, as the intertwining family of self-adjoint operators, is simultaneously diagonalizable by an orthonormal basis, say $X_1, \dots, X_n \in T_y M$. So, we infer at once that our shape operators fulfil the following conditions:

$$A_{\xi_s} X_j = r_{sj} \cdot X_j \qquad \text{for} \ \ j = 1, \dots, n; \ \ s = 1, \dots, N-n,$$

where $r_{sj} \in \mathbb{R}$ are associated eigenvalues.

Now, the Gauss equation yields the formula

$$\begin{aligned}
&\langle R(X_i, X_j)X_k, X_l\rangle \\
&= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \cdot \kappa + \sum_{s=1}^{N-n} (\langle A_{\xi_s} X_i, X_k\rangle \cdot \langle A_{\xi_s} X_j, X_l\rangle - \langle A_{\xi_s} X_i, X_l\rangle \cdot \langle A_{\xi_s} X_j, X_k\rangle) \\
&= (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \cdot (\kappa + \sum_{s=1}^{N-n} r_{si} \cdot r_{sj}).
\end{aligned}$$

The last equation shows that the curvature operator R_y can be diagonalized by the decomposable orthonormal basis $X_i \wedge X_j \in \Lambda^2 T_y M$, $i < j$.

This contradicts to the fact that $\omega(y) \in \Lambda^2 T_y M$ is the indecomposable eigenvector generating the one-dimensional eigenspace of R_y. $\square$

2. Application

In order to show that the assumptions of the above theorem are non-void, we shall give an example:

Example. Let (M, g) be the *Kaehler manifold with constant holomorphic sectional curvature* $c \neq 0$ of complex dimension $m \geq 2$ (for instance, the complex projective space $\mathbb{C}P^m$ with the *Fubini-Study metric* [G], [KN]). The spectral decomposition of the curvature operator is the following ([B], [W], [G]). There are three eigenvalues $\lambda_0 = 0$, $\lambda_1 = c(m+1)/2$, $\lambda_2 = c/2$ of ranks $m(m-1)$, 1, $(m+1)(m-1)$, respectively. The single eigenvalue λ_1 corresponds to the *Kaehler form* $\omega \in \Lambda^{1,1} M$, $\omega(X,Y) = g(X, JY)$ where by J we denote the complex structure of M; λ_2 is associated with the subspace of primitive two-forms in $\Lambda^{1,1} M$; and λ_0 corresponds to the subspace of two-forms $\eta \in \Lambda^2 M$ which satisfy the condition $\eta(JX, JY) = -\eta(X, Y)$. We immediately infer that the eigenvector

$$\omega = \sum_{j=1}^{m} X_j^* \wedge J X_j^*$$

(*Kaehler form*) is indecomposable and generates the one-dimensional eigenspace associated with the eigenvalue λ_1 , where by $\{X_1^*, \dots, X_m^*, JX_1^*, \dots, JX_m^*\}$ we denote the dual (real) basis to the orthonormal adapted basis $\{X_1, \dots, X_m, JX_1, \dots, JX_m\}$.

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