Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 281–320

NATURAL BUNDLES AND NATURAL LIFTINGS PROLONGATIONS OF GEOMETRIC STRUCTURES

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Abstract. This paper is a survey of results on prolongation functors, natural bundles and liftings of geometric structures. The general ideas, important examples and fundamental results in theory of natural bundles, product preserving functors, natural transformations and liftings of geometric objects are presented. Especially problems of regularity conditions and finite order for these notions are studied. There are also new results.

Key words: natural bundle, prolongation functor, product preserving functor, Weil functor, local algebra, geometric object, natural transformation, lifting, lift, order of natural bundle, order of transformation (of lift).

AMS classifications: 53-02, 53C15, 58A20, 58A30.

§ 0. Introduction.

Some geometric objects were known in XIX century; the notions of vector, tensor and connection were systematically used in the differential geometry and its applications. The general idea of geometric object first was formulated in 1928 by O. Veblen [120] in his plenary lecture "Differential Invariants and Geometry" at ICM in Bologna. O. Veblen has used there the notion "invariant". The expression "geometric object" was firstly used by J. A. Schouten and E. R. van Kampen [109]. Some important results were obtained by J. A. Schouten and J. Haantjes [108], A. Nijenhuis [91], J. Aczel and S. Gołąb [1], and his school (see also M. Kucharzewski, M. Kuczma [66]). The idea of fibre bundle introduced by H. Whitney [122], C. Ehresmann [12] and N. Steenrod [118] gave new possibilities of describing geometric objects. New attempts to formulate the theory of geometric objects were made by J. Haantjes and G. Laman [34], and N. K. Kuiper and K. Yano [67], with their fibre bundle approach. The more "functorial" approach to this theory was introduced by A. Nijenhuis [93]. He has presented the very important concept of natural bundle. Some interesting results in the theory of geometric objects using the language of natural bundles firstly were given by S. Salvioli [106].

The general important questions in the theory of geometric objects there are questions on classifications and regularity. Attempts to classify geometric objects have been made in book of J. Aczel and S. Gołąb [1]. First general results were given by C._L. Terng [119], and R. Palais and C._L. Terng [94]. In the last paper was given the fundamental theorem about classification of geometric objects. R. Palais and C._L. Terng formulate the modern definition of natural bundle [94].

A natural bundle over n-dimensional manifolds is a covariant functor $\mathcal{F}$ from the category of n-dimensional manifolds and their embeddings into the category of fibered manifolds satisfying the following conditions:

1. For every n-dimensional manifold M, $\mathcal{F}(M)$ is a fibered manifold over M and $\pi = \pi_M : \mathcal{F}(M) \longrightarrow M$ is its projection. For a point $x \in M$ we denote by $\mathcal{F}_x(M) = \pi_M^{-1}(x)$ the fibre over x.

2. *The naturality conditions.* For every embedding $\varphi : M \to N$ of two n-dimensional manifolds M, N, for the induced mapping $\mathcal{F}(\varphi) : \mathcal{F}(M) \to \mathcal{F}(N)$ the following diagram

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(N) \\ {\scriptstyle \pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\varphi} & N \end{array}$$

commutes and for every point $x \in M$ the restriction $\mathcal{F}(\varphi)_{|\mathcal{F}_x(M)} : \mathcal{F}_x(M) \to \mathcal{F}_{\varphi(x)}(N)$ is a diffeomorphism.

3. *The regularity condition.* If $\varphi_t : M \to N$ is a smooth family of embeddings, then $\mathcal{F}(\varphi_t) : \mathcal{F}(M) \to \mathcal{F}(N)$ is a smooth family of mappings.

D. B. A. Epstein and W. P. Thurson [11] proved that conditions (1) and (2) imply the regularity condition (3) (see also [71]).

Many natural bundles can be obtained as the restriction of *prolongation functors* to the category of n-dimensional manifolds and their embeddings. The notion of prolongation functor was introduced by I. Kolář (see also J. Janyška [36]).

A prolongation functor is a covariant functor $\mathcal{F}$ from the category of all manifolds and all smooth mappings into the category of fibered manifolds satisfying the similar conditions as a natural bundle. An essential difference between "natural bundle" and "prolongation functor" is that the induced mapping $\mathcal{F}(\varphi)$ is defined for all mapping φ in the case of "prolongation functor" and only for embeddings in the case of "natural bundle".

The tangent bundles, the bundles of p^r-velocities are well-known examples of prolongation functors. The cotangent bundle and the bundles of tensors of type (p, q), where $p, q > 0$, are examples of natural bundles which are not obtained by the restrictions of prolongation functors.

I. Kolář and J. Slovák [55] proved that in the case of prolongation functors the regularity condition is a consequence of conditions (1) and (2).

The very important problem is a finite determinancy of natural bundles. $\mathcal{F}$ is called *of finite order* if there is a number k such that the following implication

$$j_x^k \varphi = j_x^k \psi \Longrightarrow \mathcal{F}(\varphi)_{|\mathcal{F}_x(M)} = \mathcal{F}(\psi)_{|\mathcal{F}_x(M)}$$

holds for every mapping (embedding if $\mathcal{F}$ is a natural bundle) $\varphi, \psi : M \to N$ and every point $x \in M$. The smallest number k for which this implication holds, is called *order of* $\mathcal{F}$.

R. Palais and C._L. Terng (see [94]) show that every natural bundle $\mathcal{F}$ has a finite order (see also [58], [11], [131]). For prolongation functors a similar theorem is false (see Example 1.21). We can prove only that prolongation functors have locally finite order (see [77]). The theorem on finite order of natural bundles is called traditionally Palais-Terng theorem, but it can be deduce simply from a paper of K. Masuda [69].

Very important family of prolongation functors is formed by *product preserving functors* called also *Weil functors.*

Let $\mathcal{F}$ be a prolongation functor. For two manifolds M_1, M_2 we denote by $\pi_i : M_1 \times M_2 \to M_i$ the standard projection on the i-th factor, where $i = 1, 2$. $\mathcal{F}$ is called *product preserving functor* or *Weil functor* if the mapping

$$\big(\mathcal{F}(\pi_1), \mathcal{F}(\pi_2)\big) \colon \mathcal{F}(M_1 \times M_2) \longrightarrow \mathcal{F}(M_1) \times \mathcal{F}(M_2)$$

is a diffeomorphism for all manifolds M_1, M_2.

The tangent bundle and the bundle $J^r_0(\mathbb{R}^p, M)$ of r-jets at $0 \in \mathbb{R}^p$ of mappings $\mathbb{R}^p \to M$ are important examples of product preserving functors.

G. Kainz, P. Michor [41], O. O. Luciano [68] and D. J. Eck [9] have given characterization of product preserving functors (see also [54]).

In the differential geometry we are interested to study geometric objects on natural bundles and constructions transforming geometric objects from one natural bundle into geometric object on the second one. We introduce the general definition of liftings of geometric objects.

Let $\mathcal{F}, \mathcal{G}$ be natural bundles over n-dimensional manifolds, $n + k = \dim \mathcal{F}(\mathbb{R}^n)$ and let $\mathcal{H}$ be a natural bundle over $(n + k)$-dimensional manifolds. We denote by $\underline{\mathcal{G}}(M)$ and $\underline{\mathcal{H}}(\mathcal{F}(M))$ the families of local sections of $\mathcal{G}(M) \to M$ and $\mathcal{H}(\mathcal{F}(M)) \to \mathcal{F}(M)$ respectively. Elements of these families are called geometric $\mathcal{G}$- and $\mathcal{H}$-objects.

A lifting to $\mathcal{F}$ of geometric $\mathcal{G}$-objects from M to geometric $\mathcal{H}$-objects on $\mathcal{F}(M)$ is a family $\Lambda = \{\Lambda_M\}$ of mappings $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$ satisfying the following conditions:

1. If $\sigma \in \underline{\mathcal{G}}(M)$ is defined on an open subset $U \subset M$, then $\Lambda_M(\sigma) \in \underline{\mathcal{H}}(\mathcal{F}(M))$ is defined on $\mathcal{F}(U) \subset \mathcal{F}(M)$.

2. *The naturality condition.* For every embedding $\varphi : M \to N$, if objects $\sigma_1 \in \underline{\mathcal{G}}(M), \sigma_2 \in \underline{\mathcal{G}}(N)$ are φ-related, then $\Lambda_M(\sigma_1) \in \underline{\mathcal{H}}(\mathcal{F}(M))$, $\Lambda_M(\sigma_2) \in \underline{\mathcal{H}}(\mathcal{F}(N))$ are $\mathcal{F}(\varphi)$-related.

3. *The regularity condition.* If $\sigma_t \in \underline{\mathcal{G}}(M)$ is a smooth family of local fields of geometric objects on M, then $\Lambda_M(\sigma_t) \in \underline{\mathcal{H}}(\mathcal{F}(M))$ is also a smooth family of local fields of geometric objects on $\mathcal{F}(M)$.

If $\mathcal{F}(M) = M$ and $\mathcal{F}(\varphi) = \varphi$ we obtain the definition of a natural transformation of $\mathcal{G}$-object into $\mathcal{H}$-objects on M. The standard constructions of Riemannian connections, curvature tensors, Ricci tensors are important examples of natural transformations. A complete lift of vector fields, a complete lift of geometric structures and complete lift of connections are classical examples of liftings.

There are four interesting types of problems:

(1) to study a finitity of order of liftings;
(2) to study the regularity condition of liftings;
(3) for given natural functors $\mathcal{F}, \mathcal{G}, \mathcal{H}$, to give characterizations and classifications of liftings $\Lambda = \{\Lambda_M\}$, where $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$;
(4) to consider a particular lifting $\Lambda = \{\Lambda_M\}$, where $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are some special natural bundles, and to study properties of Λ from a point of view of a possibility of prolongations of geometric structures from a manifold M to $\mathcal{F}(M)$.

We suppose always that all manifolds in this paper are smooth, paracompact and second countable.

This paper is a survey of results on prolongation functors, natural bundles and liftings of geometric structures. There are also new results. Our references are not complete. For more complete references see the book of I. Kolář, P. Michor and J. Slovák [54].

§ 1. Natural bundles and prolongation functors.

We recall the definition of natural bundle proposed by Palais and Terng [94].

Definition 1.1. *A natural bundle* over n-dimensional manifolds is a covariant functor $\mathcal{F}$ from the category of n-dimensional manifolds and their embeddings into the category of fibered manifolds satisfying the following conditions:

(1) For every n-dimensional manifold M $\mathcal{F}(M)$ is a fibered manifold over M with a projection $\pi = \pi_M : \mathcal{F}(M) \longrightarrow M$. For a point $x \in M$ we denote by $\mathcal{F}_x(M) = \pi_M^{-1}(x)$ the fibre over x.

(2) *The naturality conditions.* For every embedding $\varphi : M \to N$ of two n-dimensional manifolds M, N, for the induced mapping $\mathcal{F}(\varphi) : \mathcal{F}(M) \to \mathcal{F}(N)$ the following diagram

$$\begin{array}{ccc} \mathcal{F}(M) & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(N) \\ {\scriptstyle \pi_M}\big\downarrow & & \big\downarrow{\scriptstyle \pi_N} \\ M & \xrightarrow{\varphi} & N \end{array}$$

commutes and for every point $x \in M$ the restriction $\mathcal{F}(\varphi)_{|\mathcal{F}_x(M)} : \mathcal{F}_x(M) \to \mathcal{F}_{\varphi(x)}(N)$ is a diffeomorphism.

(3) *The regularity condition.* If $\varphi_t : M \to N$ is a smooth family of embeddings, then $\mathcal{F}(\varphi_t) : \mathcal{F}(M) \to \mathcal{F}(N)$ is a smooth family of mappings.

A natural bundle $\mathcal{F}$ is called *linear* or *a natural vector bundle* if for every manifold M the bundle $\mathcal{F}(M)$ is a vector bundle and for every embedding $\varphi : M \to N$ the induced mapping $\mathcal{F}(\varphi) : \mathcal{F}(M) \to \mathcal{F}(N)$ is a vector bundle homomorphism. It means that $\mathcal{F}_x(\varphi)$ is linear for every point $x \in M$.

Remark 1.2. Let us observe that the commutativity of the diagram in condition (2) of our definition implies that for any point $x \in M$ the induced map $\mathcal{F}(\varphi)$ transforms the fibre $\mathcal{F}_x(M)$ into $\mathcal{F}_{\varphi(x)}(N)$. It means that the second part of condition (2) has a sense.

Remark 1.3. Let us observe that if $\mathcal{F}$ is a natural bundle then for any open subset $U \subset M$ of an n-dimensional manifold M the inclusion $\mathcal{F}(i)$ is a diffeomorphism of $\mathcal{F}(U)$ onto $\mathcal{F}(M)_{|U} = \pi_M^{-1}(U)$. Using this diffeomorphism we will identify $\mathcal{F}(U)$ with $\mathcal{F}(M)_{|U}$.

Remark 1.4. Epstein and Thurson [11] proved that conditions (1) and (2) from Definition 1.1 imply the regularity condition (see also [71]).

In the theory of geometric objects prolongation functors play very important role. They are defined in a similar way as natural bundles.

Definition 1.5. *A prolongation functor* is a covariant functor $\mathcal{F}$ from the category of all manifolds and all smooth mappings into the category of fibered manifolds satisfying the following conditions:

(1) For every manifold M, $\mathcal{F}(M)$ is a fibered manifold over M and $\pi = \pi_M : \mathcal{F}(M) \longrightarrow M$ is its projection. For a point $x \in M$ we denote by $\mathcal{F}_x(M) = \pi_M^{-1}(x)$ the fibre over x.

(2) For every mapping $\varphi : M \to N$ of two manifolds M, N, for the induced mapping $\mathcal{F}(\varphi) : \mathcal{F}(M) \to \mathcal{F}(N)$ the following diagram

$$\begin{tikzcd} \mathcal{F}(M) \arrow[r, "\mathcal{F}(\varphi)"] \arrow[d, "\pi_M"'] & \mathcal{F}(N) \arrow[d, "\pi_N"] \\ M \arrow[r, "\varphi"] & N \end{tikzcd}$$

commutes and if φ is an embedding, then for $x \in M$ the restriction $\mathcal{F}(\varphi)_{|\mathcal{F}_x(M)} : \mathcal{F}_x(M) \to \mathcal{F}_{\varphi(x)}(N)$ is a diffeomorphism.

(3) *The regularity condition.* If $\varphi_t : M \to N$ is a smooth family of mappings, then $\mathcal{F}(\varphi_t) : \mathcal{F}(M) \to \mathcal{F}(N)$ is a smooth family of mappings.

A prolongation functor $\mathcal{F}$ is called *linear* if for every manifold M the bundle $\mathcal{F}(M)$ is a vector bundle and for every mapping $\varphi : M \to N$ the induced mapping $\mathcal{F}(\varphi) : \mathcal{F}(M) \to \mathcal{F}(N)$ is a vector bundle homomorphism.

Let us observe that for every fixed natural number n the restriction of a prolongation functor to the category of n-dimensional manifolds and their embeddings is a natural bundle. Remarks 1.2 — 1.4 hold also for a prolongation functor (the property formulated in Remark 1.4 for prolongation functors was proved by I. Kolář and J. Slovák [55], [114].

We give the following list of examples of natural bundles and prolongation functors.

Example 1.6. *The tangent bundle.* TM denotes the tangent bundle of a manifold M and for a smooth mapping $\varphi : M \to N$ of two manifolds the differentiation $d\varphi : TM \to TN$ is the induced mapping. It is a prolongation functor.

Example 1.7. *The cotangent bundle.* For a manifold M we denote by T^*M the cotangent bundle and for a smooth mapping $\varphi : M \to N$ and a point $x \in M$ let $(d_x\varphi)^* : T^*_{\varphi(x)}N \to T^*_xM$ be the transposed mapping of the differential $d_x\varphi : T_xM \to T_{\varphi(x)}N$ of φ at x . If φ is an embedding of two n-dimensional manifolds M and N , then $d_x\varphi$ is an linear isomorphism, and we define $T^*_x\varphi = ((d_x\varphi)^*)^{-1} : T^*_xM \to T^*_{\varphi(x)}N$ and the induced mapping $T^*\varphi : T^*M \to T^*N$ by $(T^*\varphi)|T^*_xM = T^*_x\varphi$. T^*M is a natural bundle. The cotangent bundle T^*M is not the restriction of a prolongation functor to the category of n-dimensional manifolds and their embeddings because for $\omega \in T^*_0\mathbb{R}^n$, $\omega \neq 0$, the function $\mathbb{R}\backslash\{0\} \ni t \to T^*(t\, id_{\mathbb{R}^n})(\omega) \in T^*_0\mathbb{R}^n$ has not a limit for $t \to 0$. It is a contradiction with the regularity condition.

Example 1.8. *The bundle of linear frames.* For a manifold M we denote by LM the principal bundle of linear frames. The linear group $GL(n, \mathbb{R})$, where $n = \dim M$, is its structure group. Let $\varphi : M \to N$ be an embedding of two n-dimensional manifolds. We define an induced mapping $L\varphi : LM \to LN$ as follows: if $p = (v_1, \dots, v_n)$ is an element of LM, then p is a basis of T_xM for some point $x \in M$ and $(d_x\varphi(v_1), \dots, d_x\varphi(v_n))$ is a basis of $T_{\varphi(x)}N$ because $d_x\varphi$ is an isomorphism; we set $L(\varphi)(p) = (d_x\varphi(v_1), \dots, d_x\varphi(v_n))$. LM is a natural bundle. The induced mappings are defined only for embeddings of n-dimensional manifolds.

Example 1.9. *The bundle of frames of order* r. Let n and r be natural numbers. We denote by L^r_n *the differential group of order* r, it means

$$L^r_n = \{\ j^r_0\xi \mid \xi : U \to \mathbb{R}^n,\ U \subset \mathbb{R}^n \text{ is open, } \xi \text{ is an embedding, } \xi(0) = 0\ \}.$$

L^r_n is a Lie group with respect to multiplication of r-jets $j^r_0\xi_1 \cdot j^r_0\xi_2 = j^r_0(\xi_1 \circ \xi_2)$.

Let M be an n-dimensional manifold. We denote by $L^r(M)$ *the bundle of r-frames* over M. It means

$$L^r(M) = \{\ j^r_0\gamma \mid \gamma : U \to M,\ U \subset \mathbb{R}^n \text{ is open, } \gamma \text{ is an embedding } \}.$$

The group L^r_n acts on $L^r(M)$ on the right by the formula $j^r_0\gamma \cdot j^r_0\xi = j^r_0(\gamma \circ \xi)$. $L^r(M)$ is a principal fibre bundle with the structure group L^r_n. The principal bundle $L^r(M)$ is a natural bundle, where for an embedding $\varphi : M \to N$ the induced mapping $L^r(\varphi) : L^r(M) \to L^r(N)$ is given by $L^r(\varphi)(j^r_0\xi) = j^r_0(\varphi \circ \xi)$. If $r = 1$, L^1M coincides with the bundle of linear frames constructed in Example 1.8.

Let us observe that for $r = +\infty$ we can construct a group L^r_n (of infinite dimension) and an infinite dimensional principal fibre bundle $L^r(M)$.

Example 1.10. *The associated bundle.* Let F be an L^r_n-space, it means that F is a manifold and L^r_n acts on F on the left. For a manifold M we can define *associated bundle* $E^F(M)$ with $L^r(M)$, where $L^r(M)$ is defined in Example 1.9, and standard fibre F. We recall that $E^F(M)$ is the set of all orbits in $L^r(M) \times F$ with respect to the action of L^r_n given by $(j^r_0\gamma, y)\cdot j^r_0\xi = (j^r_0(\gamma\circ\xi), j^r_0(\xi^{-1})\cdot y)$. We denote by $< j^r_0\gamma, y >$ the orbit of $(j^r_0\gamma, y)$ and we define $\pi : E^F(M) \to M$, $\pi(< j^r_0\gamma, y >) = \gamma(0)$.

For an embeddings $\varphi : M \to M$ of two n-dimensional manifolds we define $E^P(\varphi) : E^F(M) \to E^F(N)$ by $E^F(\varphi)(< j^r_0\gamma, y >) =< j^r_0(\varphi\circ\gamma), y >$. It is a natural bundle.

Example 1.11. *The r-th order vector tangent and cotangent bundle.* Let $J^r(M,\mathbb{R})$ be the bundle of r-jets (at points of M) of smooth functions on M. It is a vector bundle with the projection $\pi : J^r(M,\mathbb{R}) \ni j^r_x f \to x \in M$. Let $J^r_x(M,\mathbb{R}) = \pi^{-1}(x)$ be the fiber over $x \in M$. For a smooth mapping $\varphi : M \to N$ of two manifolds and a point $x \in M$ of M we define the induced mappings of corresponding fibres $\varphi_{*|x} : J^r_{\varphi(x)}(N,\mathbb{R}) \longrightarrow J^r_x(M,\mathbb{R})$, $\varphi_{*|x}(j^r_{\varphi(x)}f) = j^r_x(f\circ\varphi)$. (Let us observe a contravariant character of this mapping.) Let $\beta : J^r(M,\mathbb{R}) \to \mathbb{R}$ be *the target projection* given by $\beta(j^r_x f) = f(x)$. Because of β is linear on fibers of $J^r(M,\mathbb{R})$, thus

$$J^r(M,\mathbb{R})_0 = \ker\beta = \{\ j^r_x f\ :\ f \text{ is a function on } M,\ x \in M, f(x) = 0\ \}$$

is a vector subbundle of $J^r(M,\mathbb{R})$. Let $J^r_x(M,\mathbb{R})_0$ be the fiber of $J^r(M,\mathbb{R})_0$ over $x \in M$. It is easy to observe that $\varphi_{*|x}(J^*_{\varphi(x)}(N,\mathbb{R})_0) \subset J^*_x(M,\mathbb{R})_0$.

If $\varphi : M \to N$ is an embeddings of two n-dimensional manifolds, then $\varphi_{*|x}$ is an isomorphism and we can define $J^r(\varphi,\mathbb{R}) : J^r(M,\mathbb{R}) \to J^r(N,\mathbb{R})$ by the formula

$$J^r(\varphi,\mathbb{R})|J^r_x(M,\mathbb{R}) = (\varphi_{*|x})^{-1}.$$

It is clear that $J^r(\varphi,\mathbb{R})\big(J^r(M,\mathbb{R})_0\big) \subset J^r(N,\mathbb{R})_0$.

$J^r(M,\mathbb{R})$ and $J^r(M,\mathbb{R})_0$ are natural bundles called *extended r-th order vector cotangent bundle* and *r-th order vector cotangent bundle* respectively. They are not prolongation functors,

Let us observe that for any manifold M we have a natural isomorphism (equivariance) of functors $J^r(M,\mathbb{R})_0 \times \mathbb{R} = J^r(M,\mathbb{R})$ given by $(j^r_x f, a) \to j^r_x(f + a)$, $f(x) = 0$.

Let

$$\widetilde{T}^r(M) = \big(J^r(M,\mathbb{R})\big)^*, \qquad T^rM = \big(J^r(M,\mathbb{R})_0\big)^*$$

be dual vector bundles. $\widetilde{T}^r(M)$ is called *extended r-th order vector tangent bundle* and T^rM — *r-th order vector tangent bundle.* T^rM is a vector subbundle of $\widetilde{T}^r(M)$ and $\widetilde{T}^r(M) = T^rM \times \mathbb{R}$. Let us observe that $T^1(M) = TM$ is the tangent bundle (the case $r = 1$).

For a mapping $\varphi : M \to N$ and a point $x \in M$ we define the induced mapping $\widetilde{T}^r(\varphi)$: $\widetilde{T}^r(M) \longrightarrow \widetilde{T}^r(N)$ by the condition that the restriction $\widetilde{T}^r(\varphi)_{\widetilde{T}^r_x(M)} : \widetilde{T}^r_x(M) \longrightarrow \widetilde{T}^r_{\varphi(x)}(N)$ is the transposed mapping to $\varphi_{*|x}$. It is easy to see that $\widetilde{T}^r(M)$ is a linear prolongation functor. Because of $\widetilde{T}^r(\varphi)(T^rM) \subset T^rN$, we can define $T^r\varphi : T^rM \to T^rN$ as the restriction of $\widetilde{T}^r(\varphi)$. T^rM is also a linear prolongation functor (cf. also [42]).

Example 1.12. *The bundle of connections.* Let n be a natural number and let $Q = R^{n^3}$. We define an action on the left of the group L^2_n on Q as follows: let $\gamma = [\gamma^i_{jk}]^{i=1,\dots,n}_{jk=1,\dots,n}$, $\bar\gamma = [\bar\gamma^i_{jk}]^{i=1,\dots,n}_{jk=1,\dots,n}$ be two elements of Q and A^i_j, A^i_{jk} be the coordinates of an element ξ of L^2_n; we define $\bar\gamma = \xi \cdot \gamma$ if and only if

$$\bar\gamma^i_{jk} = \sum_{p,q,r=1}^{n} A^i_p\, \gamma^p_{qs}\, \widetilde{A}^q_j\, \widetilde{A}^s_k + \sum_{p=1}^{n} A^i_p\, \widetilde{A}^p_{jk} \qquad \text{for } i,j,k = 1,\dots,n\,,$$

where $(\widetilde{A}^i_j, \widetilde{A}^i_{jk})$ are the coordinates of ξ^{-1} (transformation rules of Christoffel's symbols). Let $\mathcal{Q}(M)$ be the associated bundle with the principal bundle L^2M of frames of order 2 and the standard fibre Q (see Example 1.10). $\mathcal{Q}(M)$ is called *a bundle of linear connections.* Smooth sections of $\mathcal{Q}(M)$ are linear connections on M.

Example 1.13. *Jet prolongation of natural bundle.* Let $\mathcal{G}$ be a natural bundle over n-dimensional manifolds and let k be a non-negative integer. For a manifold M we denote by $\underline{\mathcal{G}}(M)$ the set of local smooth sections of $\mathcal{G}(M)$ with open domains. We define a new bundle $J^k\mathcal{G}$ as follows:

For an n-dimensional manifold M we set

$$J^k\mathcal{G}(M) = \left\{\, j^k_x\sigma \mid \sigma \in \underline{\mathcal{G}}(M),\ x \in \text{ domain of } \sigma,\ x \in M \,\right\}$$

and let $\pi^k : J^k\mathcal{G}(M) \to M$ be a projection given by $\pi^k(j^k_x\sigma) = x$. For a point $x \in M$ we denote, as usual, $J^k_x\mathcal{G}(M) = (\pi^k)^{-1}(x)$ the fibre over x.

For an embedding $\varphi : M \to N$ of two n-dimensional manifolds M, N we define the induced mapping $J^k\mathcal{G}(\varphi) : J^k\mathcal{G}(M) \to J^k\mathcal{G}(N)$ by the formula

$$(J^k\mathcal{G}(\varphi))(j^k_x\sigma) = j^k_{\varphi(x)}(\mathcal{G}(\varphi) \circ \sigma \circ \varphi^{-1})\,.$$

$J^k\mathcal{G}(M)$ is called *k-jet prolongation* of $\mathcal{G}(M)$. $J^k\mathcal{G}$ is a new natural bundle.

If $k = +\infty$ then we can also define $J^k\mathcal{G}(M)$ and $J^k\mathcal{G}(M)$ verifies all conditions from definition of natural bundles instead of the fact that $J^k\mathcal{G}(M)$ is a finite dimensional manifold.

Using the identification from Remark 1.3 we can prove

Proposition 1.14. *If $\mathcal{F}$ is a natural bundle (respectively a prolongation functor), then for two embeddings (respectively two mappings if $\mathcal{F}$ is a prolongation functor) $\varphi, \psi : M \to N$ and for any open subset $U \subset M$ we have the following implication*

$$\varphi_{|U} = \psi_{|U} \Longrightarrow \mathcal{F}\varphi_{|\pi^{-1}(U)} = \mathcal{F}\psi_{|\pi^{-1}(U)} .$$

Proof. Let $i : U \to M$ be the inclusion. Then the equality $\varphi_{|U} = \psi_{|U}$ is equivalent to $\varphi \circ i = \psi \circ i$. From the naturality condition we obtain $\mathcal{F}(\varphi) \circ \mathcal{F}(i) = \mathcal{F}(\psi) \circ \mathcal{F}(i)$. It means that $\mathcal{F}\varphi_{|\pi^{-1}(U)} = \mathcal{F}\psi_{|\pi^{-1}(U)}$. □

Remark 1.15. The property formulated in Proposition 1.14 is called often *the locality condition.* This condition is supposed often in the definition of natural bundle instead of the second part of condition (2) (see [54])

Proposition 1.16. *For every natural bundle $\mathcal{F}$, $\mathcal{F}(\mathbb{R}^n)$ is isomorphic with the trivial bundle $\mathbb{R}^n \times F$, where $F = \mathcal{F}_0(\mathbb{R}^n)$.*

Proof. Let $\tau_a : \mathbb{R}^n \to \mathbb{R}^n$ be the translation, $\tau_a(x) = x + a$. Now the mapping $I : \mathbb{R}^n \times F \to \mathcal{F}(\mathbb{R}^n)$ given by $I(a, y) = \mathcal{F}(\tau_a)(y)$ is a bundle isomorphism. □

Using the standard methods (so-called Borel Lemma) we can prove (see [94]).

Proposition 1.17. *Let $\mathcal{F}$ be either a prolongation functor or a natural bundle. For two mappings (embeddings if $\mathcal{F}$ is a natural bundle) $\varphi, \psi : M \to N$ ($\dim M = \dim N = n$ if $\mathcal{F}$ is a natural bundle) and for a point $x \in M$ we have the following implication*

$$j^\infty_x \varphi = j^\infty_x \psi \Longrightarrow \mathcal{F}(\varphi)_{|\mathcal{F}_x(M)} = \mathcal{F}(\psi)_{|\mathcal{F}_x(M)} .$$

Now we can propose a following definition. $\mathcal{F}$ is called *of finite order* if there is a number k such that the following implication

$$j^k_x \varphi = j^k_x \psi \Longrightarrow \mathcal{F}(\varphi)_{|\mathcal{F}_x(M)} = \mathcal{F}(\psi)_{|\mathcal{F}_x(M)}$$

holds for every mappings (embeddings if $\mathcal{F}$ is a natural bundle) $\varphi, \psi : M \to N$ and every point $x \in M$. The smallest number k for which this implication holds, is called *order of $\mathcal{F}$*.

All bundles constructed in Examples 1.6 — 1.13 have finite orders. Namely, the tangent and cotangent bundle are of order 1, the bundle of frames of order r is of order r, the bundle associated with $L^r M$ is at most of order r, the r-th order (extended) tangent and cotangent bundles are of order r, the bundle of linear connections is of order 2 and if $\mathcal{G}$ is of finite order r then its jet-prolongation $J^k\mathcal{G}$ is of order $r + k$.

We have the following important theorem (see [94]):

Theorem 1.18 (Palais, Terng). *Every natural bundle $\mathcal{F}$ has a finite order. The order of $\mathcal{F}$ is less than or equal $2^f + 1$, where $f = \dim\, \mathcal{F}_0(\mathbb{R}^n)$.*

In the particular case when for every manifold M $\mathcal{F}(M)$ is a principal fibre bundle the above theorem was proved earlier by D. Krupka [58].

Palais and Terng have been proved this theorem using the regularity condition. Epstein and Thurston [11] proved this theorem in the following version:

Theorem 1.19 (Epstein, Thurston). *If $\mathcal{F}$ is a covariant functor from Definition 1.1 satisfying conditions (1) and (2), then $\mathcal{F}$ satisfies also (3) (i.e. it is a natural bundle) and $\mathcal{F}$ has a finite order $k \leq 2f+1$, where $f = \dim\, \mathcal{F}_0(\mathbb{R}^n)$.*

The sharp estimation of order of natural bundles was obtained by Zajtz [131].

Theorem 1.20 (Zajtz). *Let $\mathcal{F}$ be a natural bundle over n-dimensional manifolds, $n \geq 2$, with a standard fibre of dimension k. Then the order r of $\mathcal{F}$ satisfies the inequality $r \leq \max(\frac{k}{n-1}, \frac{k}{n}+1)$.*

Let us observe that similar theorem about finite order for prolongation functors is false. We give an example (Mikulski [74]).

Example 1.21. *There exist a prolongation functor of infinite order.* We recall that $T^r(M)$ denotes the r-th order vector tangent bundle (see Example 1.11). Let $d_r = \dim T_0^r(\mathbb{R}^r)$. We set

$$\mathcal{F}(M) = \bigoplus_{k=1}^{+\infty} \bigwedge\nolimits^{d_k} T^k M$$

For every manifold M, $\mathcal{F}(M)$ is a finite dimensional manifolds because for $k > \dim M$ the bundle $\bigwedge^{d_k} T^k M$ is zero-bundle. Hence the direct sum in the definition of $\mathcal{F}(M)$ is in reality a finite sum. For a mapping $\varphi : M \to N$ the induced mapping $\mathcal{F}(\varphi) : \mathcal{F}(M) \to \mathcal{F}(N)$ is defined in the natural way from $\bigwedge^{d_k} T^k(\varphi)$.

The prolongation functor $\mathcal{F}$ is of infite order because its restriction to the category of k-dimensional manifolds is at least of order k (see [74]).

A prolongation functor has locally a finite order in the sense of the following theorem (see Mikulski [77]).

Theorem 1.22. *Every prolongation functor $\mathcal{F}$ has locally a finite order. More precisely, for any manifolds M, N, any point $x \in M$ and any mappings $f, g : M \to N$ the condition $j_x^{r(\dim M+1)} f = j_x^{r(\dim M+1)} g$ implies $\mathcal{F}_x(f) = \mathcal{F}_x(g)$, where $r(m)$ denotes the order of the natural bundle obtained by the restrictions of $\mathcal{F}$ to the category of m-dimensional manifolds and their embeddings.*

Theorem 1.18 (or 1.19) gives a characterization of natural bundles. Before presenting this characterization we introduce the following notations.

Let $\mathcal{F}$ be a natural bundle over n-dimensional manifolds. According to Theorem 1.18, $\mathcal{F}$ has a finite order r. Let $F = \mathcal{F}_0(\mathbb{R}^n)$. F is called *standard fibre* of $\mathcal{F}$. We can define an action on the left of L_n^r on F by the formula

$$j_0^r(\xi) \cdot y = \mathcal{F}(\xi)(y)\,, \tag{1.1}$$

where $y \in F$ and $j_0^r \xi \in L_n^r$.

Now we can consider the associated bundle $E^F(M)$ (see Example 1.10). This new natural bundle is equivalent to the given bundle $\mathcal{F}$. A diffeomorphism $\Omega_M : E^F(M) \longrightarrow \mathcal{F}(M)$ is given by

$$\Omega_M(< j_0^r\gamma, y >) = \mathcal{F}(\gamma)(y)\,.$$

For every embedding $\varphi : M \to N$ the following diagram

$$\begin{array}{ccc} E^F(M) & \xrightarrow{E^F(\varphi)} & E^F(N) \\ \Omega_M \big\downarrow & & \big\downarrow \Omega_N \\ \mathcal{F}(M) & \xrightarrow{\mathcal{F}(\varphi)} & \mathcal{F}(N) \end{array}$$

commutes.

Thus, we have the following theorem (Palais, Terng [94], see also [21]):

Theorem 1.23 (Palais, Terng). *Every natural bundle $\mathcal{F}$ is equivalent with the associated bundle E^F, where $F = \mathcal{F}_0(\mathbb{R}^n)$ is the standard fibre of $\mathcal{F}$ with the action of L^r_n given by (1.1) and r is order of $\mathcal{F}$.*

For an arbitrary prolongation functor there is no complete characterization analogical to the characterization given in Theorem 1.23. We know the following result (see [55] and [76]).

Proposition 1.24. *If $\mathcal{F}$ is a prolongation functor such that for every natural number n $\mathcal{F}_0(\mathbb{R}^n)$ is compact, then for every manifold M $\mathcal{F}(M)$ is isomorphic with the trivial bundle $M \times F$, where $F = \mathcal{F}(\{0\})$. The natural isomorphism $i_M : \mathcal{F}(M) \to M \times F$ is given by $i_M(y) = (\pi_M(y), \mathcal{F}(0_M)(y))$, where $0_M : M \to \{0\}$ is the constant mapping. For every smooth mapping $\varphi : M \to N$ we have $i_N \circ \mathcal{F}(\varphi) \circ i_M^{-1} = \varphi \times id_F$.*

§ 2. Product preserving functors.

The concept of product preserving functor is very important in differential geometry and in theory of natural bundles especially. Many functors fulfils the property of product preserving.

First we recall the definition of product preserving functor.

Definition 2.1. Let $\mathcal{F}$ be a prolongation functor. For two manifolds M_1, M_2 we denote by $\pi_i : M_1 \times M_2 \to M_i$ the standard projection onto the i-th factor, where $i = 1, 2$. $\mathcal{F}$ is called *product preserving functor* if the mapping

$$\big(\mathcal{F}(\pi_1), \mathcal{F}(\pi_2)\big) : \mathcal{F}(M_1 \times M_2) \longrightarrow \mathcal{F}(M_1) \times \mathcal{F}(M_2)$$

is a diffeomorphism for all manifolds M_1, M_2.

For a product preserving functor $\mathcal{F}$ we will always identify $\mathcal{F}(M_1 \times M_2)$ with $\mathcal{F}(M_1) \times \mathcal{F}(M_2)$ by the natural diffeomorphism from the definition. After this identification we have

$$\mathcal{F}(f_1 \times f_2) = \mathcal{F}f_1 \times \mathcal{F}f_2$$
$$\mathcal{F}(f, g) = (\mathcal{F}f, \mathcal{F}g)$$

for smooth mappings f, f_1, f_2, g. We have also that $\mathcal{F}(point) = point$.

Next we present some important examples of product preserving functors.

Example 2.2. *The tangent bundle.* It is a product preserving functor.

Example 2.3. *The tangent bundle of p^r-velocities.* For a manifold M we define the set $T_p^r M = J_0^r(\mathbb{R}^p, M)$ of r-jets at 0 of mappings $\mathbb{R}^p \to M$. $T_p^r M$ is a fibre bundle with projection $\pi : T_p^r M \to M$ given by $\pi(j_0^r \gamma) = \gamma(0)$. It is called *the tangent bundle of p^r-velocities.* If $\varphi : M \to N$ is a smooth mapping then the induced mapping $T_p^r \varphi : T_p^r M \to T_p^r N$ is defined by $T_p^r(\varphi)(j_0^r \gamma) = j_0^r(\varphi \circ \gamma)$. It is a product preserving functor.

The most important and general example of product preserving functor is the functor called *Weil functor.*

To construct the Weil functor first we present the definition of local algebra or Weil algebra.

Let A be an associative algebra with unit over $\mathbb{R}$. We call A *a local algebra* or *a Weil algebra* if A is commutative and of finite dimension over $\mathbb{R}$ admitting a unique maximal ideal $\mathfrak{m}$ such that $A/\mathfrak{m}$ is of dimension 1 over $\mathbb{R}$ and that $\mathfrak{m}^{h+1} = 0$ for a non-negative integer h. The smallest h such that $\mathfrak{m}^{h+1} = 0$ will be called the height of A. In other words the Weil algebra is a real commutative and associative algebra with unit which is of the form $A = \mathbb{R} \cdot 1 \oplus N$, where N is a finite dimensional ideal of nilpotent elements.

Let $\mathbb{R}[p] = \mathbb{R}[[X_1, ..., X_p]]$ be the algebra of all formal power series of p indeterminantes $X_1, ..., X_p$ and let $\mathfrak{m}_p$ be the maximal ideal of $\mathbb{R}[p]$ consisting of all formal power series without constant terms. Let $\mathfrak{a}$ be an ideal of $\mathbb{R}[p]$ such that $\dim \mathbb{R}[p]/\mathfrak{a} < \infty$. The algebra $A = \mathbb{R}[p]/\mathfrak{a}$ is a local algebra with the maximal ideal $\mathfrak{m} = \mathfrak{m}_p/a$. A. Weil has proved in [121] that every Weil algebra is isomorphic to some algebra of this type.

It is possible to formulate some other equivalent definitions of Weil algebra (see [54]).

Theorem 2.4. *The following assertions are equivalent:*

(1) *A is a Weil algebra;*
(2) *A is a finite dimensional quotient of the algebra $\mathbb{R}[p]$ of formal power series;*
(3) *A is a finite dimensional quotient of the algebra $\mathbb{R}[X_1, ...X_p]$ of polynomials;*
(4) *A is a finite dimensional quotient of the algebra $J_0^k(\mathbb{R}^p, \mathbb{R})$ of jets at 0 of mappings from $\mathbb{R}^p$ into $\mathbb{R}$;*
(5) *A is a finite dimensional quotient of the algebra $\mathcal{E}_p = C_0^\infty(\mathbb{R}^p, \mathbb{R})$ of germs at zero of mappings from $\mathbb{R}^p$ into $\mathbb{R}$.*

Proposition 2.5. *If M is a smooth manifold and I is an ideal of finite codimension in the algebra $C^\infty(M, \mathbb{R})$ then $C^\infty(M, \mathbb{R})/I$ is a direct sum of finitely many Weil algebras.*

Now we construct the *Weil functor.* Let $A = \mathbb{R}[p]/\mathfrak{a}$ be a Weil algebra with the maximal ideal $\mathfrak{m} = \mathfrak{m}_p/a$ and let $\xi_A : \mathbb{R}[p] \to A$ be the natural projection.

We denote by $\tau = \tau_p : C^\infty(\mathbb{R}^p, \mathbb{R}) \to \mathbb{R}$ the formal Taylor expansion at the origin $t = 0$, i.e.

$$\tau(f) = \sum_\nu \frac{1}{\nu!}[(\frac{\partial}{\partial t})^\nu f]_{t=0} X^\nu$$

for $f \in C^\infty(\mathbb{R}^p, \mathbb{R})$, where $\nu = (\nu_1, ..., \nu_p)$ is a p-tuple of non-negative integers.

Now we define a relation in $\mathcal{C}^\infty(\mathbb{R}^p, M)$ similar to the relation of jets. Let φ and ψ be smooth mappings of $\mathbb{R}^p$ into M. We shall call φ is A-equivalent to ψ if

$$\xi_A(\tau(f \circ \varphi)) = \xi_A(\tau(f \circ \psi))$$

for every $f \in \mathcal{C}^\infty(M)$ and we denote this by $\varphi \sim_A \psi$. The relation $\sim_A$ is clearly an equivalence relation in the set $C^\infty(\mathbb{R}^p, M)$. We denote by $j^A\varphi$ or $[\varphi]_A$ the equivalence class containing φ and call it "A-point". By $T^A M$ we denote the set of all equivalence classes in $C^\infty(\mathbb{R}^p, M)$ and by $\pi_A : T^A M \to M$, $\pi_A(j^A\varphi) = \varphi(0)$ the natural projection. For $x \in M$ we write $\pi_A^{-1}(x) = T^A_x M$.

The space $T^A M$ admits a structure of differentiable manifold:

Take a coordinate system $(U, x^1, \dots, x^n)$ in M and a basis $B_1, \dots, B_K$ of A over $\mathbb{R}$. We shall define a coordinate system $(\pi_A^{-1}(U), x^{i,\nu} : i = 1, \dots, n; \nu = 1, \dots, K)$ by

$$\xi_A(\tau(x^i \circ \varphi)) = \sum_{\nu=1}^{K} a^{i,\nu} B_\nu , \qquad x^{i,\nu}(j^A\varphi) = a^{i,\nu}$$

for $j^A\varphi \in \pi_A^{-1}(U)$. Hence the set $T^A M$ becomes a structure of differentiable manifold of dimension $n \dim A$. This structure does not depend on the chosen basis $\{B_\nu\}$ of A.

The differentiable manifold $T^A M$ with the projection $\pi_A : T^A M \to M$ will be called *the Weil bundle* of M or *the bundle of A-points* of M (see Morimoto [90]).

Let $f : M \to M'$ be a mapping of a manifold M into a manifold M'. Then we have induced mapping $T^A f : T^A M \to T^A M'$ defined by $T^A f(j^A\varphi) = j^A(f \circ \varphi)$. Clearly, $T^A f$ is differentiable.

The correspondence $M \to T^A M,\ f \to T^A f$ is a product preserving functor called *Weil functor.*

The concept of Weil functor generalizes the concept of tangent bundle functor and, more generally the concept of functor of p^r-velocities.

Remark 2.6. Let we take $\mathfrak{a} = (\mathfrak{m}_p)^{r-1}$, where $\mathfrak{m}_p$ is the maximal ideal in $\mathbb{R}[p]$, and $A = \mathbb{R}[p]/\mathfrak{a}$. Then we see that the notion of A-points is nothing but the notion of p^r-velocities. In particular, if $\mathfrak{a} = (\mathfrak{m}_1)^2$, $D = \mathbb{R}[1]/\mathfrak{a}$, then the notion of D-points is nothing but the notion of tangent vectors on M. The algebra D is called the algebra of dual numbers.

There are other descriptions of the space $T^A M$ presented for example in papers of Morimoto (see [84]) and the book of Kolář, Michor, and Slovák [54].

Take an element $j^A\varphi \in T^A_x M$ and consider the map $\varphi' : \mathcal{C}^\infty(M) \to A$ defined by

$$\varphi'(f) = \xi_A(\tau(f \circ \varphi)), \qquad \text{for } f \in \mathcal{C}^\infty(M) .$$

The mapping φ' is an homomorphism from the algebra $C^\infty(M)$ into A, i.e. φ' is an element of $Hom(C^\infty(M), A)$. The correspondence $j^A\varphi \to \varphi'$ is a bijective mapping of $T^A M$ to $Hom(\mathcal{C}^\infty(M), A)$. Let $\kappa_{A,M}$ be the inverse mapping. We have the following proposition [54].

Proposition 2.7. *There are bijective mappings $\kappa_{A,M} : Hom(C^\infty(M), A) \to T^A M$ for all smooth manifolds and Weil algebras A, which are natural in M and A.*

Now we present the main properties of Weil functors. First we note some simple propositions.

Proposition 2.8. *1. For a manifold M the triple $(T^A M, \pi_A, M)$ is a smooth fibre bundle with standard fibre $N^{\dim M}$, where N is the maximal ideal of nilpotents in A. T^A is a product preserving functor.*

2. The functor T^A maps the following classes of mappings into itself: immersions, embeddings, submersions, surjective submersions, constant mappings and fibre bundle projections.

Let us remark that an arbitrary prolongation functor preserve immersions and submersions (see [55]).

For vector spaces, vector bundles, Lie groups and its actions, and principal fibre bundles we have

Proposition 2.9. *1. Let V be a finite dimensional real vector space. Then the space T^AV becomes canonically a real vector space of dimension $\dim V \cdot \dim A$. The sum $+_{T^AV}$ and the scalar multiplication $(m_a)_{T^AV}$ in T^AV are $T^A(+_V)$ and $T^A((m_a)_V)$ respectively, where $+_V : V \times V \ni (v, w) \to v + w \in V$ and $(m_a)_V : V \ni v \to av \in V$ are the corresponding mappings in V. $T^A(0)$ is the zero of T^AV, where we identify the constant mappings with their images.*

2. If $f : V \to W$ is a linear mapping of two finite dimensional vector spaces, then $T^Af : T^AV \to T^AW$ is also linear.

3. If $\pi : E \to M$ is a vector bundle, then $T^A\pi : T^AE \to T^AM$ is a vector bundle too. If $\Psi : E \to E'$ is a vector bundle homomorphism, then $T^A\Psi : T^AE \to T^AE'$ is also a vector bundle homomorphism.

4. If $E \oplus E'$ is the direct sum of vector bundles E, E' with the same base M, then $T^A(E' \oplus E') = T^AE \oplus T^AE'$ is the direct sum of vector bundles with the base T^AM.

Proposition 2.10. *1. Let G be a Lie group with group multiplication $\mu : G \times G \to G$. Then the manifold T^AG becomes a Lie group with group multiplication $T^A(\mu)$ and $T^A(1_G)$ is the unity of T^AG.*

2. If $f : G \to G'$ is a Lie group homomorphism, then $T^Af : T^AG \to T^AG'$ is also a Lie group homomorphism

3. If G is a Lie group and $\mathfrak{g}$ is its Lie algebra with the bracket $[\ ,\ ]$ then $T^A(\mathfrak{g})$ is Lie algebra of the Lie group T^AG with the bracket $T^A([\ ,\ ])$.

4. If $\alpha : G \times M \to M$ is an action of a Lie group G on M, then $T^A(\alpha) : T^AG \times T^AM \to T^AM$ defines an action of T^AG on T^AM. Particularly, if $\rho : GL(n) \times \mathbb{R}^n \to \mathbb{R}^n$ is the usual action of $GL(n)$ on $\mathbb{R}^n$, then $T^A(GL(n))$ acts on the vector space $T^A\mathbb{R}^n$ by $T^A\rho$ as a linear transformation group.

5. If a Lie group G operates on a manifold M transitively (respectively freely), then the group T^AG operates in T^AM also transitively (respectively freely).

Proposition 2.11. *1. If $P(M, G, \pi)$ is a principal fibre bundle with base M, structure group G and projection π, then T^AP is a principal fibre bundle with base T^AM, structure group T^AG and projection $T^A\pi$.*

2. If $f : P(M, G) \to P'(M', G')$ is a principal fibre bundle homomorphism, then $T^Af : T^AP(T^AM, T^AG) \to T^AP'(T^AM', T^AG')$ is a principal fibre bundle homomorphism.

3. If $P_0(M, H)$ is a reduced fibre bundle in a principal fibre bundle $P(M, G)$ with structure subgroup $H \subset G$, then $T^AP_0(T^AM, T^AH)$ is a reduced fibre bundle in $T^AP(T^AM, T^AG)$.

4. If $P_1 \oplus P_2$ is the Whitney sum (fibre product) of two principal fibre bundles with the same base M, then $T^A(P_1 \oplus P_2)$ is the Whitney sum of fibre bundles T^AP_1 and T^AP_2 with the base T^AM.

5. If $E = E(M, F, G, P, \pi)$ is the associated fibre bundle with a principal fibre bundle $P(M, G)$, then T^AE is associated with $T^AP(T^AM, T^AG)$.

For two Weil algebras we can write the following remarks

Proposition 2.12. *Let A and B be Weil algebras. Then we have:*

1. $T^A\mathbb{R}$ can be identified with A. $T^A\mathbb{R}$ is the algebra with addition $T^A(+_{\mathbb{R}})$, multiplication $T^A(\cdot_{\mathbb{R}})$ and scalar multiplication $T^A(m_t) : A \to A$. If A is a quotient of

$J^\infty_0(\mathbb{R}^p,\mathbb{R})$ with the projection $p : J^\infty_0(\mathbb{R}^p,\mathbb{R}) \to A$, then $T^A\mathbb{R} \ni j^A\gamma \to p(j^\infty_0\gamma) \in A$ is the identification.

2. Any algebra homomorphism $\varphi : A \to B$ induces a natural transformation $T_\varphi : T^A \to T^B$ between Weil functors. For every manifold M the following diagram

$$\begin{array}{ccc}
T^AM & \xrightarrow{T_{\varphi,M}} & T^BM \\
{\scriptstyle\kappa_{A,M}}\uparrow & & \uparrow{\scriptstyle\kappa_{B,M}} \\
Hom(\mathcal{C}^\infty(M),A) & \xrightarrow{\xi\to\varphi\circ\xi} & Hom(\mathcal{C}^\infty(M),B)
\end{array}$$

commutes. If φ is injective then $T_{\varphi,M} : T^AM \to T^BM$ is an immersion for each manifold M. If φ is surjective, then $T_{\varphi,M}$ is a fibre bundle epimorphism.

3. A natural transformation $\Psi : T^A \to T^B$ corresponds exactly to the algebra homomorphisms $\Psi_\mathbb{R} : T^A\mathbb{R} = A \to T^B\mathbb{R} = B$.

4. The algebra T^AB is identified with $A \otimes B$.

5. There is a canonical natural equivalence $\eta_{B,A} : T^A \circ T^B \to T^B \circ T^A$ corresponding to the isomorphism $A \otimes B \to B \otimes A$ and the following diagram is commutative

$$\begin{array}{ccc}
T^AT^B(M) & \xrightarrow{\eta_{B,A}(M)} & T^BT^A(M) \\
{\scriptstyle T^AT^Bf}\downarrow & & \downarrow{\scriptstyle T^BT^Af} \\
T^AT^B(M') & \xrightarrow{\eta_{B,A}(M')} & T^BT^A(M')
\end{array}$$

We will use the isomorphism $\eta_{A,B}$ in the case when one of functors T^A, T^B is the tangent bundle. In this case we have a natural isomorphism $\eta_M = \eta_{D,A} : T^A(TM) \to T(T^AM)$ of vector bundles such that the diagram is commutative

$$\begin{array}{ccc}
T^A(TM) & \xrightarrow{\eta_M} & T(T^AM) \\
T^A(\pi_M) \searrow & & \swarrow \pi_{T^AM} \\
 & T^AM &
\end{array},$$

where π denotes the tangent bundle projection and D is the Weil algebra corresponding to the tangent bundle (see Remark 2.6).

In 1986 Eck [9], Kainz, Michor [41] and Luciano [68] have proved independently that any product preserving functor is in fact equivalent to some Weil functor.

Theorem 2.13. *If $\mathcal{F}$ is a product preserving functor, then there exists a Weil algebra A such that $\mathcal{F} = T^A$.*

The main idea of the proof of this theorem is to extract a Weil algebra from a product preserving functor $\mathcal{F}$. If we denote the addition and the multiplication on $\mathbb{R}$ by $+, \cdot : \mathbb{R}^2 \to \mathbb{R}$ and for $\lambda \in \mathbb{R}$ denote $m_\lambda : \mathbb{R} \to \mathbb{R}$ the scalar multiplication by λ then we can formulate the key lemma.

Lemma 2.14. *Let $\mathcal{F}$ be a product preserving functor. Then $\mathcal{F}(\mathbb{R})$ is a finite dimensional real commutative algebra with operations $\mathcal{F}(+)$, $\mathcal{F}(\cdot)$, $\mathcal{F}(m_\lambda)$, zero $\mathcal{F}(0)$ and unit $\mathcal{F}(1)$. If $\varphi : \mathcal{F}_1 \to \mathcal{F}_2$ is a natural transformation between two such functors, then $\varphi_\mathbb{R} : \mathcal{F}_1(\mathbb{R}) \to \mathcal{F}_2(\mathbb{R})$ is an algebra homomorphism.*

Using Theorem 2.13 we can formulate Propositions 2.8 — 2.12 for any product preserving functor $\mathcal{F}$ instead of T^A. For example, from Proposition 2.12(5) and Theorem 2.13 we obtain

Proposition 2.15. *For two product preserving functors $\mathcal{F}_1$ and $\mathcal{F}_2$ there exists a natural isomorphism $\eta : \mathcal{F}_1 \circ \mathcal{F}_2 \to \mathcal{F}_2 \circ \mathcal{F}_1$.*

There is an interesting characterization of product preserving functors given by Kolář and Slovák [55].

Theorem 2.16. *If $\mathcal{F}$ is a prolongation functor with the point property (i.e. $\mathcal{F}(point) = point$), then there exists a sequence $k_n, (n = 1, 2, ...)$ of natural numbers such that*

1) $\mathcal{F}_0\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^{k_n}$;

2) $k_{n_1+n_2} \geq k_{n_1} + k_{n_2}$ for all $n_1, n_2 \in \mathbb{N}$

3) $k_{n_1+n_2} = k_{n_1} + k_{n_2}$ for all $n_1, n_2 \in \mathbb{N}$ if and only if $\mathcal{F}$ is a product preserving functor.

In particular, $\mathcal{F}$ is a product preserving functor if and only if for all n we have $\dim(\mathcal{F}_0\mathbb{R}^n) = n \dim(\mathcal{F}_0\mathbb{R})$.

For product preserving functors there is possible to construct liftings of many geometric objects for example tensor fields, connections and others. These constructions are very natural and generalize special constructions of liftings for some functors. For more details see [84], [121], [54].

§3. Liftings of geometric objects. Formulation of problems.

Let $\mathcal{F}$ be a natural bundle over n-dimensional manifolds. Let r be the order of $\mathcal{F}$, $F = \mathcal{F}_0(\mathbb{R}^n)$ be the standard fibre of $\mathcal{F}$ and let $k = \dim F$. (Now, $\dim \mathcal{F}(M) = n + k$.) We are interested in *liftings (prolongations)* of some type of geometric objects from M to geometric objects (of the same or other type) on $\mathcal{F}(M)$. We introduce the following definition of liftings.

Sections of a natural bundle $\mathcal{G}(M)$ are called *fields of geometric $\mathcal{G}$-objects*, or shortly *geometric objects*, on a manifold M. We denote by $\underline{\mathcal{G}}(M)$ the set of local fields of geometric $\mathcal{G}$-objects on M, i.e. $\underline{\mathcal{G}}(M)$ is the set of all smooth sections of $\mathcal{G}(M)$ with open domains in M.

Let $\varphi : M \to N$ be an embedding. The geometric objects $\sigma_1 \in \underline{\mathcal{G}}(M)$ and $\sigma_2 \in \underline{\mathcal{G}}(N)$ with domains U_1, U_2 of σ_1 and σ_2 respectively are called *φ-related* (or *related with respect to φ*) if we have $\varphi(U_1) \subset U_2$ and the following diagram

$$\begin{array}{ccc} \mathcal{G}(U_1) & \xrightarrow{\mathcal{G}(\varphi)} & \mathcal{G}(U_2) \\ {\scriptstyle\sigma_1}\uparrow & & \uparrow{\scriptstyle\sigma_2} \\ U_1 & \xrightarrow{\varphi} & U_2 \end{array}$$

commutes.

Let us observe that if $\mathcal{G}$ is a prolongation functor, then we can define φ-related objects for an arbitrary mapping $\varphi : M \to N$.

Definition 3.1. Let $\mathcal{F}$ be a natural bundle over n-dimensional manifolds and $n + k = \dim \mathcal{F}(\mathbb{R}^n)$. Let $\mathcal{G}, \mathcal{H}$ be natural bundles respectively over n- and $(n+k)$-dimensional manifolds. *A lifting to $\mathcal{F}$* of geometric $\mathcal{G}$-objects from M to geometric $\mathcal{H}$-objects on $\mathcal{F}(M)$ is a family $\Lambda = \{\Lambda_M\}$ of mappings $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$ satisfying the following conditions:

(1) If $\sigma \in \underline{\mathcal{G}}(M)$ is defined on an open subset $U \subset M$, then $\Lambda_M(\sigma) \in \underline{\mathcal{H}}(\mathcal{F}(M))$ is defined on $\mathcal{F}(U) \subset \mathcal{F}(M)$. Often we will write $\Lambda(\sigma)$ instead of $\Lambda_M(\sigma)$.

(2) *The naturality condition.* For every embedding $\varphi : M \to N$, if objects $\sigma_1 \in \underline{\mathcal{G}}(M), \sigma_2 \in \underline{\mathcal{G}}(N)$ are φ-related, then $\Lambda_M(\sigma_1) \in \underline{\mathcal{H}}(\mathcal{F}M)$, $\Lambda_M(\sigma_2) \in \underline{\mathcal{H}}(\mathcal{F}N)$ are $\mathcal{F}(\varphi)$-related.

We say that a lifting $\Lambda = \{\Lambda_M\}$ to $\mathcal{F}$ satisfies the regularity condition if

(3) *The regularity condition.* If $\sigma_t \in \underline{\mathcal{G}}(M)$ is a smooth family of local fields of geometric objects on M, then $\Lambda_M(\sigma_t) \in \underline{\mathcal{H}}(\mathcal{F}(M))$ is also a smooth family of local fields of geometric objects on $\mathcal{F}(M)$.

If $\mathcal{F}$ is the identity functor, i.e. $\mathcal{F}(M) = M$ and $\mathcal{F}(\varphi) = \varphi$, then a lifting for this particular bundle $\mathcal{F}$ is called *natural transformation* of $\mathcal{G}$-objects into $\mathcal{H}$-objects.

Let us observe that the symbol $\underline{\mathcal{H}}(\mathcal{F}(M))$ denotes the set of all local sections of $\mathcal{H}(\mathcal{F}(M)) \to \mathcal{F}(M)$ but not sections of the composed bundle $\mathcal{H}(\mathcal{F}(M)) \to \mathcal{F}(M) \to M$.

Our definition implies immediately *the locality condition.*

Proposition 3.2. *If Λ is a lifting of geometric objects to a natural bundle $\mathcal{F}$ and $\sigma_1, \sigma_2 \in \underline{\mathcal{G}}(M)$ are two local objects on M such that $(\sigma_1)_{|U} = (\sigma_2)_{|U}$ for some open subset $U \subset M$, then $\Lambda_M(\sigma_1)_{|\mathcal{F}(U)} = \Lambda_M(\sigma_2)_{|\mathcal{F}(U)}$.*

Proof. Let $\sigma = (\sigma_1)_{|U} = (\sigma_2)_{|U}$ and let $i : U \to M$ be the inclusion. Now, for $a = 1, 2$, the objects σ and σ_a are i-related. By the naturality condition, $\Lambda_M(\sigma)$ and $\Lambda_M(\sigma_a)$ are $\mathcal{F}(i)$-related. Because of $\mathcal{F}(i)$ is the inclusion, thus $\Lambda_M(\sigma) = \Lambda_M(\sigma_a)_{|\mathcal{F}(U)}$. □

Remark 3.3. There is some "philosophical" problem of definition of liftings. We denote by $\underline{\mathcal{G}}_{glob}(M)$ the set of all global sections of $\mathcal{G}(M) \to M$ (global geometric $\mathcal{G}$-objects) on M. We can define a lifting as a family of maps $\Lambda_M : \underline{\mathcal{G}}_{glob}(M) \longrightarrow \underline{\mathcal{H}}_{glob}(\mathcal{F}(M))$ satisfying the conditions of Definition 3.1. It means that $\Lambda_M(\sigma)$ is defined only for global geometric objects on M. We can prolong the given mapping $\Lambda_M : \underline{\mathcal{G}}_{glob}(M) \longrightarrow \underline{\mathcal{H}}_{glob}(\mathcal{F}(M))$ to one and only one mapping $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$ satisfying the conditions (1), (2) and eventually (3) because any $\sigma \in \underline{\mathcal{G}}(M)$ belongs to $\underline{\mathcal{G}}_{glob}(dom\ \sigma)$.

We give now a few of standard examples.

Example 3.4. *Complete lift of vector fields.* Let $\mathcal{F}$ be a natural bundle and $\mathcal{G}, \mathcal{H}$ be the tangent bundle. We denote by $\mathcal{X}(M) = \underline{T}(M)$ the space of local vector fields on M.

We will define a lifting of vector fields from M to a vector fields on TM as follows. If X is a vector field on an open subset $U \subset M$ and φ_t is its local 1-parameter group, then $\mathcal{F}(\varphi_t)$ is an 1-parameter group defined on $\mathcal{F}(U) = \mathcal{F}(N)|_U$ which defines a vector field X^C on $TM|_U$. The family of mappings $\mathcal{X}(M) \ni X \longrightarrow X^C \in \mathcal{X}(\mathcal{F}(M))$ is a lifting because if X and Y are f-related, where $f : M \to N$ is an embedding, then for local 1-parameter groups φ_t, ψ_t of X and Y respectively we have $f \circ \varphi_t = \psi_t \circ f$, and in consequence $\mathcal{F}(f) \circ \mathcal{F}(\varphi_t) = \mathcal{F}(\psi_t) \circ \mathcal{F}(f)$, i.e. the vector fields X^C and Y^C are $\mathcal{F}(f)$-related. X^C is called *complete lift of X from M to $\mathcal{F}(M)$.*

The above lift is used in all papers concerning the geometry of bundles (see [84] — [88], [123] — [126], [14], [16] — [20], [26] — [33] and others).

The complete lift of vector fields have the following properties (see Salvioli [106] cf. [14], [21]):

Proposition 3.5. *If $\mathcal{F}$ is an arbitrary natural bundle then for vector fields $X, Y \in \mathcal{X}(M)$ and reals a, b we have $(a\,X + b\,Y)^C = a\,X^C + b\,Y^C$, $[X^C, Y^C] = [X, Y]^C$.*

Example 3.6. *Vertical lift of sections* (see [20]). Let $\mathcal{F}$ be a natural vector bundle, $\mathcal{G}(M) = \mathcal{F}(M)$ and $\mathcal{H}(\mathcal{F}(M)) = T(\mathcal{F}M)$ be the tangent bundle. We will define a lifting $\underline{\mathcal{F}}(M) \to \mathcal{X}(\mathcal{F}(M))$ of section of $\mathcal{F}(M)$ into vector fields on $\mathcal{F}(M)$. At first we observe that the vertical space $V_y(\mathcal{F}M)$ is the tangent space at $y \in \mathcal{F}(M)$ to a vector space $\mathcal{F}_{\pi(y)}(M)$. Thus we have the isomorphism $\Psi_y : \mathcal{F}_{\pi(y)}(M) \to V_y(\mathcal{F}M)$. Now we define the isomorphism $\Psi_M : \mathcal{F}(M) \times_M \mathcal{F}(M) \ni (y, z) \longrightarrow \Psi_y(z) \in V(\mathcal{F}M)$. Ψ_M is natural in the sense that for every embedding $\varphi : M \to N$ the following diagram

$$\begin{array}{ccc}
\mathcal{F}(M) \times_M \mathcal{F}(M) & \xrightarrow{\Psi_M} & V(\mathcal{F}M) \\
{\scriptstyle \mathcal{F}(\varphi)\times\mathcal{F}(\varphi)}\big\downarrow & & \big\downarrow{\scriptstyle d\mathcal{F}(\varphi)} \\
\mathcal{F}(N) \times_N \mathcal{F}(N) & \xrightarrow{\Psi_N} & V(\mathcal{F}N)
\end{array}$$

commutes.

Let $\sigma \in \underline{\mathcal{F}}(M)$ be a local section. We define a vertical vector field

$$\sigma^V(y) = \Psi_M(y, \sigma(\pi(y))) = \Psi_y(\sigma(\pi(y))) \in V_y(\mathcal{F}M)$$

on $\mathcal{F}(M)|_U$. From the commutativity of the last diagram we obtain immediately that the family of mappings $\underline{\mathcal{F}}(M) \ni \sigma \longrightarrow \sigma^V \in \mathcal{X}(\mathcal{F}M)$ is a lifting. For a section $\sigma \in \underline{\mathcal{F}}(M)$ the vector field σ^V is called *vertical lift of σ to vertical vector fields on $\mathcal{F}(M)$.*

Let us observe that the mapping $\sigma \to \sigma^V$ is linear (over $\mathbb{R}$) and for two sections σ_1, σ_2 we have $[\sigma_1^V, \sigma_2^V] = 0$.

If $\mathcal{F}(M) = TM$ is the tangent bundle, then we obtain the vertical lift of vector fields from M to vector fields on TM (see Yano and Kobayashi [125], see also [123]). If $\mathcal{F}(M) = T^*M$ is the cotangent bundle, then we obtain the vertical lift of 1-forms from M to vector fields on T^*M (see Yano and Paterson [126]; cf. [123]).

Example 3.7. *Vertical lift of functions.* Let $\mathcal{F}$ be a natural bundle. For a function f on an open subset $U \subset M$ we define the function $f^V = f \circ \pi$ on $\mathcal{F}(M)|_U$ called *vertical lift* of function f from M to a function on $\mathcal{F}(M)$. (Functions on M are considered as sections of the trivial bundle $\mathcal{G}(M) = M \times \mathbb{R}$.) It is easy to verify that the family of mappings $C^\infty(M) \ni f \longrightarrow f^V \in C^\infty(\mathcal{F}M)$ is a lifting.

Let us observe that vertical lift of functions have the following property. Let $\widetilde{X}$ be a vector field on $\mathcal{F}(M)$. $\widetilde{X}$ is projectable on M and X is its projection if and only if for each function $f \in C^\infty(M)$ we have $\widetilde{X}(f^V) = (Xf)^V$. Particularly, $\widetilde{X}$ is a vertical vector field on $\mathcal{F}(M)$ if and only if $\widetilde{X}(f^V) = 0$ for all functions $f \in C^\infty(M)$.

In the presented examples we considered liftings of very simple objects to an arbitrary natural bundle. Next we give a few examples of liftings to a natural bundle $\mathcal{F}$ of more complicated geometrical objects but $\mathcal{F}$ will be a more particular bundle.

Example 3.8. *Lifts of tensor fields to the tangent bundle* (see Yano and Kobayashi [125], see also [123]). Let $\mathcal{F}(M) = TM$ be the tangent bundle and let $T^p_q M = \bigotimes^p TM \otimes \bigotimes^q T^*M$ be the bundle of tensors of type (p, q). We denote by $\mathcal{X}^p_q(M) = \underline{T^p_q}(M)$ the space of local tensor fields of type (p, q) on M (local sections of $T^p_q M$). We will define liftings $\mathcal{X}^p_q(M) \longrightarrow \mathcal{X}^p_q(TM)$.

If $p = 1$ and $q = 0$, then $\mathcal{X}^1_0(M) = \mathcal{X}(M)$ is the space of vector fields on M. In this case we have two liftings of vector fields from M to vector fields on TM. Namely, the complete lift $X \to X^C$ (see Example 3.4) and vertical lift $X \to X^V$ (see Example 3.6).

If $p = q = 0$, then $\mathcal{X}_0^0(M) = C^\infty(M)$ is the space of functions on M. In this case we can define two liftings of functions from M to functions on TM. Namely, the vertical lift $f \to f^V$ (see Example 3.7) and the complete lift $f \to f^C = df$.

If $p = 0$ and $q = 1$, then $\mathcal{X}_1^0(M) = \mathcal{X}^*(M)$ is the space of 1-forms on M. In this case we can define two liftings of 1-forms from M to 1-forms on TM. Namely, if ω is an 1-form on M, then we can prove (see [125]):

(1) there exists one and only one 1-form ω^C on TM such that for every vector field X on M we have $\omega^C(X^C) = (\omega(X))^C$. ω^C is called *complete lift of ω from M to 1-forms on TM*.

(2) there exists one and only one 1-form ω^V on TM such that for every vector field X on M we have $\omega^V(X^C) = (\omega(X))^V$. ω^V is called *vertical lift of ω from M to 1-forms on TM*. (Let us observe that ω^V denotes here 1-form on TM. On the other hand, applying to a section ω of T^*M the construction given in Example 3.6 we obtain a vertical vector field on T^*M denoted also by ω^V.)

It is easy to observe that the family of mappings $\mathcal{X}^*(M) \ni \omega \longrightarrow \omega^V \in \mathcal{X}^*(TM)$, $\mathcal{X}^*(M) \ni \omega \longrightarrow \omega^C \in \mathcal{X}^*(TM)$ are liftings.

Next we can prove that for any manifold M and numbers p, q there are uniquely determined mappings

$$\mathcal{X}_q^p(M) \ni t \longrightarrow t^V \in \mathcal{X}_q^p(TM)\,, \tag{3.1}$$

$$\mathcal{X}_q^p(M) \ni t \longrightarrow t^C \in \mathcal{X}_q^p(TM) \tag{3.2}$$

such that the following conditions hold:

(1) if $p + q \leq 1$, then this mappings coincide with the mappings defined sooner;
(2) they are linear over $\mathbb{R}$;
(3) for two tensor fields $t \in \mathcal{X}_q^p(M)$ and $t' \in \mathcal{X}_{q'}^{p'}(M)$ we have

$$(t \otimes t')^C = t^V \otimes t'^C + t^C \otimes t'^V\,, \qquad (t \otimes t')^V = t^V \otimes t'^V\,.$$

t^C is called *complete lift of tensor field t from M to tensor fields on TM*. t^V is called *vertical lift of tensor field t from M to tensor fields on TM*. The above constructions was due by Yano and Kobayashi [125], (see also [123]).

The constructed families of mappings (3.1) and (3.2) are liftings.

Example 3.9. *Complete lift of linear connections to the tangent bundle.* Let $\mathcal{Q}(M)$ be the bundle of linear connections on M (see Example 1.12). A linear connection on M is a section of $\mathcal{Q}(M)$ (an element of $\underline{\mathcal{Q}}(M)$). We will define a lifting of linear connections from M to linear connections on TM.

We can prove that for a linear connection $\nabla \in \underline{\mathcal{Q}}(M)$ there is one and only one linear connection $\nabla^C \in \underline{\mathcal{Q}}(TM)$ such that

$$\nabla^C_{X^C} Y^C = (\nabla_X Y)^C\,, \qquad \nabla^C_{X^C} Y^V = \nabla^C_{X^V} Y^C = (\nabla_X Y)^V\,, \qquad \nabla^C_{X^V} Y^V = 0\,,$$

for all vector fields X, Y on M, where X^C and X^V denote the complete and the vertical lift of X (see Yano and Paterson [126] cf. also [123]).

The connection ∇^C is called *complete lift of ∇ from M to connections on TM*. The family of mappings $\underline{\mathcal{Q}}(M) \ni \nabla \longrightarrow \nabla^C \in \underline{\mathcal{Q}}(TM)$ is a lifting.

In the differential geometry enormous numbers of natural liftings was used. In the examples given in our short list all liftings verify the regularity condition. According to Theorem 1.19, there is a natural question: *Is the regularity condition (3) a consequence of conditions (1) and (2) in Definition 3.1?*

In general, the answer for this question is "No". There exist liftings which do not verify the regularity condition. In some particular cases (for some particular functors $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$), it is possible to prove that conditions (1) and (2) imply (3).

Example 3.10. *There exists a lifting* $\Lambda = \{\Lambda_M\}$, $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$, *which does not satisfy the regularity condition (3)* (see Mikulski [75]). Let $\mathcal{F}$ be the identity functor ($\mathcal{F}(M) = M$, $\mathcal{F}(\varphi) = \varphi$) and $\mathcal{G}(M) = TM \times \mathbb{R}$, $\mathcal{H}(M) = M \times \mathbb{R}$. Now $\underline{\mathcal{G}}(M) = \mathcal{X}(M) \times C^\infty(M)$ and $\underline{\mathcal{H}}(M) = C^\infty(M)$. We will construct our example in the category of 1-dimensional manifolds.

We consider a function $\beta : \mathbb{R}^2 \to \mathbb{R}$ of class C^∞ such that

$$\beta(x,y) = \begin{cases} 1, & \text{for } \ x = 0, \ y \in \mathbb{R} \\ 0, & \text{for } \ \|x\| \geq 1, \ y \in \mathbb{R} \end{cases}$$

and let I be the inversion with respect to the circ of a center $(0,1)$ and radius 1. We define a function $f : \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} (\beta \circ I)(x,y), & \text{for } \ (x,y) \neq (0,1) \\ 0, & \text{for } \ (x,y) = (0,1) \end{cases}$$

The function f is smooth on $\mathbb{R}^2 - \{(0,1)\}$. Let us observe that $f(x,y) = 0$ for any point (x,y) which belongs to the two closed disks of radius $\frac{1}{2}$ and centers $(\pm\frac{1}{2}, 1)$.

For an element $(X,h) \in \underline{\mathcal{G}}(M)$ we define a function $\Lambda_M(X,h)$ on M (a section of $\mathcal{H}(M)$) by

$$\Lambda_M(X,h) = f \circ (h, X(h)) .$$

We will show that the family $\Lambda = \{\Lambda_M\}$ is a lifting (natural transformation) which does not satisfy the regularity condition.

It is easy to see that Λ satisfies (formally) the naturality condition. To show that the family Λ is a lifting we need to verify that for every element $(X,h) \in \underline{\mathcal{G}}(M)$ and for every point $x_0 \in M$ the function $H = \Lambda_M(X,h)$ is of class C^∞ in some neighborhood of x_0.

It is clear that if x_0 is a point such that $(h, X(h))(x_0) \neq (0,1)$ then H is of class C^∞ in a neighborhood of x_0. Let now $x_0 \in M$ be a point such that $(h, X(h))(x_0) = (0,1)$. Since $X_{x_0}(h) = 1$, there is an embedding $\varphi : \mathbb{R} \to M$ such that $\varphi(0) = x_0$, and the vector fields $\frac{\partial}{\partial x} = \frac{d}{dx}$ and X are φ-related. Let $\bar{h} = h \circ \varphi$. Now from $\bar{h}(0) = 0$ and $\frac{d\bar{h}}{dx}(0) = 1$ we obtain $\bar{h}(x) = x + x^2 g(x)$, where g is of class $\mathcal{C}^\infty$. (Now the assumption that $\dim M = 1$ is important.) From the naturality condition we obtain

$$\big(\Lambda_M(X,h) \circ \varphi\big)(x) = \big(\Lambda_{\mathbb{R}}(\frac{d}{dx}, \bar{h})\big)(x) = f(x + x^2 g(x), 1 + 2xg(x) + x^2 g'(x))$$

Because of φ is an embedding and $\varphi(0) = x_0$, thus it is sufficient to show that the function $f(x + x^2 g(x), 1 + 2xg(x) + x^2 g'(x))$ is of class C^∞ in a neighborhood of $0 \in \mathbb{R}$. Since $(x + x^2 g(x) - \frac{1}{2})^2 + (2xg(x) + x^2 g'(x))^2 = \frac{1}{4} - x(1 + x\eta(x))$, where η is smooth, thus for sufficient small $x > 0$ we have

$$(x + x^2 g(x) - \frac{1}{2})^2 + (2xg(x) + x^2 g'(x))^2 \leq \frac{1}{4} .$$

Similarly, for sufficiently small $x < 0$ we have $(x+x^2g(x)+\frac{1}{2})^2+(2xg(x)+x^2g'(x))^2 \leq \frac{1}{4}$. It means that $(x + x^2g(x), 1 + 2xg(x) + x^2g'(x))$ belongs to one of the closed disks of radius $\frac{1}{2}$ and centers $(\pm\frac{1}{2}, 1)$. Therefore $f(x+x^2g(x), 1+2xg(x)+x^2g'(x)) = 0$ on some neighborhood of $0 \in \mathbb{R}$. We have proved that Λ is a lifting.

Λ do not satisfy the regularity condition because the mapping

$$\mathbb{R} \times \mathbb{R} \ni (t, y) \to \varphi_t(y) = ((1+t)\frac{d}{dx}(y), y) \in T\mathbb{R} \times \mathbb{R}$$

is of class C^∞ and the induced mapping

$$\Lambda_{\mathbb{R}}(\varphi_t)(0) = f(0, 1+t) = \begin{cases} 1, & \text{for } t \neq 0 \\ 0, & \text{for } t = 0 \end{cases}$$

is not of class $\mathcal{C}^\infty$.

A more complicated example of non-regular lifting is given by Epstein in [10].

In [72] it is proved the following theorem on regularity of liftings.

Theorem 3.11. *Let $\{\Lambda_M : \underline{\mathcal{G}}(M) \to \underline{\mathcal{H}}(\mathcal{F}M)\}$ be a lifting. Let $\sigma_m, \sigma \in \underline{\mathcal{G}}(M)$ $(m = 1, 2, ...)$, $x \in M$, $v_m, v \in (\pi_M^{\mathcal{F}})^{-1}(x)$ be such that $v_m \to v$ and $j_x^\infty(\sigma_m) \to j_x^\infty(\sigma)$ as $m \to \infty$. If there exists a vector field X on M such that $X(x) \neq 0$ and $j_x^\infty(L_X\sigma) = j_x^\infty(0)$, then $\Lambda_M(\sigma_m)(v_m) \to \Lambda_M(\sigma)(v)$ as $m \to \infty$, where $L_X\sigma$ denotes the Lie derivative of σ with respect to X.*

Remark 3.12. If $\{\Lambda_M\}$ satisfies the regularity condition, then $\Lambda_M(\sigma_m)(v_m) \to \Lambda_M(\sigma)(v)$ as $m \to \infty$ for all sequences $\sigma_m, \sigma \in \underline{\mathcal{G}}(M)$ and v_m, v over x with $j_x^\infty(\sigma_m) \to j_x^\infty(\sigma)$ and $v_m \to v$. Example 3.10 shows that the assumption " there exists a vector field X ... " in Theorem 3.11 is essential.

Using so-called Borel Lemma we obtain the following proposition:

Proposition 3.13. *Let $\Lambda = \{\Lambda_M\}$ be a lifting. For every point $x \in M$ and every sections $\sigma_1, \sigma_2 \in \underline{\mathcal{G}}(M)$ we have the following implication*

$$j_x^\infty\sigma_1 = j_x^\infty\sigma_2 \Longrightarrow \Lambda_M(\sigma_1)_{|\mathcal{F}_x(M)} = \Lambda_M(\sigma_2)_{|\mathcal{F}_x(M)}.$$

Proof. According to the naturality condition it is sufficient to show the implication for $M = \mathbb{R}^n$ and $x = 0 \in \mathbb{R}^n$. We denote by $G = \mathcal{G}_0(\mathbb{R}^n)$ the standard fibre.

Let σ_1, σ_2 be two section of $\mathcal{G}(\mathbb{R}^n) = \mathbb{R}^n \times G$ such that $j_0^\infty\sigma_1 = j_0^\infty\sigma_2$. From Borel Lemma (see [94]) we find two open subsets $U_1, U_2 \subset \mathbb{R}^n$ such that $\overline{U_1} \cap \overline{U_2} = \{0\}$ and a section $\sigma \in \underline{\mathcal{G}}\mathbb{R}^n$ such that $\sigma|_{U_a} = \sigma_a|_{U_a}$ for $a = 1, 2$. From Proposition 3.2 we obtain $\Lambda_{\mathbb{R}^n}(\sigma)|_{\mathcal{F}(U_a)} = \Lambda_{\mathbb{R}^n}(\sigma_a)|_{\mathcal{F}(U_a)}$. Since all sections are smooth and $\mathcal{F}_0(\mathbb{R}^n) = \overline{\mathcal{F}U_1} \cap \overline{\mathcal{F}U_2}$, thus $\Lambda_{\mathbb{R}^n}(\sigma)|_{\mathcal{F}_0(\mathbb{R}^n)} = \Lambda_{\mathbb{R}^n}(\sigma_a)|_{\mathcal{F}_0(\mathbb{R}^n)}$. Let us observe that we do not use the regularity condition in this proof. □

This proposition means that for a point $y \in \mathcal{F}(M)$ a value $\Lambda_M(\sigma)(y)$ depends only on the infinite jet of σ at the point $x = \pi(y)$. We propose a definition.

A lifting $\Lambda = \{\Lambda_M\}$ is called *of finite order* if there exists a number k such that for every point $x \in M$ and every two sections $\sigma_1, \sigma_2 \in \underline{\mathcal{G}}(M)$ defined on neighborhoods of x the following implication

$$j_x^k\sigma_1 = j_x^k\sigma_2 \Longrightarrow \Lambda_M(\sigma_1)_{|\mathcal{F}_x(M)} = \Lambda_M(\sigma_2)_{|\mathcal{F}_x(M)}$$

holds. The smallest number k for which this implication holds is called *order* of Λ. If there is no number k for which the implication holds, then Λ is of *infinite order*.

All examples of liftings given in this section are finite order. The complete lift of vector fields to a natural bundle $\mathcal{F}$ (see Example 3.4) is of order r, where r is the order of $\mathcal{F}$. Vertical lifts of sections (Example 3.6) and functions (Example 3.7) are liftings of order 0. Complete lift of tensor fields to the tangent bundle (Example 3.8) is of order 1 and the vertical lift to TM is of order 0. The complete lift of connections to the tangent bundle TM (Example 3.9) is of order 1.

According to Theorems 1.18 and 1.19 we have a natural question: *Does a lifting Λ have a finite order?* In general, the answer is "No", but in some particular cases (for some particular natural bundles $\mathcal{F}, \mathcal{G}, \mathcal{H}$) the answer could be "Yes" (see Proposition 3.16). There are some general theorems which give statements under which all liftings have finite order (see [111]).

Example 3.14. *There is a lifting of infinite order.* Let $\mathcal{F}$ be the identity functor ($\mathcal{F}(M) = M$, $\mathcal{F}(\varphi) = \varphi$) and $\mathcal{G}(M) = TM \times \mathbb{R}$, $\mathcal{H}(M) = M \times \mathbb{R}$. Now $\underline{\mathcal{G}}(M) = \mathcal{X}(M) \times C^\infty(M)$ and $\underline{\mathcal{H}}(M) = C^\infty(M)$. We can construct a lifting (natural transformation) $\Lambda = \{\Lambda_M\}$, $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(M)$, of infinite order.

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $f(t) = 0$ for $t \leq 0$ and $f(t) > 0$ for $t > 0$. For a vector field $X \in \mathcal{X}(M)$ we note $X^i = X \circ \cdots \circ X$ (i-times). If $(X, h) \in \underline{\mathcal{G}}(M)$ then we define

$$\big(\Lambda_M(X,h)\big)(x) = \sum_{k=1}^{+\infty} f\big(h(x) - \sum_{i=1}^{k}([X^i(h)(x)]^2 + 1)\big) .$$

For every point of M there exist a neighborhood U and a number k_0 such that $h(x) - \sum_{i=1}^k([X^i(h)(x)]^2 + 1) < 0$ for all $x \in U$ and $k \geq k_0$. Now we have

$$\big(\Lambda_M(X,h)\big)(x) = \sum_{k=1}^{k_0} f\big(h(x) - \sum_{i=1}^{k}([X^i(h)(x)]^2 + 1)\big) \qquad \text{for } x \in U.$$

It means that $\Lambda_M(X,h)$ is of class C^∞ on U. It is easy to observe that the family of mappings $\Lambda_M : \underline{\mathcal{G}}(M) \to \underline{\mathcal{H}}(M)$ satisfies the naturality and regularity conditions.

The natural transformation $\Lambda = \{\Lambda_M\}$ is of infite order because for every point $x \in M$ and $k \in \mathbb{N}$ there exist a vector field $X \in \mathcal{X}(M)$ and a function h on M such that $h(x) - \sum_{i=1}^k\big([X^i h(x)]^2 + 1\big) > 0$ and $X^k(h)(x) \neq 0$. It means that $\Lambda_M(X,h)(x)$ depends at least on $j^k_x(X,h)$.

Another example of a lifting of infinite order is given by Slovák [111].

There are two interesting type of problems:

(1) For given natural bundles $\mathcal{F}, \mathcal{G}, \mathcal{H}$, to give a complete characterizations of liftings $\Lambda = \{\Lambda_M\}$, where $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$.
(2) To consider a particular lifting $\Lambda = \{\Lambda_M\}$, where $\mathcal{F}, \mathcal{G}, \mathcal{H}$ are some special natural bundles, and to study properties of Λ from a point of view of a possibility of prolongations of geometric structures from a manifold M to $\mathcal{F}(M)$.

To solve these problems, particularly problem 1, we can use some general procedures.

Let $\Lambda = \{\Lambda_M\}$ be a lifting, where $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$. If Λ is of order k, $k \in \mathbb{N} \cup \{+\infty\}$, then we define a mapping $\lambda : \mathcal{F}_0(\mathbb{R}^n) \times J^k_0\mathcal{G}(\mathbb{R}^n) \longrightarrow (\mathcal{H}\mathcal{F})_0(\mathbb{R}^n)$ by

$$\lambda(y, j^k_0\sigma) = \Lambda_{\mathbb{R}^n}(\sigma)(y) , \tag{3.3}$$

where $J^k\mathcal{G}$ denote the jet-prolongation of $\mathcal{G}$ (see Example 1.13). From the definition of order of lifting it follows that the mapping λ is well-defined. The mapping λ has the following properties:

The following diagram

$$\begin{array}{ccc} \mathcal{F}_0(\mathbb{R}^n) \times J^k_0\mathcal{G}(\mathbb{R}^n) & \xrightarrow{\lambda} & (\mathcal{H}\mathcal{F})_0(\mathbb{R}^n) \\ p_1 \searrow & & \swarrow \pi_H \\ & \mathcal{F}_0(\mathbb{R}^n) & \end{array} \tag{3.4}$$

commutes, where p_1 is the standard projection onto the first factor and $\pi_{\mathcal{H}}$ is the projection for the natural bundle $\mathcal{H}$. If Λ satisfies the regularity condition and $k < \infty$, then by the Boman Theorem (see [2]) λ is of class C^∞.

To formulate a second property we need to introduce some actions of Lie groups. Let $r_{\mathcal{F}}, r_{\mathcal{G}}, r_{\mathcal{H}}$ be the orders of $\mathcal{F}, \mathcal{G}$ and $\mathcal{H}$ respectively.

Let $r = \max(k + r_{\mathcal{G}}, r_{\mathcal{H}} + r_{\mathcal{F}})$. The group L^r_n acts on $\mathcal{F}_0(\mathbb{R}^n) \times J^k_0\mathcal{G}(\mathbb{R}^n)$ by the formula

$$j^r_0\xi \cdot (y, j^k_0\sigma) = (\mathcal{F}(\xi)(y), j^k_0(\mathcal{G}(\xi) \circ \sigma \circ \xi^{-1}))\,, \tag{3.5}$$

where $\xi : \mathbb{R}^n \to \mathbb{R}^n$, $\xi(0) = 0$, and on $(\mathcal{H}\mathcal{F})_0(\mathbb{R}^n)$ by

$$j^r_0\xi \cdot z = \mathcal{H}(\mathcal{F}(\xi))(z)\,, \tag{3.6}$$

where $z \in (\mathcal{H}\mathcal{F})_0(\mathbb{R}^n)$. The mapping λ is L^r_n-equivariant, i.e. for $(y, j^k_0\sigma) \in \mathcal{F}_0(\mathbb{R}^n) \times J^k_0\mathcal{G}(\mathbb{R}^n)$ and $j^r_o\xi \in L^r_n$ we have

$$\lambda(j^r_0\xi \cdot (y, j^k_0\sigma)) = j^r_0\xi \cdot \lambda(y, j^k_0\sigma). \tag{3.7}$$

Inversely, if $k < \infty$, then every smooth mapping λ satisfying conditions (3.4) and (3.7) defines a lifting $\Lambda = \{\Lambda_M\}$ of order k satisfying the regularity condition. For n-dimensional manifold M the mapping Λ_M is defined by the formula

$$\Lambda_M(\sigma)(y) = (\mathcal{H} \circ \mathcal{F})(\varphi^{-1})\big(\lambda(\mathcal{F}(\varphi)(y), j^k_0(\mathcal{F}(\varphi) \circ \sigma \circ \varphi^{-1}))\big)\,, \tag{3.8}$$

where $\sigma \in \underline{\mathcal{G}}(M)$, $y \in \mathcal{F}(M)$ and φ is a chart on M defined on a neighborhood of $x = \pi_{\mathcal{F}}(y)$ such that $\varphi(x) = 0$.

The above remarks we can formulate in the form of the following proposition (the second part of the proposition is due by Krupka see [39]).

Proposition 3.15. *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be natural bundles of orders $r_{\mathcal{F}}, r_{\mathcal{G}}, r_{\mathcal{H}}$ respectively.*

For every lifting $\Lambda = \{\Lambda_M\}$ of order $k \in \mathbb{N} \cup \{\infty\}$, where $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$, the mapping

$$\lambda : \mathcal{F}_0(\mathbb{R}^n) \times J^k_0\mathcal{G}(\mathbb{R}^n) \longrightarrow (\mathcal{H}\mathcal{F})_0(\mathbb{R}^n)$$

given by (3.3) is L^r_n-equivariant with respect to the actions (3.5) and (3.6) (not necessarily continuous) and diagram (3.4) commutes, where $r = max(k + r_{\mathcal{G}}, r_{\mathcal{F}} + r_{\mathcal{H}})$. The correspondence $\Lambda \to \lambda$ is injective.

Moreover, if $k < \infty$, then there exists one-to-one correspondence between liftings $\Lambda = \{\Lambda_M\}$ of order k and satisfying the regularity condition, where $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}(M))$, and smooth L^r_n-equivariant mappings

$$\lambda : \mathcal{F}_0(\mathbb{R}^n) \times J^k_0\mathcal{G}(\mathbb{R}^n) \longrightarrow (\mathcal{H}\mathcal{F})_0(\mathbb{R}^n)\,,$$

where $r = max(k + r_{\mathcal{G}}, r_{\mathcal{F}} + r_{\mathcal{H}})$, such that diagram (3.4) commutes.

The above proposition reduces the problem of classifications of liftings of order k to classifications of equivariant mappings. This is, in general, a first step to solve of a problem of type (1).

If we want to classify liftings $\Lambda = \{\Lambda_M\}$, $\Lambda_M : \underline{\mathcal{G}}(M) \longrightarrow \underline{\mathcal{H}}(\mathcal{F}M)$, then firstly, we try to show that in the considered cases of natural bundles $\mathcal{F}, \mathcal{G}, \mathcal{H}$ all liftings are of finite order. There are some general theorems about finite order of liftings. A theorem of Slovák is the more general (see [111]).

Let $\mathcal{F}$ be a natural bundle of order r. We say that $\mathcal{F}$ satisfies the condition (C) if the standard fibre $\mathcal{F}_0(\mathbb{R}^n)$ together with the restriction of the corresponding L^r_n-action to the subgroup $GL(n, \mathbb{R}) = L^1_n \subset L^r_n$ can be identified with an invariant linear subspace in a finite direct sum of vector spaces V_j, where V_j is a space of tensors of type (p_j, q_j) and $p_j < q_j$. Now we can formulate theorem of Slovák [111].

Theorem 3.16 (Slovák). *Let $\mathcal{F}, \mathcal{G}, \mathcal{H}$ be three natural bundles. Assume that either $\mathcal{F}$ satisfies condition (C) or the standard fibre $\mathcal{F}_0(\mathbb{R}^n)$ of $\mathcal{F}$ is compact. If $\mathcal{G}$ satisfies condition (C), then every lifting $\Lambda_M : \underline{\mathcal{G}}(M) \to \underline{H}(\mathcal{F}M)$ has finite order.*

In the above theorem we do not suppose that a lifting Λ satisfies the regularity condition.

This theorem is very convenient if we study lifts of connections or Riemannian metrics but not applicable if we lift tensors of type $(1, 1)$.

§ 4. Liftings of vector fields.

Let $\mathcal{F}$ be a natural bundle. We will study liftings of vector fields from a manifold M to $\mathcal{F}(M)$. We always suppose that all liftings verify the regularity condition.

In the investigations of natural bundles the possibility of liftings of vector fields from a base to vector fields on a total space of bundle play an important role. Different lifts of vector fields are used practically in all papers in which problems of prolongations of geometric structures were studied. Thus the problem of characterization of such liftings is very important. We give here a complete classification of liftings of vector fields to $\mathcal{F}$ in two cases: (1) if $\mathcal{F}$ is a product preserving functor (it is a very nice result of Kolář [49]), and (2) if $\mathcal{F}$ is the r-order vector tangent bundle (see Mikulski [80]). The second case is completely different from (1) and the proof due by Mikulski contains many important, interesting and universal properties usable in an investigations of natural vector bundles.

At first, we recall that we have one standard example of a lifting of vector fields to any natural bundle $\mathcal{F}$ called *complete lift* (see Example 3.4).

In the case of liftings of vector fields we have the following proposition about order (Lemma 1 in [49], cf. also [19]).

Proposition 4.1. *Let $\mathcal{F}$ be a natural bundle of order r. Then every lifting Λ of vector fields to $\mathcal{F}$ satisfying the regularity condition is of order $\leq r$.*

Proof. It is sufficient to show that for a vector fields $X, Y \in \mathcal{X}(\mathbb{R}^n)$ the implication

$$j^r_0 X = j^r_0 Y \Longrightarrow \forall\ y \in \mathcal{F}_0(\mathbb{R}^n) :\ \Lambda(X)(y) = \Lambda(Y)(y) \tag{4.1}$$

holds, where $\Lambda = \Lambda_{\mathbb{R}^n}$.

At first, we suppose that $X(0) \neq 0$. There exists a chart on $\mathbb{R}^n$ such that $X = \frac{\partial}{\partial x^1}$. By the naturality condition the implication (4.1) is equivalent to

$$j_0^r Y = j_0^r \frac{\partial}{\partial x^1} \Longrightarrow \forall\ y \in \mathcal{F}_0(\mathbb{R}^n)\ :\ \Lambda(Y)(y) = \Lambda(\frac{\partial}{\partial x^1})(y)\,. \tag{4.2}$$

Let us observe that the function

$$f(x^1, \dots, x^n) = (exp_{x^1} Y)(0, x^2, \dots, x^n)$$

is a diffeomorphism on some neighborhood of $0 \in \mathbb{R}^n$ such that $f_* \frac{\partial}{\partial x^1} = Y$ near of $0 \in \mathbb{R}^n$ (see [35]). The diffeomorphism f satisfies the conditions

$$\frac{\partial f}{\partial x^1} = Y \circ f \tag{4.3}$$

$$f(0, x^2, \dots, x^n) = (0, x^2, \dots, x^n)\,. \tag{4.4}$$

From (4.3), (4.4) and $j_0^r Y = j_0^r(1, 0, \dots, 0)$ we deduce that $j_0^{r+1} f = j_0^{r+1} id$. Because of $\mathcal{F}$ is of order r, thus $\mathcal{F}(f^{-1})$ and $d\mathcal{F}(f)$ restricted to $\mathcal{F}_0(\mathbb{R}^n)$ are identities. Now from the naturality condition we obtain (4.2). The general case is a consequence of the regularity condition. □

Let $\mathcal{F}$ be a product preserving functor. We will discuss a classification of liftings of vector fields from M to $\mathcal{F}(M)$. According to Proposition 2.13 $\mathcal{F}$ is equivalent to a Weil functor T^A, where $A = \mathcal{F}(\mathbb{R}^n)$ is a Weil algebra.

Firstly, we define some examples of such liftings of vector fields to a product preserving functor different from the complete lift.

We consider the mapping $\Psi : \mathbb{R} \times TM \to TM$ given by $\Psi(t, v) = t\, v$. Using the natural isomorphism $\eta_M : \mathcal{F}(TM) \longrightarrow T(\mathcal{F}M)$, the induced mapping $\mathcal{F}(\Psi) : \mathcal{F}(\mathbb{R}) \times \mathcal{F}(TN) \to \mathcal{F}(TM)$ determines the mapping $\widetilde{\Psi} : A \times T(\mathcal{F}M) \to T(\mathcal{F}M)$. For an element $a \in A$ and a vector $\widetilde{v} \in T(\mathcal{F}M)$ we define

$$a \cdot \widetilde{v} = \widetilde{\Psi}(a, \widetilde{v})\,. \tag{4.5}$$

Now, it is easy to verify that for any element $a \in A$ the mapping

$$\mathcal{X}(M) \ni X \longrightarrow a \cdot X^C \in \mathcal{X}(\mathcal{F}M)$$

is a linear lifting of vector fields from M to vector fields on $\mathcal{F}(M)$. Moreover, for all vector fields X, Y on M and all $a, b \in A$ we have (see [29])

$$[a \cdot X^C, b \cdot Y^C] = ab \cdot [X, Y]^C\,. \tag{4.6}$$

A family $\widetilde{X} = \{\widetilde{X}_M\}$ of vector fields on $\mathcal{F}(M)$ (where M is an n-dimensional manifold) is called *invariant vector fields* if for any embedding $\varphi : M \to N$ the diagram

$$\begin{tikzcd}
T(\mathcal{F}M) \arrow[r, "d\mathcal{F}(\varphi)"] & T(\mathcal{F}N) \\
\mathcal{F}M \arrow[u, "\widetilde{X}_M"] \arrow[r, "\mathcal{F}(\varphi)"] & \mathcal{F}N \arrow[u, "\widetilde{X}_N"']
\end{tikzcd}$$

commutes. If $\widetilde{X} = \{\widetilde{X}_M\}$ is an invariant vector fields on $\mathcal{F}(M)$, then the family of constant mappings $\mathcal{X}(M) \ni X \longrightarrow \widetilde{X}_M \in \mathcal{X}(\mathcal{F}M)$ is a lifting.

Let $G = Aut(A)$ be the group of all algebra homomorphism of A. It is a Lie group because G is a closed subgroup of $GL(A)$. We denote by $\mathcal{L}(G)$ the Lie algebra of G.

Let D be an element of $\mathcal{L}(G)$ and let d_t be the 1-parameter subgroup of D. For every manifold M d_t defines a vertical vector field D_M on $\mathcal{F}(M)$ because $\{d_t\}$ can be interpreted as a flow on $\mathcal{F}(M)$. D_M is an invariant vertical vector field on $\mathcal{F}(M)$ and the constant mapping

$$op(D) : \mathcal{X}(M) \ni X \longrightarrow D_M \in \mathcal{X}(\mathcal{F}M) \tag{4.7}$$

is a lifting of vector fields to $\mathcal{F}$.

Now we can formulate the following very nice theorem proved by Kolář (see [49]).

Theorem 4.2 (Kolář). *Let $\mathcal{F}$ be a product preserving functor and let $A = \mathcal{F}(\mathbb{R})$ be its local algebra. Every lifting $\Lambda = \{\Lambda_M\}$ satisfying the regularity condition of vector fields $\Lambda_M : \mathcal{X}(M) \longrightarrow \mathcal{X}(\mathcal{F}(M))$ from a manifolds M to $\mathcal{F}(M)$ can be written in the form*

$$\Lambda_M(X) = a \cdot X^C + op(D)\,,$$

where $a \in A$ and $D \in \mathcal{L}(Aut(A))$

Schema of Proof. It is easy to see that $\Lambda_M(0)$ is an invariant vector field on $\mathcal{F}(M)$. There are two difficulties: (1) to show that for an invariant vector field $\widetilde{X}$ on $\mathcal{F}(M)$ there is $D \in \mathcal{L}(Aut(A))$ such that $\widetilde{X} = op(D)$, and (2) there is $a \in A$ such that $\Lambda_M(X) - \Lambda_M(0) = a \cdot X^C$ for all vector fields X. For all details see [49]. □

From Theorem 4.2 we obtain immediately the result of Sekizawa [110].

Proposition 4.3. *If $\Lambda = \{\Lambda_M\}$ is a lifting satisfying the regularity condition of vector fields from manifolds to vector fields on its tangent bundles, then there are reals a, b, c such that for every vector field X on M we have $\Lambda_M(X) = a\,X^C + b\,X^V + c\,L_M$, where X^C, X^V are respectively the complete and vertical lift of X to TM and L_M is the Liouville vector field on TM.*

From Theorem 4.2 we can deduce an interesting characterization of the complete lift of vector fields.

Proposition 4.4. *Let $\mathcal{F}$ be a product preserving functor. If a lifting $\Lambda = \{\Lambda_M\}$ of vector fields $\Lambda_M : \mathcal{X}(M) \longrightarrow \mathcal{X}(\mathcal{F}(M))$ from a manifolds M to vector fields on $\mathcal{F}(M)$ is a nonzero Lie algebra homomorphism, then $\Lambda_M(X) = X^C$ is the complete lift.*

Proof. Let $A = \mathcal{F}(\mathbb{R})$. Since Λ is linear, then, by [70] Λ satisfies the regularity condition, and then, by Theorem 4.2, there is $a \in A$ such that $\Lambda_M(X) = a \cdot X^C$ for all $X \in \mathcal{X}(M)$. From (4.6) we have $[\Lambda_M(X), \Lambda_M(Y)] = a^2\,[X,Y]^C$. Therefore, the equality $[\Lambda_M(X), \Lambda_M(Y)] = \Lambda_M([X,Y])$ implies $a^2 = a$. The element a can be written in the form $a = \lambda \cdot 1 + \eta$, where $\lambda \in \mathbb{R}$ and η is a nilpotent element. From the equality $a^2 = a$ we deduce that λ is either 0 or 1, and $\eta = 0$. It means that Λ is either identically zero or the complete lift. The proof is finished. □

We have also very interesting characterizations of liftings of vector fields to the vector r-th order tangent bundle T^rM (for the definition of T^rM see Example 1.11). This characterization is due by Mikulski [80]. The vector r-th order tangent bundle is not product preserving functor.

At first, we define some liftings of vector fields from M to vector fields on $T^r M$.

Let X be a vector field defined on an open subset $U \subset M$. For a number s, $1 \leq s \leq r$, we define a section $\tilde{X}^{(s)} : U \longrightarrow T^r M$; if x is a point of U then the element $\tilde{X}^{(s)}(x) = \tilde{X}^{(s)}_x : J^r_x(M, \mathbb{R})_0 \longrightarrow \mathbb{R}$ is a linear mapping given by $\tilde{X}^{(s)}_x(j^r_x f) = X^s(f)(x)$, where $X^s = X \circ \cdots \circ X$ (s times). Now, $X^{(s)} = (\tilde{X}^{(s)})^V$ is a vertical vector field on $T^r M$ (see Example 3.6) and it is easy to see that the mapping $X \longrightarrow X^{(s)}$ is a lifting of vector fields from n-manifold M to vector fields on the r-th order vector tangent bundle $T^r M$. We have the following theorem (see Mikulski [80]):

Theorem 4.5. *Let $n \geq 2$. Every lifting $\Lambda_M : \mathcal{X}(M) \longrightarrow \mathcal{X}(T^r M)$ satisfying the regularity condition of vector fields from a n-manifold M to its r-th order vector tangent bundle $T^r M$ can be written in the form*

$$\Lambda_M(X) = aX^C + \sum_{s=1}^{r} a_s X^{(s)} + b\, L\,,$$

where a, a_s, b are some reals and L is the Liouville vector field on $T^r M$.

Schema of proof. Let us observe that if Λ_1, Λ_2 are two liftings of vector fields from M to vector fields on a natural bundle $\mathcal{F}$ such that $(\Lambda_1)_{\mathbb{R}^n}(\frac{\partial}{\partial x^1}) = (\Lambda_2)_{\mathbb{R}^n}(\frac{\partial}{\partial x^1})$ on $\mathcal{F}_0(\mathbb{R}^n)$, then $\Lambda_1 = \Lambda_2$.

To prove the theorem we consider the mapping $g : \mathbb{R} \times T^r_0 \mathbb{R}^n \to T_0 \mathbb{R}^n$ defined by

$$g(t, w) = dp(\Lambda_{\mathbb{R}^n}(t\frac{\partial}{\partial x^1})(w))\,,$$

where $p : T^r \mathbb{R}^n \to \mathbb{R}^n$ is the bundle projection. Using the naturality condition for $\varphi = t\, id_{\mathbb{R}^n}$, where $t \in \mathbb{R} - \{0\}$, we obtain that g is linear.

Let $0_M : M \to T^r(M)$ be the zero section of $T^r(M)$. It is clear that the family $C_M : \mathcal{X}(M) \to \mathcal{X}(M)$ given by $C_M(X) = dp \circ \Lambda_M(X) \circ 0_M : M \to TM$, where $X \in \mathcal{X}(M)$, is a natural transformation of vector fields on M into itself. It implies (see [80]) that there exists $a \in \mathbb{R}$ such that $C_M(X) = aX$, for all $X \in \mathcal{X}(M)$. In particular, we have $g(1, 0) = a \frac{\partial}{\partial x^1}|_0$.

On the other hand the family $A_M : T^r(M) \to TM$, $A_M(w) = (dp \circ \Lambda_M(0))(w)$, is a natural transformation of the functor T^r into the tangent bundle functor T. For $r \geq 2$, we have $A_M = 0$. It means that $g(0, w) = 0$ for all $w \in T^r(\mathbb{R}^n)$.

We consider a new lifting $\overline{\Lambda}_M : \mathcal{X}(M) \to \mathcal{X}(T^r M)$

$$\overline{\Lambda}_M(X) = \Lambda_M(X) - a\, X^C\,.$$

From precedent remarks we obtain that

$$\big(dp \circ \overline{\Lambda}_{\mathbb{R}^n}(\frac{\partial}{\partial x^1})\big)(w) = g(1, w) - a\,\frac{\partial}{\partial x^1}|_0 = 0\,,$$

and in consequence, $\overline{\Lambda}_M(X)$ is a vertical vector field on $T^r M$ for all $X \in \mathcal{X}(M)$.

We have a natural isomorphism $\Psi_M : T^r M \times_M T^r M \to V(T^r M)$ (see Example 3.6). Let $q_M : V(T^M) \to T^r M$ be the composition of the inverse isomorphism with the projection $T^rM \times_M T^rM \to T^rM$ on the second factor. We consider the family $B_M : T^rM \to T^rM$ defined by $B_M(w) = \big(q_M \circ \overline{\Lambda}_M(0)\big)(w)$. It is possible to show that there is $b \in \mathbb{R}$ such that $B_M(w) = b\,w$ (see [57]).

Because of $\overline{\Lambda}_M(X)$ is a vertical vector field on T^rM for all $X \in \mathcal{X}(M)$, thus the family

$$D_M : \mathcal{X}(M) \longrightarrow \underline{T^r}(M), \qquad D_M(X) = q_M \circ \overline{\Lambda}_M(X) \circ 0_M$$

is a natural transformation of vector fields on M into sections of T^rM. It is proved in [80] that any such transformation is a linear combination of $\tilde{X}^{(s)}$, $s = 1, ..., r$.

Finally we consider the lifting $\overline{\overline{\Lambda}}_M : \mathcal{X}(M) \to \mathcal{X}(T^rM)$ defined by

$$\begin{aligned}\overline{\overline{\Lambda}}_M(X) &= \overline{\Lambda}_M(X) - b\,L - (D_M(X))^V \\ &= \Lambda_M(X) - a\,X^C - b\,L - (D_M(X))^V .\end{aligned}$$

We can verify that the family $\{\overline{\overline{\Lambda}}_M\}$ satisfies the following conditions:

$$\begin{gathered} dp \circ \overline{\overline{\Lambda}}_{\mathbb{R}^n}(\tfrac{\partial}{\partial x^1})|T_0^r\mathbb{R}^n = 0 \\ \overline{\overline{\Lambda}}_{\mathbb{R}^n}(\tfrac{\partial}{\partial x^1})(0) = 0 \qquad \text{for } 0 \in T_0^r\mathbb{R}^n \\ \overline{\overline{\Lambda}}_{\mathbb{R}^n}(0)|T_0^r\mathbb{R}^n = 0 . \end{gathered} \tag{4.8}$$

To finish the proof we need to show that if a lifting $\overline{\overline{\Lambda}}_M : \mathcal{X}(M) \to \mathcal{X}(T^rM)$ satisfies (4.8), then $\overline{\overline{\Lambda}}_M = 0$ (see, [80]). It is a very important element of this proof. □

Using similar arguments as in [80], one can prove

Theorem 4.6. *Let $n \geq 2$. Every lifting $\mathcal{X}(M) \longrightarrow \mathcal{X}(\widetilde{T}^r(M)) = \mathcal{X}(T^rM \times \mathbb{R})$ satisfying the regularity condition of vector fields from a manifold M to vector fields on the extended r-th order vector tangent bundle $\widetilde{T}^r(M)$ can be written in the form*

$$\Lambda_M(X) = aX^C + \sum_{s=1}^{r} a_s\,X^{(s)} + b\,L + c\,\frac{d}{dt}\,,$$

where $a, a_s, b : \mathbb{R} \to \mathbb{R}$ are smooth functions, L is the Liouville vector field on T^rM and $\frac{d}{dt}$ is the canonical vector field.

A classification of liftings of vector fields from a manifold M to the bundle $T(T^*M)$ is given by Kobak [43]. A classification of liftings of vector fields to jet-bundles was also given by Kolář and Slovák [56] and to the r-th order cotangent bundle by Mikulski [83].

§ 5. Prolongation of foliations to some natural bundles.

Let $\mathcal{F}$ be a natural bundle. We will study possibilities to construct foliations on $\mathcal{F}(M)$ from given foliation on M. At first we introduce basic definitions and notations.

A foliation on a manifold M is an integrable distribution (of the same dimension at each point) on M. We denote by $Fol_q(M)$ the set of all q-dimensional foliations on M. We denote by $Distr(M)$ the set of distributions (of a constant dimension) on M.

Let $D \subset TM$ be a foliation on M of dimension q. If $x \in M$ then we denote by D_x the fibre of D over x. A vector fields X is called *D-vector field* on M if $X(x)$ belongs to D_x for all $x \in M$.

Definition 5.1. Let $\mathcal{F}$ be a natural bundle over n-dimensional manifolds. A family $\Lambda = \{\Lambda_M\}$ of mappings $\Lambda_M : Fol_q(M) \longrightarrow Distr(\mathcal{F}(M))$, $\dim(M) = n$, is called *a natural lifting (prolongation)* of foliations on n-manifolds to distributions from $\mathcal{F}$ if the following *naturality condition* holds:

for every embedding $\varphi : M \to N$ of two n-dimensional manifolds M, N and every q-dimensional foliations D_1, D_2 on M and N respectively such that $d\varphi((D_1)_x) = (D_2)_{\varphi(x)}$ for all $x \in M$ we have $d\mathcal{F}(\varphi)\big(\Lambda_M(D_1)_y\big) = \Lambda_N(D_2)_{\mathcal{F}(\varphi)(y)}$ for all $y \in \mathcal{F}(M)$.

The above definition implies immediately *the locality condition* for natural prolongations of foliations.

Proposition 5.2. *Let $\Lambda = \{\Lambda_M\}$ be a natural prolongation of q-dimensional foliations from n-manifolds to distributions on $\mathcal{F}$. If $U \subset M$ is an open subset and $D_1, D_2 \in Fol_q(M)$ are two foliations such that $D_1|U = D_2|U$, then $\Lambda_M(D_1)|\mathcal{F}(U) = \Lambda_M(D_2)|\mathcal{F}(U)$.*

Proof. It follows from the naturality condition for the inclusion $i : U \to M$. □

Theorem 5.3. *Let $\Lambda_M : Fol_q(M) \to Distr(T^AM)$, $\dim(M) = n$, be a natural prolongation of foliations from n-manifolds to distributions on a Weil functor T^A. Then $\Lambda_M(D)$ is a foliation on T^AM for every n-dimensional manifold M and every foliation $D \in Fol_q(M)$.*

Proof. We fix a basis in the Weil algebra A. Let $\partial_{i,\nu}$, $i = 1, ..., n$, $\nu = 1, ..., \dim(A)$, be the canonical frame on $T^A\mathbb{R}^n$ with respect to the coordinates system $x^{i,\nu}$ induced by the canonical coordinates $x^1, ..., x^n$ on $\mathbb{R}^n$ and the basis in A (see Section 2). Of course,

$$dT^A(t\, id_{\mathbb{R}^n}) \circ \partial_{i,\nu} = t\, \partial_{i,\nu} \circ T^A(t\, id_{\mathbb{R}^n}) \tag{5.1}$$

for every $t \in \mathbb{R} - \{0\}$ (because $T^A(t\, id_{\mathbb{R}^n})(x^{i,\nu}) = (tx^{i,\nu})$). Let D^q be the standard q-dimensional foliation on $\mathbb{R}^n$ spanned by $\frac{\partial}{\partial x^1}, ..., \frac{\partial}{\partial x^q}$. Consider $\zeta \in T^A\mathbb{R}^n$. There exist linearly independent constant (with respect to the coordinate system $x^{i,\nu}$) vector fields ${}^\zeta X_1, ..., {}^\zeta X_s$ on $T^A\mathbb{R}^n$ such that

$$\Lambda_{\mathbb{R}^n}(D^q)_\zeta = span\{{}^\zeta X_1(\zeta), ..., {}^\zeta X_s(\zeta)\}.$$

Using the naturality condition with respect to the homotheties $t\, id_{\mathbb{R}^n}$, $t \neq 0$, (preserving D^q) and formula (5.1), we deduce that

$$\Lambda_{\mathbb{R}^n}(D^q)_{T^A(t\, id_{\mathbb{R}^n})(\zeta)} = span\{{}^\zeta X_1(T^A(t\, id_{\mathbb{R}^n})(\zeta)), \ldots, {}^\zeta X_s(T^A(t\, id_{\mathbb{R}^n})(\zeta))\}$$

for any $t \in \mathbb{R} - \{0\}$. Then putting $t \to 0$ we get

$$\Lambda_{\mathbb{R}^n}(D^q)_{j^A0} = span\{{}^\zeta X_1(j^A0), ..., {}^\zeta X_s(j^A0)\}.$$

Hence

$$\Lambda_{\mathbb{R}^n}(D^q)_\eta = span\{{}^\zeta X_1(\eta), ..., {}^\zeta X_s(\eta)\}$$

for any $\eta \in T^A\mathbb{R}^n$ because

$$span\{{}^\zeta X_1(j^A0), ..., {}^\zeta X_s(j^A0)\} = \Lambda_{\mathbb{R}^n}(D^q)_{j^A0} = span\{{}^\eta X_1(j^A0), ..., {}^\eta X_s(j^A0)\}$$

and ${}^\eta X_i, {}^\zeta X_j$ are constants. Thus $\Lambda_{\mathbb{R}^n}(D^q)$ is a foliation on $T^A\mathbb{R}^n$. By the naturality condition, Proposition 5.2 and the Frobenius theorem we complete the proof. □

We give a complete characterization of natural prolongations of foliations from n-manifolds to the tangent and cotangent bundles.

At first we consider the case of the tangent bundle. We construct some examples of natural prolongations of foliations to the tangent bundle.

Example 5.4. Let D be a q-dimensional foliation on a n-manifold M. Using D we construct some distributions on TM.

For $v \in TM$ we set

$$\begin{aligned}
D_v^{(1)} &= \{ \ X^V(v) \mid X \text{ is a } D\text{-vector field on } M \ \} \\
D_v^{(2)} &= \{ \ X^C(v) \mid X \text{ is a } D\text{-vector field on } M \ \} + D_v^{(1)} \\
D_v^{(3)} &= D_v^{(1)} + V_v(TM) \\
D^{(a)} &= \bigcup_{v \in TM} D_v^{(a)} \qquad \text{for } a = 1,2,3.
\end{aligned}$$

$D^{(1)}$ is a foliation on TM of dimension q. $D^{(2)}$ is a foliation on TM of dimension $2q$. $D^{(3)}$ is a foliation on TM of dimension $n+q$. Of course, the correspondences $D \to D^{(a)}$ are natural prolongations of q-foliations from n-manifolds to the distributions on the tangent bundle.

Using similar arguments as in [78] or [79] we can prove the following theorem.

Theorem 5.5. *If $\Lambda = \{\Lambda_M\}$ is a natural prolongation of q-dimensional foliations from n-manifolds to distributions on tangent bundles, where $0 < q < n$, then Λ_M is equal to one of the list:*

(1) $\Lambda_M(D) = \{0\}$ *is the zero distribution,*
(2) $\Lambda_M(D) = T(TM)$ *is the full distribution,*
(3) $\Lambda_M(D) = V(TM)$ *is the vertical distribution,*
(4) $\Lambda_M(D) = D^{(a)}$, $a = 1,2,3$, *are constructed in Example 5.4.*

From the proof of Theorem 5.3 it follows easily the following lemma which we will use to prove Theorem 5.5.

Lemma 5.6. *Let Λ and Λ' be two liftings of q-dimensional foliations from n-dimensional manifolds to distributions on the tangent bundle and let $v \in T_0\mathbb{R}^n$. Then $\Lambda = \Lambda'$ if and only if $\Lambda_{\mathbb{R}^n}(D^q)_v = \Lambda'_{\mathbb{R}^n}(D^q)_v$, where D^q is the standard q-dimensional foliation on $\mathbb{R}^n$.*

Moreover, if X is a constant vector field on $T\mathbb{R}^n$ (with respect to the standard coordinates on $T\mathbb{R}^n$), then X is a $\Lambda_{\mathbb{R}^n}(D^q)$-vector field if and only if $X(v) \in \Lambda_{\mathbb{R}^n}(D^q)_v$.

Schema of proof of Theorem 5.5. Let $D = D^q$ be the standard q-dimensional foliation on $\mathbb{R}^n$. Let us observe that we have

$$\begin{aligned}
&D^{(1)} \text{ is spanned by } \partial_1^V, \dots, \partial_q^V, \\
&D^{(2)} \text{ is spanned by } \partial_1^V, \dots, \partial_q^V, \partial_1^C, \dots, \partial_q^C, \\
&D^{(3)} \text{ is spanned by } \partial_1^V, \dots, \partial_n^V, \partial_1^C, \dots, \partial_q^C, \\
&VT\mathbb{R}^n \text{ is spanned by } \partial_1^V, \dots, \partial_n^V, \\
&TT\mathbb{R}^n \text{ is spanned by } \partial_1^V, \dots, \partial_n^V, \partial_1^C, \dots, \partial_n^C,
\end{aligned}$$

We consider two cases:

(I) We assume that $\Lambda_{\mathbb{R}^n}(D)_0 \subset V_0T\mathbb{R}^n$, where $0 \in T_0\mathbb{R}^n$.

If there exists $v \in \Lambda_{\mathbb{R}^n}(D)_0 \cap D_0^{(1)}$ (respectively $v \in \Lambda_{\mathbb{R}^n}(D)_0 \cap \left[V_0T\mathbb{R}^n \backslash D_0^{(1)}\right]$), then for every $w \in D_0^{(1)}$ (respectively $w \in V_0T\mathbb{R}^n \backslash D_0^{(1)}$) there is a linear isomorphism

$\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $d\varphi(v) = w$ and $d\varphi(D) = D$. Now, from the naturality condition we deduce that $w \in \Lambda_{\mathbb{R}^n}(D)_0$. Hence, $D_0^{(1)} \subset \Lambda_{\mathbb{R}^n}(D)_0$ (respectively $V_0T\mathbb{R}^n \backslash D_0^{(1)} \subset \Lambda_{\mathbb{R}^n}(D)_0$). Therefore $\Lambda_{\mathbb{R}^n}(D)_0$ is either $\{0\}$ or $D_0^{(1)}$ or $V_0T\mathbb{R}^n$. From Lemma 5.6 we deduce that Λ is from the list presented in Theorem 5.5.

(II) It remains to assume that there are constant vector fields Y, Z on $\mathbb{R}^n$ such that $Y \neq 0$ and $Y^C(0) + Z^V(0) \in \Lambda_{\mathbb{R}^n}(D)_0$. From Lemma 5.6 we obtain that for $e_n = \partial_n(0)$ we have $Y^C(e_n) + Z^V(e_n) \in \Lambda_{\mathbb{R}^n}(D)_{e_n}$.

If Y is a D-vector field (respectively Y is not a D-vector field), then for every constant nonzero D-vector field U on $\mathbb{R}^n$ (respectively for every constant non D-vector field U on $\mathbb{R}^n$), there is a linear isomorphism $\psi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi_* Y = U$ and $d\psi(D) = D$. Now, by the naturality condition, we obtain $U^C(0) + (\psi_* Z)^V(0) \in \Lambda_{\mathbb{R}^n}(D)_0$. Hence, it is sufficient to consider two possibilities:

$$(a) \qquad Y = \partial_1 \text{ and } \Lambda_{\mathbb{R}^n}(D)_{e_n} \subset D^{(3)}_{e_n}$$

$$(b) \qquad Y = \partial_n$$

In (a) using the (local) diffeomorphism $\Phi(y^1, \dots, y^n) = (y^1 + y^1y^n, y^2, \dots, y^n)$ preserving D and e_n, by the naturality condition, we deduce that

$$\partial_1^C(e_n) + Z^V(e_n) + \partial_1^V(e_n) = dd\Phi(\partial_1^C(e_n) + Z^V(e_n)) \in \Lambda_{\mathbb{R}^n}(D)_{e_n}$$

It implies $\partial_1^V(e_n) \in \Lambda_{\mathbb{R}^n}(D)_{e_n}$. By Lemma 5.6 we obtain $\partial_1^V(0) \in \Lambda_{\mathbb{R}^n}(D)_0$. Using a similar arguments as in (I) we get $D_0^{(1)} \subset \Lambda_{\mathbb{R}^n}(D)_0$. From the naturality condition for $(x^1, \dots, x^n) \to (x^1, \dots, x^q, t\,x^{q+1}, \dots, t\,x^n)$, $t \neq 0$, and putting $t \to 0$ we can assume that $Z^V(0) \in D_0^{(1)}$. Thus $\partial_1^C(0) \in \Lambda_{\mathbb{R}^n}(D)_0$. Now similar to (I) we have $\Lambda_{\mathbb{R}^n}(D)_0 \supset D_0^{(2)}$. If $\Lambda_{\mathbb{R}^n}(D)_0 \backslash D_0^{(2)}$ is not empty, then we can assume that $Z^V(0) \in V_0T\mathbb{R}^n \backslash D_0^{(1)}$ and, in consequence, we have $\Lambda_{\mathbb{R}^n}(D)_0 \supset D_0^{(3)}$. Therefore $\Lambda_{\mathbb{R}^n}(D)_0$ is either $D_0^{(2)}$ or $D_0^{(3)}$.

In (b), using the (local) diffeomorphism $\Psi(y^1, \dots, y^n) = (y^1, \dots, y^{n-1}, y^n + \frac{1}{2}(y^n)^2)$ we deduce that $\partial_n^V(0) \in \Lambda_{\mathbb{R}^n}(D)_0$. Then, similar as in (a), we obtain $V_0T\mathbb{R}^n \subset \Lambda_{\mathbb{R}^n}(D)_0$. It implies that $\partial_n^C(0) \in \Lambda_{\mathbb{R}^n}(D)_0$, and in consequence, we have $\Lambda_{\mathbb{R}^n}(D)_0 = T_0T\mathbb{R}^n$.

Now, Lemma 5.6 implies Theorem 5.5. The proof is finished. □

In the case of the cotangent bundle we have the following examples of liftings of foliations [79]:

Example 5.7. Let D be a q-dimensional foliation on a n-manifold M. We construct the following distributions on T^*M. For $\omega \in T^*M$ we set

$$\begin{aligned}
D_\omega^{(1)} &= \{\frac{d}{dt}(\omega + t\sigma)_{t=0} \in T_\omega T^*M : \sigma \in Anih(D_{\pi(\omega)})\},\\
D_\omega^{(2)} &= \{X^C(\omega) \ : \ X \text{ is a } D\text{-vector field}\} + V_\omega(T^*M),\\
D^{(a)} &= \bigcup_{\omega \in T^*M} D_\omega^{(a)}, \qquad \text{for } a = 1, 2.
\end{aligned}$$

$D^{(1)}$ is a foliation on T^*M of dimension $n - q$. $D^{(2)}$ is a foliation on T^*M of dimension $n + q$. Of course, the correspondences $D \to D^{(a)}$ are natural prolongations from q-foliations on n-manifolds to cotangent bundles.

We have the following theorem (see [79])

Theorem 5.8. *Let q, n be such that $0 < q < n$. If $\Lambda = \{\Lambda_M\}$ is a natural prolongation of q-dimensional foliations from n-manifolds to distributions on cotangent bundles such that $\Lambda_M(D)$ is a foliation for any n-manifold M and any $D \in Fol_q(M)$, then Λ_M is equal to one of the list:*

(1) $\Lambda_M(D) = \{0\}$ *is the zero distribution,*
(2) $\Lambda_M(D) = T(T^*M)$ *is the full distribution,*
(3) $\Lambda_M(D) = V(T^*M)$ *is the vertical distribution,*
(4) $\Lambda_M(D) = D^{(a)}$, $a = 1, 2$, *are constructed in Example 5.7*

§ 6. Prolongation of geometric structures to product preserving functors.

Let $\mathcal{F}$ be a product preserving functor. In this section we will study problems of prolongations of geometric structures from a manifold M to $\mathcal{F}(M)$.

At first we define a complete lift of tensor fields of type $(1, k)$ from manifolds to $\mathcal{F}$.

Let S be a tensor field of type $(1, k)$ on a manifold M, $k \geq 1$. We interpret S as a mapping $S : TM \times_M \cdots \times_M TM \longrightarrow TM$ of the bundle product over M of k copies of the tangent bundle into TM covering the identity on M such that its restrictions to fibres $(TM \times_M \cdots \times_M TM)_x = T_xM \times \cdots \times T_xM \to T_xM$ are k-linear for all $x \in M$. We apply the product preserving functor $\mathcal{F}$ to S. If we compose the induced mapping $\mathcal{F}(TM) \times_{\mathcal{F}(M)} \cdots \times_{\mathcal{F}(M)} \mathcal{F}(TM) \to \mathcal{F}(TM)$ with the natural isomorphism $\eta_M : \mathcal{F}(TM) \longrightarrow T(\mathcal{F}M)$ we obtain a mapping

$$S^C : T(\mathcal{F}M) \times_{\mathcal{F}(M)} \cdots \times_{\mathcal{F}(M)} T(\mathcal{F}M) \longrightarrow T(\mathcal{F}M)$$

called *complete lift of S from M to $\mathcal{F}(M)$*. S^C is a tensor field of type $(1, k)$ on $\mathcal{F}(M)$ (see [29]).

Let $\mathcal{T}^1_k(M)$ be the space of all smooth tensor fields of type $(1, k)$ on M. We have the following proposition:

Proposition 6.1. *If $\mathcal{F}$ is a product preserving functor, then the family of mappings*

$$\mathcal{T}^1_k(M) \ni S \longrightarrow S^C \in \mathcal{T}^1_k(\mathcal{F}M)$$

(where M is an n-dimensional manifold) is a lifting.

Proof. We need to show only the naturality condition. In order of this let $S \in \mathcal{T}^1_k(M)$ and $S' \in \mathcal{T}^1_k(N)$ be φ-related, where $\varphi : M \to N$ ia an embedding. It means that the following diagram

$$\begin{array}{ccc} TM \times_M \cdots \times_M TM & \xrightarrow{S} & TM \\ {\scriptstyle d\varphi\times\cdots\times d\varphi}\downarrow & & \downarrow{\scriptstyle d\varphi} \\ TN \times_N \cdots \times_N TN & \xrightarrow{S'} & TN \end{array}$$

commutes. Applying $\mathcal{F}$ to this diagram and using the isomorphism $\eta_M : \mathcal{F}(TM) \leftrightarrow T(\mathcal{F}M)$ we obtain the commutative diagram

$$\begin{array}{ccc} T(\mathcal{F}M) \times_{\mathcal{F}(M)} \cdots \times_{\mathcal{F}(M)} T(\mathcal{F}M) & \xrightarrow{S^C} & T(\mathcal{F}M) \\ {\scriptstyle d\mathcal{F}(\varphi)\times\cdots\times d\mathcal{F}(\varphi)}\downarrow & & \downarrow{\scriptstyle d\mathcal{F}(\varphi)} \\ T(\mathcal{F}N) \times_{\mathcal{F}(N)} \cdots \times_{\mathcal{F}(N)} T\mathcal{F}N) & \xrightarrow{S'^C} & T(\mathcal{F}N) \end{array}$$

so S^C and S'^C are $\mathcal{F}(\varphi)$-related. $\square$

For $\mathcal{F}(M) = TM$ the constructed lifting coincides with the complete lifting from Example 3.8.

For two tensor fields S, S' on M respectively of type $(1,k)$ and $(1,k')$ we define a tensor fields SS' of type $(1, k+k'-1)$ on M by

$$(SS')(v_1, \ldots, v_{k+k'-1}) = S(S'(v_1, \ldots, v_{k'}), v_{k'+1}, \ldots, v_{k+k'-1}) \tag{6.1}$$

for $v_1, \ldots, v_{k+k'-1} \in T_xM$, $x \in M$. If S, S' are tensor fields of type $(1,1)$, then SS' is the composition $SS' : TM \xrightarrow{S'} TM \xrightarrow{S} TM$.

We have the following properties of complete lift of tensor fields to a Weil functor $\mathcal{F}$.

Proposition 6.2. *Let $\mathcal{F}$ be a product preserving functor.*

(a) The mapping $\mathcal{T}^1_k(M) \ni S \longrightarrow S^C \in \mathcal{T}^1_k(\mathcal{F}M)$ is linear over $\mathbb{R}$.

(b) For tensor fields S, S" respectively of type $(1,k)$ and $(1,k')$ on M we have $(SS')^C = S^C S'^C$, where SS' is defined by (6.1).

(c) If $W(t)$ is a polynomial with real coefficients and S is a tensor fields of type $(1,1)$ on M, then $W(S^C) = (W(S))^C$.

Proof. (a) is evident. (b) is a consequence of the formula $SS' = S \circ (S' \times id)$. From (b) we show by induction that $(S^n)^C = (S^C)^n$. According to (a) it implies immediately (c) (see [29]). $\square$

From the above proposition we obtain immediately

Corollary 6.3. *Let $\mathcal{F}$ be a product preserving functor. If t is an almost complex structure (respectively an almost tangent structure, a f-structure) on M, then so is t^C.*

The analogous properties was proved in the case of tangent bundle, r-order tangent bundle, p^r-velocities bundle in [123], [84], [87], [89].

We prove now

Proposition 6.4. *Let $\mathcal{F}$ be a product preserving functor, $A = \mathcal{F}(R)$ be its local algebra and let S be a tensor field of type $(1,k)$ on M.*

If $X_1, \ldots, X_k \in \mathcal{X}(M)$ are vector fields on M and $a_1, \ldots, a_k \in A$, then

$$S^C(a_1 \cdot X_1^C, \ldots, a_k \cdot X_k^C) = a_1 \cdots a_k \cdot (S(X_1, \ldots, X_k))^C,$$

where "$\cdot$" is defined by (4.5).

Proof. The equality $(t_1X_1, \ldots, t_kX_k) = t_1 \ldots t_k S(X_1, \ldots, x_k)$ and definitions of S^C, X_i^C and of the product "$\cdot$" imply immediately our proposition. $\square$

To prove the next proposition we will need the following lemma:

Lemma 6.5. *Let $\mathcal{F}$ be a product preserving functor and $A = \mathcal{F}(\mathbb{R})$. If $\widetilde{S}, \widetilde{S}'$ are two fields of tensors of type $(1,k)$ on $\mathcal{F}(M)$ such that for all vector fields $X_1, \ldots, X_k$ on M and all elements $a_1 \ldots, a_k$ of A the equality*

$$\widetilde{S}(a_1 \cdot X_1^C, \ldots, a_k \cdot X_k^C) = \widetilde{S}'(a_1 \cdot X_1^C, \ldots, a_k \cdot X_k^C)$$

holds, then $\widetilde{S} = \widetilde{S}'$.

Proof. Let (U, φ) be a chart on M, $\partial_1, \dots, \partial_n$ be its adapted frame and let $a_1, \dots, a_K$ be a basis of A. Analyzing the definition of the product "$\cdot$" we deduce that $\{ a_\nu \cdot \partial_i^C \mid i = 1, \dots, n,\ \nu = 1, \dots, K \}$ is the adapted frame to the induced chart $(\mathcal{F}(U), \mathcal{F}(\varphi))$ on $\mathcal{F}(M)$. It implies our lemma. □

Remark 6.6. From the proof of Lemma 6.5 it follows that the analogous lemma is also true for fields of tensors of type $(0, k)$.

Proposition 6.7. *Let $\mathcal{F}$ be a product preserving functor. If S is a field of tensors of type $(1,1)$ on M, then*

$$N_{t^C} = (N_t)^C,$$

where N denotes the Nijenhuis tensor.

Proof. Let X, Y be vector fields on M and a, b be elements of $A = \mathcal{F}(\mathbb{R})$. From Proposition 6.4 and equality (4.6) we obtain

$$N_{t^C}(a \cdot X^C, b \cdot Y^C) = a \cdot b \cdot (N_t(X,Y))^C = (N_t)^C(a \cdot X^C, b \cdot Y^C)$$

Now Lemma 6.5 implies our proposition. □

Because of the integrability for considered structures is equivalent to vanishing of the Nijenhuis tensor, thus we obtain

Theorem 6.8. *Let $\mathcal{F}$ be a product preserving functor. Let t be an almost complex structure (respectively a almost tangent structure, a f-structure) on M. Then t^C is a complex structure (respectively a tangent structure, an integrable f-structure) on $\mathcal{F}(M)$ if and only if t is a complex structure (respectively a tangent structure, an integrable f-structure) on M.*

We can also present two propositions on the integrability of structures deferent from Theorem 6.8. We recall that a tensor field S on M is integrable if in a neighborhood of any point of M there exists a chart (U, φ) in which coordinates of S are constant.

Proposition 6.9. *Let $\mathcal{F}$ be a product preserving functor. If S is an integrable tensor field of type $(1, k)$ on M, then S^C is also integrable.*

Proof. There is an atlas on M such that for every its chart (U, φ) the coordinates of S are constant. It means that there exists an k-linear mapping $S_0 : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}^n$ such that the following diagram

$$\begin{array}{ccccccc}
 & & TM \times_M \cdots \times_M TM & \xrightarrow{S} & TM & & \\
 & & \big\downarrow{\scriptstyle d\varphi \times \cdots \times d\varphi} & & \big\downarrow{\scriptstyle d\varphi} & & \\
\mathbb{R}^n \times \cdots \times \mathbb{R}^n & = & T\mathbb{R}^n \times_{\mathbb{R}^n} \cdots \times_{\mathbb{R}^n} T\mathbb{R}^n & \xrightarrow{\widetilde{S}} & TM & = & \mathbb{R}^n \times \mathbb{R}^n
\end{array}$$

commutes, where $\widetilde{S}(x, v_1, \dots, v_k) = (x, S_0(v_1, \dots, v_k))$. Applying the functor $\mathcal{F}$ to this diagram and using the natural identifications $\eta_M : \mathcal{F}(TM) \to T\mathcal{F}(M)$, $\mathcal{F}(\mathbb{R}^n) = A^n$ we obtain the commutative diagram

$$\begin{array}{ccc}
T\mathcal{F}(M) \times_{\mathcal{F}(M)} \cdots \times_{\mathcal{F}(M)} T\mathcal{F}(M) & \xrightarrow{S^C} & T\mathcal{F}(M) \\
\big\downarrow{\scriptstyle d\mathcal{F}(\varphi) \times \cdots d\mathcal{F}(\varphi)} & & \big\downarrow{\scriptstyle d\mathcal{F}(\varphi)} \\
(A^n)^{k+1} & \xrightarrow{\widetilde{\widetilde{S}}} & A^n \times A^n
\end{array}$$

where $\widetilde{\widetilde{S}}$ is defined by $\widetilde{\widetilde{S}}(a, a_1, \dots, a_k) = (a, \mathcal{F}(S_0)(a_1, \dots, a_k))$. Since $\mathcal{F}(S_0) : A^n \times \dots \times A^n \to A^n$ is k-linear, S^C has constant coordinates with respect to the induced chart $(\mathcal{F}(U), \mathcal{F}(\varphi))$ on $\mathcal{F}(M)$. The proof is finished. □

Proposition 6.10. *Let $\mathcal{F}$ be a product preserving functor and let a family $\Lambda = \{\Lambda_M\}$ of mappings $\Lambda_M : \mathcal{T}_1^1(M) \longrightarrow \mathcal{T}_1^1(\mathcal{F}(M))$ be a lifting of tensor fields of type $(1,1)$ from M to $\mathcal{F}(M)$. If t is an integrable tensor field on M, so is $\Lambda_M(t)$.*

Proof. By the naturality condition we can assume that $M = \mathbb{R}^n$ and

$$t = \sum_{i,j=1}^{n} a_j^i dx^j \otimes \frac{\partial}{\partial x^i},$$

where a_j^i are real constants and $(x^1, \dots, x^n)$ are the canonical coordinates on $\mathbb{R}^n$.

We fix a basis of $A = \mathcal{F}(\mathbb{R})$. Then we have the induced coordinates $x^{i,\nu}$, in $T^A\mathbb{R}^n$, where $i = 1, \dots, n$, $\nu = 1, \dots, K = \dim A$. Suppose that

$$\Lambda_{\mathbb{R}^n}(t) = \sum_{i,j=1}^{n} \sum_{\nu,\mu=1}^{K} a_{j,\mu}^{i,\nu} dx^{j,\mu} \otimes \frac{\partial}{\partial x^{i,\nu}},$$

where $a_{j,\mu}^{i,\nu} : T^A\mathbb{R}^n \to \mathbb{R}$ are smooth functions. Since the homotheties $\tau\, id$, where $\tau \neq 0$, preserve t, then by the naturality condition $T^A(\tau\, id)$ preserves $\Lambda_{\mathbb{R}^n}(t)$. $T^A(\tau\, id)$ preserves $dx^{j,\mu} \otimes \frac{\partial}{\partial x^{i,\nu}}$ because $T^A(\tau id)$ is the homothety on $T^A\mathbb{R}^n$. Thus, for $\tau \neq 0$ and all $\eta \in T^A\mathbb{R}^n$ we have $a_{j,\mu}^{i,\nu}(\tau\eta) = a_{j,\mu}^{i,\nu}(\eta)$. Putting $\tau \to 0$ we obtain that $a_{j,\mu}^{i,\nu}$ are constants. Then $\Lambda_{\mathbb{R}^n}(t)$ is integrable. □

Let us observe that using the same arguments we can prove the similar proposition for: (1) tensor fields of type (p,p), and (2) tensor fields of an arbitrary type (p,q) if Λ is linear over $\mathbb{R}$.

We can define also liftings of tensor fields of type $(0,k)$ but there are no "complete" lift (see [29]).

Now we present an other point of view for prolongations of structures. Let $P(M,G,\pi)$ be a principal fibre bundle over M with the group structure G and the projection π and let H be a closed subgroup of G. We recall that a *H-structure* in $P(M,G,\pi)$ is a reduced fibre bundle $P_0(M,H)$ in $P(M,G,\pi)$ with the structure group H. From Proposition 2.11(3) we obtain immediately that if $P_0(M,H)$ is a H-structure in $P(M,G,\pi)$ and $\mathcal{F}$ is a product preserving functor, then $\mathcal{F}(P_0)(\mathcal{F}(M), \mathcal{F}(H))$ is a $\mathcal{F}(H)$-structure in $\mathcal{F}(P)(\mathcal{F}(M), \mathcal{F}(G), \mathcal{F}(\pi))$. We will study the case when $P(M,G,\pi)$ is the principal bundle of linear frames. We prove.

Proposition 6.11. *Let $\mathcal{F}$ be a product preserving functor.*

There exist a natural monomorphism $I_M : \mathcal{F}(LM) \to L(\mathcal{F}(M))$ of principal fibre bundles such that for every embedding $\varphi : M \to N$ of two n-dimensional manifolds M, N the diagram

$$\begin{array}{ccc}
\mathcal{F}(LM) & \xrightarrow{I_M} & L(\mathcal{F}(M)) \\
{\scriptstyle \mathcal{F}(L(\varphi))}\downarrow & & \downarrow{\scriptstyle L(\mathcal{F}(\varphi))} \\
\mathcal{F}(LN) & \xrightarrow{I_N} & L(\mathcal{F}(N))
\end{array}$$

commutes.

If (U,φ) is a chart on M and $\sigma : U \to LM$ is the associated local section of LM, then $I_M \circ \mathcal{F}(\sigma) : \mathcal{F}(U) \to L\mathcal{F}(M)$ is a local section of $L\mathcal{F}(M)$ associated with the induced chart $(\mathcal{F}(U), \mathcal{F}(\varphi))$ on $\mathcal{F}(M)$.

Proof. According to Proposition 2.11(1) $\mathcal{F}(LM)$ is a principal fibre bundle with base $\mathcal{F}(M)$ and the structure group $\mathcal{F}(GL(n,\mathbb{R}))$. By Proposition 2.10(4) we can consider $\mathcal{F}(GL(n,\mathbb{R}))$ as a subgroup of $GL(\mathcal{F}(R^n))$.

We consider an atlas $\{(U_i,\varphi_i)\}$ on M. Let $\sigma_i : U_i \to LM$ and $\widetilde{\sigma}_i : \mathcal{F}(U_i) \to L\mathcal{F}(M)$ be the associated sections with (U_i,φ_i) and $(\mathcal{F}(U_i), \mathcal{F}(\varphi_i))$ respectively. There exists a principal fibre bundle homomorphism $I_M : \mathcal{F}(LM) \to L\mathcal{F}(M)$ covering the identity on M with the inclusion of structure groups such that $I_M \circ \mathcal{F}(\sigma_i) = \widetilde{\sigma}_i$. It is clear that the diagram from the proposition is commutative. □

Let H be a closed subgroup of $GL(n,\mathbb{R})$. We recall that a H-structure on M (i.e. an H-structure P_0 in LM) is called *integrable* if there exists an atlas on M such that sections associated with all charts of the atlas have values in P_0. From Proposition 6.11 it follows immediately

Proposition 6.12. *Let $\mathcal{F}$ be a product preserving functor and let $H \subset GL(n,\mathbb{R})$. If P_0 is a H-structure on M, then $I_M(\mathcal{F}(P_0))$ is a $\mathcal{F}(H)$-structure on $\mathcal{F}(M)$. If P_0 is integrable, then so is $I_M(\mathcal{F}(P_0))$.*

To finish this paragraph we define a complete lift of connections. We start from a general situations of connections in principal fibre bundles (see Pogoda [99]).

Let $\mathcal{F}$ be a product preserving functor and $P(M,G)$ be a principal fibre bundle with a connection $\omega : TP \to TG$ on P. We define $\omega^C : T\mathcal{F}(P) \to T\mathcal{F}(G)$ as the composition of the induced mapping $\mathcal{F}(\omega) : \mathcal{F}(TP) \to \mathcal{F}(TG)$ with the canonical isomorphisms $\eta_P : \mathcal{F}(TP) \to T\mathcal{F}(P)$, $\eta_G : \mathcal{F}(TG) \to T\mathcal{F}(G)$, i.e.

$$\omega^C = \eta_G \circ \mathcal{F}(\omega) \circ \eta_P^{-1}.$$

ω^C is called *complete lift* of ω from P to $\mathcal{F}(P)$.

It is easy to prove (see [99]).

Proposition 6.13. *Let $\mathcal{F}$ be a product preserving functor and $P(M,G)$ be a principal fibre bundle. If $\omega : TP \to TG$ is a connection in $P(M,G)$ then the complete lift $\omega^C : T\mathcal{F}(P) \to T\mathcal{F}(G)$ is a connection in the principal fibre bundle $\mathcal{F}(P)(\mathcal{F}(M), \mathcal{F}(G))$.*

Let $f : P \to P'$ be a homomorphism of principal fibres and let $\rho_f : G \to G'$ be the corresponding Lie group homomorphism. Connections $\omega : TP \to TG$ and $\omega' : TP' \to TG'$ are called *f-related*, where $f : P \to P'$ is a homomorphism of principal fibre bundle, if the following diagram

$$\begin{array}{ccc} TP & \xrightarrow{df} & TP' \\ {\scriptstyle\omega_1}\downarrow & & \downarrow{\scriptstyle\omega_2} \\ TG & \xrightarrow{d\rho_f} & TG' \end{array}$$

commutes.

The complete lift of connections has the following property (see Pogoda [99]).

Proposition 6.14. *Let $\mathcal{F}$ be a product preserving functor, $f : P(M,G) \to P'(M',G')$ be a homomorphism of principal fibre bundles and let ω, ω' be connections in $P(M,G)$ and $P'(M',G')$ respectively. If ω and ω' are f-related, then ω^C and ω'^C are $\mathcal{F}(f)$-related.*

The above proposition means that the correspondence $\omega \to \omega^C$ satisfy the naturality condition but only for mappings $f : P \to P'$ being homomorphisms of principal fibre bundles. The family of mappings $\omega \longrightarrow \omega^C$ is a lifting in the sense of more general definition as Definition 3.1. It is called "gauge-natural" transformation (see [8]).

We can prove many properties of the complete lift of connections. For instance we have the following results about the Whitney product (see [99], for the definition of Whitney product see [44]).

Proposition 6.15. *Let $\mathcal{F}$ be a product preserving functor, $P_1(M,G_1)$ and $P_2(M,G_2)$ be two principal fibre bundles and let ω_1 and ω_2 be connections in $P_1(M_1,G_1)$ and $P_2(M_2,G_2)$ respectively. If ω is the Whitney product of these connections, then ω^C is the Whitney product of the connections ω_1^C and ω_2^C .*

We can apply our construction to linear connections. If $\omega : LM \to TG(n,\mathbb{R}^n)$ is a linear connection on M then, according to our general definition, $\omega^C : \mathcal{F}(LM) \to T\mathcal{F}(GL(n,\mathbb{R}))$ is a connection in $\mathcal{F}(LM)$. Using the inclusion $I_M : \mathcal{F}(LM) \to L(\mathcal{F}(M))$ we can prolong ω^C to one and only one linear connection $L(\mathcal{F}(M)) \to GL(\mathcal{F}(\mathbb{R}^n))$. This prolongation we denote also by ω^C. In the case of the tangent bundle $\mathcal{F}(M) = TM$ the constructed connection coincides with the connection defined in Example 3.9. We can prove (see [29]).

Proposition 6.16. *Let $\mathcal{F}$ be a product preserving functor.*

(1) If ω is a linear connection on a manifold M then ω^C is a linear connection on $\mathcal{F}(M)$.

(2) The family of mappings $\underline{Q}(M) \ni \omega \longrightarrow \omega^C \in \underline{Q}(\mathcal{F}(M))$ is a lifting.

(3) If ∇ and ∇^C are the covariant derivation of ω and ω^C respectively, then for all vector fields X, Y on M and all $a, b \in A = \mathcal{F}(\mathbb{R})$ we have

$$\nabla^C_{a\cdot X^C}\, b \cdot Y^C = a \cdot b \left(\nabla_X Y\right)^C .$$

Other properties of liftings of linear connections and geometric objects will be formulated in [29].

Natural prolongations of generalized connections were studied by Slovák [115].

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Received 21 September 1992