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On the order reduction of differential invariants*

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Abstract: The notion of a basis of differential invariants (natural mappings) is discussed. A formula is shown determining a basis of invariants of arbitrary order by means of the lowest order basis of invariants. As an example, all invariants of n-tuples of vector fields are determined. It is shown that each such a differential invariant is generated by the Lie bracket.

Key words: jet, velocity, differential group, prolongation of a jet bundle, differential invariant, basis of invariants, Lie bracket.

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1. Introduction

There exist a few methods of searching differential invariants. One of the most relevant and effective methods is the *factorization*. Let G, H be Lie groups, $\pi: G \to H$ a surjective homomorphism, P a left G-manifold. It is well-known that the problem of a characterization of all π-equivariant mappings (*π-invariants*) of P can be solved as follows. The set $P/\ker\pi$ is naturally endowed with an action of the group H. To each π-invariant $g: P \to Q$, where Q is an arbitrary left H-manifold, there exists an H-equivariant mapping $g_0: P/\ker\pi \to Q$ such that the diagram

$$\begin{tikzcd} P \arrow[d] \arrow[dr, "g"] & \\ P/\ker\pi \arrow[r, "g_0"] & Q \end{tikzcd}$$

commutes. If moreover, $P/\ker\pi$ is endowed with a smooth structure such that the projection $P \to P/\ker\pi$ is a submersion, g_0 is a smooth mapping.

This situation leads naturally to the notion of a basis of invariants. Let P_0 be a left H-manifold. A π-invariant mapping $f: P \to P_0$ is called a *basis of π-invariants* of P if to every π-invariant mapping $g: P \to Q$ there exists a mapping $g_0: \operatorname{Im} f \to Q$ (not

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necessarily continuous) such that the diagram

$$\begin{tikzcd} P \arrow[d, "f"'] \arrow[dr, "g"] & \\ \mathrm{Im}\, f \arrow[r, "g_0"] & Q \end{tikzcd}$$

commutes. We note that in many cases f is not a submersion and then g_0 may not be smooth, even if g is smooth. Many of these cases are described by the notion of *nearly-regular* basis of invariants.

We use the concept of a basis of invariants in a sense similar to Ovsjannikov [5].

In the most important case where the groups G, and H are equal to some differential groups ($G = L^r_n, H = L^s_n$, where $s < r$) and the homomorphism π is their canonical projection we get a theory of natural differential operators. In this case π-invariants of P are in a bijective correspondence with natural transformations from certain bundle functor F^r_P of order r to bundle functors F^s_Q of order s. These natural transformations define differential operators transforming, for every manifold X, sections of $F^r_P X$ into sections of $F^s_Q X$ and depending only on 0-jets of these sections. If we find a basis of π-invariants of P we can characterize all such natural transformations with the help of natural transformations of s-th order functors. Thus we can *reduce* an order of the searched natural transformations. If we have an invariant of the k-th order jet prolongation $T^k_n P$ of P we get a differential operator on sections of $F^r_P X$ which depends on k-jets of these sections.

The main result of this paper (Theorem 4) together with Theorem 2 gives us a method how to derive from a basis of invariants of $T^1_n P$ and some other spaces of the same order, a basis of invariants of $T^k_n P$ for arbitrary k. This method gives a simplification of the proofs of many well-known results (see e.g. Krupka [3], and examples given in [2]). In this paper we derive as an example a new result on n-tuples of vector fields.

From now on, we use the standard terminology and results of the theory of natural differential operators ([6],[4],[2]) with the following notations:

L^r_n denotes the r-th *differential group* of invertible r-jets with source and target at $0 \in R^n$. The first (resp. second) canonical coordinates on L^r_n are denoted by $(a^i_{j_1}, \dots a^i_{j_1 \dots j_r})$ (resp. $(b^i_{j_1}, \dots, b^i_{j_1 \dots j_r})$). By definition, $a^i_{j_1 \dots j_u}(J^r_0 g) = D_{j_1} \dots D_{j_u} g^i(0)$, and $b^i_{j_1 \dots j_u}(J^r_0 g) = D_{j_1} \dots D_{j_u}(g^{-1})^i(0)$. $\pi^{r,s}_n \colon L^r_n \to L^s_n$, $r \geq s$, is the natural group homomorphism.

The bundle of (r,n)-*velocities* $T^r_n P$ of a manifold P is a bundle of r-jets with source $0 \in R^n$ and target in P. If (x^i) are the coordinates on P then we have the induced coordinates $(x^i, x^i_{,k_1}, \dots, x^i_{,k_1 \dots k_r})$ on $T^r_n P$, where $x^i(J^r_0 g) = x^i(g(0))$, and $x^i_{k_1 \dots k_u}(J^r_0 g) = D_{k_1} \dots D_{k_u} x^i \circ g(0)$.

Let G be a Lie group. The (r,n)-*prolongation* G^r_n of G is the semi-direct product $L^r_n \times_s T^r_n G$. The group multiplication on G^r_n is defined by the formula $(A, S) \cdot (B, T) = (A \cdot B, S \cdot (T \circ A^{-1}))$.

If a manifold P is endowed with a left action of a group G, then the manifold P^r_n is endowed with the induced action of G^r_n.

There is a canonical embedding and Lie group homomorphism $\nu \colon L^{r+s}_n \to (L^s_n)^r_n$.

All mappings considered in this paper are smooth; exceptions are evident.

2. Basis of invariants

Let G_1, and G_2 be Lie groups, $\pi: G_1 \to G_2$ their surjective homomorphism, P_1 (resp. P_2) a left G_1-manifold (resp. G_2-manifold). A smooth mapping $f: P_1 \to P_2$ is called a *π-invariant* (of P_1) if for any $a \in G_1$, and $p \in P_1$ it holds $f(a \cdot p) = \pi(a) \cdot f(p)$. Evidently, the set $\operatorname{Im} f$ is an H-invariant subset of P_2.

Let $f_1: P_1 \to P_2$ (resp. $f_2: P_2 \to P_3$) be a π_1-invariant (resp. π_2-invariant). Then the mapping $f_2 \circ f_1: P_1 \to P_3$ is evidently a $(\pi_2 \circ \pi_1)$-invariant.

Let $f: P_1 \to P_2$, and $g: P_1 \to P_3$ be two π-invariants of P_1. A *morphism* $\gamma: f \to g$ is a mapping $\gamma: \operatorname{Im} f \to P_3$ such that for every $p \in P_1$ it holds

$$g(p) = \gamma(f(p)).$$

Evidently, the mapping γ is a uniquely determined G_2-invariant mapping of the sets $\operatorname{Im} f$, and P_3. The *composition* $\delta \circ \gamma$ of two morphisms $\gamma: f \to g$, and $\delta: g \to h$ is defined by a natural way.

We get a category $\mathcal{C}(P_1, \pi)$ of π-invariants of P_1 and their morphisms. The initial object of this category (e.g. the object $f \in \operatorname{Ob}\mathcal{C}(P_1, \pi)$ such that for every $g \in \operatorname{Ob}\mathcal{C}(P_1, \pi)$ there exists a unique $\gamma \in \operatorname{Mor}\mathcal{C}(P_1, \pi)$, $\gamma: f \to g$) is called a *basis of π-invariants* of P_1. If this object exists it is unique up to an isomorphism.

Lemma 1. *Let $f: P_1 \to P_2$ be a π-invariant and let for any $p, q \in P_1$ such that $f(p) = f(q)$ there exists $a \in \ker \pi$ such that $p = a \cdot q$. Then f is a basis of π-invariants of P_1.*

Proof. Let $g: P_1 \to P_3$ be a π-invariant. If for $p, q \in P_1$ it holds $f(p) = f(q)$ then $g(p) = \pi(a) \cdot g(q) = g(q)$, and we can define $\gamma(f(p)) = g(p)$.

Lemma 2. *Let X, Y, and Z be smooth manifolds, $f: X \to Y$ a surjective submersion and $g: X \to Z$ a smooth mapping. If there exists a mapping $\gamma: Y \to Z$ such that $g = \gamma \circ f$, then it is a smooth mapping.*

Lemma 1 and Lemma 2 give us an inspiration to the following definition. A basis $f: P_1 \to P_2$ of π-invariants ($\pi: G_1 \to G_2$) is called *regular*, if the set $\operatorname{Im} f \subset P_2$ is an immersed submanifold, and $(P_1, \operatorname{Im} f, \pi)$ is a fibered manifold whose fibres are equal to orbits of the group $\ker \pi$.

Theorem 1. *Let $f: P_1 \to P_2$ be a regular basis of π-invariants ($\pi: G_1 \to G_2$), and $g: P_1 \to P_3$ be an arbitrary π-invariant. Then the corresponding morphism $\gamma_g: \operatorname{Im} f \to P_3$ is a smooth id_{G_2}-invariant mapping and the correspondence $g \to \gamma_g$ between π-invariants of P_1 and id_{G_2}-invariants of $\operatorname{Im} f$ is a bijection.*

Proof. Let $p_2 = f(p_1) \in \operatorname{Im} f$. Then for any $a_2 = \pi(a_1) \in G_2$ it holds $\gamma_g(a_2 \cdot p_2) = \gamma_g(\pi(a_1) \cdot f(p_1)) = \gamma_g(f(a_1 \cdot p_1)) = g(a_1 \cdot p_1) = a_2 \cdot \gamma_g(p_2)$. Thus, $\gamma_g: \operatorname{Im} f \to P_3$ is an id_{G_2}-invariant. Smoothness of γ_g follows from Lemma 2. The last assertion follows from the fact that a composition of a π-invariant and an id_{G_2}-invariant is a π-invariant.

Theorem 2. *Let $\pi_1: G_1 \to G_2$ (resp. $\pi_2: G_2 \to G_3$) be a surjective Lie group homomorphism and let $f_1: P_1 \to P_2$ (resp. $f_2: P_2 \to P_3$) be a regular basis of π_1-invariants (resp. π_2-invariants) of P_1 (resp. P_2). Then the composition $f_2 \circ f_1$ is a regular basis of $(\pi_2 \circ \pi_1)$-invariants of P_1.*

Proof. As we already know the mapping $f_2 \circ f_1$ is a $(\pi_2 \circ \pi_1)$-invariant. The set $\mathrm{Im}\, f_1 \subset P_2$ is a G_2-invariant submanifold. Hence, the triple $(\mathrm{Im}\, f_1, \mathrm{Im}\,(f_2|\mathrm{Im}\, f_1), f_2|\mathrm{Im}\, f_1)$ is a fibered manifold. Since the composition of surjective submersions is a surjective submersion, the triple $(P_1, \mathrm{Im}\,(f_2 \circ f_1), f_2 \circ f_1)$ is a fibered manifold. Further, let p, and $\bar{p}$ be elements of P_1 such that $f_2(f_1(p)) = f_2(f_1(\bar{p}))$. Then there exists an element $a \in \ker(\pi_2 \circ \pi_1)$ such that $f_1(\bar{p}) = \pi_1(a) \cdot f_1(p)$. But $\pi_1(a) \cdot f_1(p) = f_1(a \cdot p)$ and we can find an element $b \in \ker \pi_1$ transforming $a \cdot p$ to $\bar{p}$. Thus we have $\bar{p} = (b \cdot a) \cdot p$ where evidently $b \cdot a$ belongs to $\ker(\pi_2 \circ \pi_1)$.

Theorem 3. *Let $\pi: G \to H$ be a Lie group homomorphism, $f: P \to Q$ a regular basis of π-invariants of P and let r, n be integers. Then the mapping $T^r_n f: P^r_n \to Q^r_n$ is a regular basis of π^r_n-invariants of P^r_n.*

Proof. We obviously can assume that $\mathrm{Im}\, f = Q$. In this case the triple (P, Q, f) is a fibered manifold. Then the coordinate expression of f with respect to a fiber chart on P is a projection and the expression of $T^r_n f$ with respect to the induced fiber chart on $T^r_n P$ is also a projection. Hence the mapping $T^r_n f$ is a surjective submersion.

Since the mapping f is a π-invariant then for every $(A, S) \in G^r_n$, $A = J^r_0 \alpha$, $S = J^r_0 g$, and $T \in P^r_n$, $T = J^r_0 \tau$ it holds $T^r_n f((A, S) \cdot T) = T^r_n f(S \cdot (T \circ A^{-1})) = J^r_0(f(g \cdot (\tau \circ \alpha^{-1}))) = J^r_0(\pi(g) \cdot f(\tau \circ \alpha^{-1})) = (A, J^r_0 \pi \circ g) \cdot J^r_0 f \circ \tau = \pi^r_n(A, S) \cdot T^r_n f(T)$, and the mapping $T^r_n f$ is a π^r_n-invariant.

Finally, for $J^r_0 \tau_1, J^r_0 \tau_2 \in T^r_n P$ such that $T^r_n f(J^r_0 \tau_1) = T^r_n f(J^r_0 \tau_2)$ it holds $J^r_0 f \circ \tau_1 = J^r_0 f \circ \tau_2$. Moreover since the mapping f is a submersion we can suppose that $f \circ \tau_1 = f \circ \tau_2$. Then there exists $S = J^r_0 g \in G$ such that $\tau_1 = g \cdot \tau_2$. Now, $(J^r_0 \mathrm{id}_{R^n}, S) \cdot J^r_0 \tau_2 = S \cdot J^r_0 \tau_2 = J^r_0(g \cdot \tau_2) = J^r_0 \tau_1$.

Thus the theorem is proved.

Lemma 3. *Let Y and X be manifolds, $\tau: Y \to X$ a smooth mapping of constant rank. If for any $x \in X$ the set $\tau^{-1}(x)$ is connected then* $\mathrm{Im}\, \tau$ *is an immersed submanifold of X and the triple* $(Y, \mathrm{Im}\, \tau, \tau)$ *is a fibered manifold.*

Proof. The mapping τ is a subimmersion, i.e. it holds $\tau = i \circ s$, where s is a submersion, and i is an immersion (see [1]). From the condition of the lemma it follows that i is injective, which means that $\mathrm{Im}\, \tau$ is an immersed submanifold of X.

Now we shall prove our main result.

Theorem 4. *Let n, r, s be positive integers, P (resp. Q) a left L^r_n-manifold (resp. L^{r+s-1}_n-manifold). If a mapping $f^s: T^s_n P \to Q$ is a regular basis of $\pi^{r+s, r+s-1}_n$-invariants of $T^s_n P$, then for any $t > s$ the composition $f^t: T^t_n P \to T^{t-s}_n Q$ of the canonical embedding $\tau: T^t_n P \to T^{t-s}_n T^s_n P$ and the prolonged mapping $T^{t-s}_n f^s: T^{t-s}_n T^s_n P \to T^{t-s}_n Q$ is a regular basis of $\pi^{r+t, r+t-1}_n$-invariants of $T^t_n P$.*

Proof. We shall prove this theorem only for $t = s+1$ since the other cases are immediate consequences of this case. In this proof we use two different notations for the canonical coordinates on L^r_n: the traditional notation $(a^k_{l_1 \ldots l_u})$, (resp. $(b^k_{l_1 \ldots l_u})$) for the first (resp. second) canonical coordinates, and the special notation (a^I), where $I = 1, \ldots, n + n\binom{n+r}{n}$

for the first canonical coordinates. The special coordinates on L_n^{r+s} are then of the form $(a^I, a^I_{k_1\dots k_\sigma})$, $\sigma = 1\dots s$, $1 \le k_1 \le \dots \le k_\sigma \le n$. The coordinates on $(L_n^{r+s})_n^1$ are $(c_l^k, (a^I, a^I_{k_1\dots k_\sigma}, a^I_{,q}, a^I_{k_1\dots k_\sigma,q}))$ (we denote the inverse matrix of (c_l^k) by (d_l^k)).

Let $p \in T_n^1 T_n^s P$, $a \in (L_n^{r+s})_n^1$, $a = (c_l^k, (a^I, a^I_{k_1\dots k_\sigma}, a^I_{,q}, a^I_{k_1\dots k_\sigma,q}))$, $\bar{p} = a\cdot p$. Further let (U,φ), $\varphi = (x^1,\dots,x^m)$ be a chart of P such that $\varrho(p), \varrho(\bar{p}) \in U$ (ϱ is the canonical projection $\varrho\colon T_n^1 T_n^s P \to P$), and there is an integer μ, $0 \le \mu \le \dim P$ such that the orbit of p in the action of L_n^r has the equations $x^i = 0$ for $i > \mu$ (it is evident that $\mu < \dim L_n^r$), and for arbitrary constants c^i, $i > \mu$ the set given by $x^i = c^i$ is a subset of some orbit (existence of such a chart follows from properties of Lie group action orbits. See [2]). Denote the induced coordinates of p (resp. $\bar{p}$) by $(p^i, p^i_{k_1\dots k_\sigma}, p^i_{,q}, p^i_{k_1\dots k_\sigma,q})$ (resp. $(\bar{p}^i, \bar{p}^i_{k_1\dots k_\sigma}, \bar{p}^i_{,q}, \bar{p}^i_{k_1\dots k_\sigma,q})$).

Let $\bar{x}^j = \alpha^j(a^I, x^i)$ be a coordinate expression of the action of L_n^r on P with respect to (U,φ). In particular, it holds $\bar{p}^j = \alpha^j(a^I, p^i)$. We get analogous equations of the action of L_n^{r+s} on $T_n^s P$ by a formal differentiation of this equation (we write simply α^j instead $\alpha^j(a^I, p^i)$):

$$
\begin{aligned}
\bar{p}^j &= \alpha^j \\
\bar{p}^j_k &= \left(\frac{\partial \alpha^j}{\partial a^I} a^I_h + \frac{\partial \alpha^j}{\partial p^i} p^i_h\right) b^h_k \\
\bar{p}^j_{k_1 k_2} &= \left(\frac{\partial \alpha^j}{\partial a^I} a^I_h + \frac{\partial \alpha^j}{\partial p^i} p^i_h\right) b^h_{k_1 k_2} + \left(\frac{\partial^2 \alpha^j}{\partial a^{I_1}\partial a^{I_2}} a^{I_1}_{h_1} a^{I_2}_{h_2} + 2\frac{\partial^2 \alpha^j}{\partial p^i \partial a^I} p^i_{h_1} a^I_{h_2}\right. \\
&\quad \left. + \frac{\partial^2 \alpha^j}{\partial p^{i_1}\partial p^{i_2}} p^{i_1}_{h_1} p^{i_2}_{h_2} + \frac{\partial \alpha^j}{\partial a^I} a^I_{h_1 h_2} + \frac{\partial \alpha^j}{\partial p^i} p^i_{h_1 h_2}\right) b^{h_1}_{k_1} b^{h_2}_{k_2} \\
&\;\;\vdots \\
\bar{p}^j_{k_1\dots k_s} &= \sum_{K^s}\sum_{H^s}\sum_{\gamma} C_{K^s,H^s,\gamma} \frac{\partial^\kappa \alpha^j}{\partial a^{I_1}\dots\partial a^{I_\gamma}\partial p^{i_1}\dots\partial p^{i_{\kappa-\gamma}}} \\
&\quad \cdot a^{I_1}_{H_1}\cdot\ldots\cdot a^{I_\gamma}_{H_\gamma} p^{i_1}_{H_{\gamma+1}}\cdot\ldots\cdot p^{i_{\kappa-\gamma}}_{H_\kappa} b^{h_1}_{K_1}\cdot\ldots\cdot b^{h_\iota}_{K_\iota}
\end{aligned}
$$

(summation runs over all decompositions $K^s = \{K_1,\dots,K_\iota\}$ of the set $\{k_1,\dots,k_s\}$, decompositions $H^s = \{H_1,\dots,H_\kappa\}$ of the sets $\{h_1,\dots,h_\iota\}$, and integers $\gamma \in \{0,\dots,\kappa\}$. $C_{K,H,\gamma}$ are integers which we do not specify exactly)

According to [4], we have in addition the following equations for the action of $(L_n^{r+s})_n^1$ on $T_n^1 T_n^s P$:

$$
\begin{aligned}
\bar{p}^j_{,q} &= \frac{\partial \alpha^j}{\partial a^I} a^I_{,q} + \frac{\partial \alpha^j}{\partial p^i} p^i_{,u} d^u_q \\
\bar{p}^j_{k,q} &= \left(\frac{\partial \alpha^j}{\partial a^I} a^I_h + \frac{\partial \alpha^j}{\partial p^i} p^i_h\right) b^h_{k,q} + \left(\frac{\partial^2 \alpha^j}{\partial a^{I_1}\partial a^{I_2}} a^{I_1}_h a^{I_2}_{,q} + \frac{\partial^2 \alpha^j}{\partial p^i \partial a^I} p^i_h a^I_{,q}\right. \\
&\quad \left. + \frac{\partial^2 \alpha^j}{\partial a^I \partial p^i} a^I_h p^i_{,u} d^u_q + \frac{\partial^2 \alpha^j}{\partial p^{i_1}\partial p^{i_2}} p^{i_1}_h p^{i_2}_{,u} d^u_q + \frac{\partial \alpha^j}{\partial a^I} a^I_{h,q} + \frac{\partial \alpha^j}{\partial p^i} p^i_{h,u} d^u_q\right) b^h_k \\
&\;\;\vdots
\end{aligned}
$$

$$\bar{p}^j_{k_1\dots k_s,q} = \sum_{K^s}\sum_{H^s}\sum_{\gamma} C_{K^s,H^s,\gamma}\Bigg(\frac{\partial^{\kappa+1}\alpha}{\partial a^{I_1}\dots\partial a^{I_\gamma}\partial a^{I_{\gamma+1}}\partial p^{i_1}\dots\partial p^{i_{\kappa-\gamma}}} a^{I_{\gamma+1}}_{,q}$$

$$\cdot a^{I_1}_{H_1}\cdot\ldots\cdot a^{I_\gamma}_{H_\gamma} p^{i_1}_{H_{\gamma+1}}\cdot\ldots\cdot p^{i_{\kappa-\gamma}}_{H_\kappa} b^{h_1}_{K_1}\cdot\ldots\cdot b^{h_\iota}_{K_\iota}$$

$$+\frac{\partial^{\kappa+1}\alpha}{\partial a^{I_1}\dots\partial a^{I_\gamma}\partial p^{i_1}\dots\partial p^{i_{\kappa-\gamma}}\partial p^{i_{\kappa-\gamma+1}}} p^{i_{\kappa-\gamma+1}}_{,h_{\iota+1}} d^{h_{\iota+1}}$$

$$\cdot a^{I_1}_{H_1}\cdot\ldots\cdot a^{I_\gamma}_{H_\gamma} p^{i_1}_{H_{\gamma+1}}\cdot\ldots\cdot p^{i_{\kappa-\gamma}}_{H_\kappa} b^{h_1}_{K_1}\cdot\ldots\cdot b^{h_\iota}_{K_\iota}$$

$$+\frac{\partial^{\kappa}\alpha^j}{\partial a^{I_1}\dots\partial a^{I_\gamma}\partial p^{i_1}\dots\partial p^{i_{\kappa-\gamma}}}$$

$$\cdot\Bigg(\sum_{\sigma=1}^{\gamma} a^{I_1}_{H_1}\cdot\ldots\cdot a^{I_\sigma}_{H_\sigma,q}\cdot\ldots\cdot a^{I_\gamma}_{H_\gamma} p^{i_1}_{H_{\gamma+1}}\cdot\ldots\cdot p^{i_{\kappa-\gamma}}_{H_\kappa} b^{h_1}_{K_1}\cdot\ldots\cdot b^{h_\iota}_{K_\iota}$$

$$+\sum_{\sigma=\gamma+1}^{\kappa} a^{I_1}_{H_1}\cdot\ldots\cdot a^{I_\gamma}_{H_\gamma} p^{i_1}_{H_{\gamma+1}}\cdot\ldots\cdot p^{i_{\sigma-\gamma}}_{H_\sigma,u} d^u_q\cdot\ldots\cdot p^{i_{\kappa-\gamma}}_{H_\kappa} b^{h_1}_{K_1}\cdot\ldots\cdot b^{h_\iota}_{K_\iota}$$

$$+\sum_{\sigma=1}^{\iota} a^{I_1}_{H_1}\cdot\ldots\cdot a^{I_\gamma}_{H_\gamma} p^{i_1}_{H_{\gamma+1}}\cdot\ldots\cdot p^{i_{\kappa-\gamma}}_{H_\kappa} b^{h_1}_{K_1}\cdot\ldots\cdot b^{h_\sigma}_{K_\sigma,q}\cdot\ldots\cdot b^{h_\iota}_{K_\iota}\Bigg)\Bigg)$$

The equations of the kernel of the canonical projection

$$(\pi^{r+s,r+s-1}_n)^1_n\colon (L^{r+s}_n)^1_n \to (L^{r+s-1}_n)^1_n$$

are $c^k_l = \delta^k_l$, $a^k_l = \delta^k_l$, $a^k_{l_1\dots l_u} = 0$ $(u = 2,\dots,r+s-1)$, $a^k_{l_1\dots l_v,q} = 0$ $(v = 1,\dots,r+s-1)$. In addition, we have the equations of the image of the kernel $K^{r+s+1,r+s}_n$ in the canonical homomorphism $\nu\colon L^{r+s+1}_n \to (L^{r+s}_n)^1_n$, $a^k_{l_1\dots l_{r+s}} = 0$. The equations of the image of the canonical embedding $\tau\colon T^{s+1}_n P \to T^1_n T^s_n P$ are $x^i_{,q} = x^i_q$, $x^i_{k_1\dots k_\sigma,q} = x^i_{k_1\dots k_\sigma q}$.

Now suppose $p,\bar{p}\in \operatorname{Im}\tau$. Since for coordinates (e^I) of the identity element of L^r_n it holds

$$\frac{\partial\alpha^j(e^I,p^i)}{\partial p^i} = \delta^j_i,$$

and for $\iota > 1$

$$\frac{\partial^\iota\alpha^j}{\partial p^{i_1}\dots\partial p^{i_\iota}} = 0$$

we have the following coordinate expression of $\bar{p} = a\cdot p$:

$$\bar{p}^j = p^j$$

$$\bar{p}^j_k = p^j_k$$

$$\vdots$$

$$\bar{p}^j_{k_1\dots k_{s-1}} = p^j_{k_1\dots k_{s-1}}$$

$$\bar{p}^j_{k_1\dots k_s} = \frac{\partial\alpha^j}{\partial a^i_{l_1\dots l_s}} a^i_{l_1\dots l_s k_1\dots k_s} + p^j_{k_1\dots k_s}$$

$$\bar{p}^j_{,q} = p^j_{,q}$$
$$\bar{p}^j_{k,q} = p^j_{k,q}$$
$$\vdots$$
$$\bar{p}^j_{k_1\dots k_{s-1},q} = p^j_{k_1\dots k_{s-1},q}$$
$$\bar{p}^j_{k_1\dots k_s,q} = \frac{\partial^2\alpha^j}{\partial a^{i_1}_{l_1\dots l_s}\partial p^{i_2}} a^{i_1}_{l_1\dots l_s k_1\dots k_s} p^{i_2}_{,q} + \frac{\partial\alpha^j}{\partial a^i_{l_1\dots l_s}} a^i_{l_1\dots l_s k_1\dots k_s,q} + p^j_{k_1\dots k_s,q}$$

The matrix

$$\left(\frac{\partial\alpha^j}{\partial a^I}\right), \qquad j = 1,\dots,\mu,$$

has a maximal rank. Hence there exists a square regular matrix A^J_I such that for $j = 1,\dots,\mu$ it holds

$$A^j_I = \frac{\partial\alpha^j}{\partial a^I}.$$

Denote by B^I_J its inverse matrix. Further let $p^J_{k_1\dots k_s,q}$, and $\bar{p}^J_{k_1\dots k_s,q}$ be for $J > \mu$ arbitrary numbers symmetric in the indices $k_1\dots k_s, q$. Put

$$\tilde{a}^I_{k_1\dots k_s} = 0$$
$$\tilde{a}^I_{k_1\dots k_s,q} = B^I_J(\bar{p}^J_{k_1\dots k_s,q} - p^J_{k_1\dots k_s,q}).$$

Since the right-hand side of the second equation is symmetric in the indices $k_1\dots k_s, q$, $\tilde{a}^I$ are coordinates of an element $\tilde{a} \in \ker(\pi^{r+s,r+s-1}_n)^1_n$ such that $\tilde{a} \in \operatorname{Im}\nu$. Let $\tilde{p} = \tilde{a}\cdot p$. Since $\bar{p}^j_{k_1\dots k_s} = \bar{p}^j_{k_1\dots k_{s-1},k_s}$, and $p^j_{k_1\dots k_s} = p^j_{k_1\dots k_{s-1},k_s}$ it holds

$$\frac{\partial\alpha^j}{\partial a^i_{l_1\dots l_s}} a^i_{l_1\dots l_s k_1\dots k_s} = 0.$$

Moreover for $j = 1,\dots,\mu$ it holds

$$\tilde{p}^j_{k_1\dots k_s,q} = \frac{\partial^2\alpha^j}{\partial a^I\partial p^i}\tilde{a}^I_{k_1\dots k_s} p^i_{,q} + \frac{\partial\alpha^j}{\partial a^I}\tilde{a}^I_{k_1\dots k_s,q} + p^j_{k_1\dots k_s,q}$$
$$= A^j_I B^I_J \bar{p}^J_{k_1\dots k_s,q} = \bar{p}^j_{k_1\dots k_s,q}.$$

Since

$$\frac{\partial\alpha^j}{\partial a^I} = 0,$$
$$\frac{\partial^2\alpha^j}{\partial a^I\partial p^i} = 0$$

then for $j > \mu$ it holds

$$\bar{p}^j_{k_1\dots k_s,q} = \tilde{p}^j_{k_1\dots k_s,q} = 0.$$

Then $\tilde{p} = \bar{p}$ and the orbits of the group $\nu(\ker(\pi_n^{r+s,r+s-1})_n^1)$ on $\operatorname{Im}\tau$ are equal to the intersections of the orbits of the group $\ker(\pi_n^{r+s,r+s-1})_n^1$ on $T_n^1 T_n^s P$, and $\operatorname{Im}\tau$. Then by Theorem 3 and Lemma 1 the mapping f^{s+1} is a basis of $\pi_n^{r+s+1,r+s}$-invariants of $T_n^{s+1}P$.

Since the mapping f^s is a regular basis of invariants then the matrix

$$\left(\frac{\partial \alpha^j}{\partial a^i_{l_1\dots l_s}}\right)$$

has a constant rank. Thus the orbits of $\nu(\ker(\pi_n^{r+s,r+s-1})_n^1)$ on $\operatorname{Im}\tau$ have a constant dimension. Moreover, these orbits are evidently connected sets. Then by Lemma 3 and Theorem 2 the mapping f^{s+1} is a regular basis of $\pi_n^{r+s+1,r+s}$-invariants of $T_n^{s+1}P$. This is the end of the proof of Theorem 4.

Now we give a generalization of the previous case. We start by an elementary topological assertion.

Lemma 4. *Let X, Y be topological spaces, Z a Hausdorff topological space, $\varphi: X \to Y$ an open surjective mapping, $\psi: X \to Z$ a continuous mapping. If there exist a subset $X_0 \subset X$ such that $\varphi(X_0)$ is a dense subset of Y, and a mapping $\rho_0: \varphi(X_0) \to Z$ such that $\rho_0 \circ \varphi = \psi | X_0$ then there exists a unique mapping $\rho: Y \to Z$ such that $\rho \circ \varphi = \psi$. This mapping is continuous.*

Let $\pi: G \to H$ be a surjective Lie group homomorphism. A basis of π-invariants $f: P \to Q$ is called *nearly-regular* if there exists a dense open G-invariant subset $P_0 \subset P$ such that the restriction $f|P_0$ is a regular basis of π-invariants. We have the following simple result:

Theorem 5. *Let $\pi: G \to H$ be a surjective Lie group homomorphism, and $f: P \to Q$ a mapping. If there exists a dense open G-invariant subset $P_0 \subset P$ such that the restriction $f|P_0$ is a regular basis of π invariants then f is a nearly-regular basis of π invariants of P.*

Proof. From the continuity of considered mappings it follows that f is a π-invariant. Consider the set $\operatorname{Im} f$ with a topology induced by the mapping f. Then this mapping is open and the set $f(P_0) \subset \operatorname{Im} f$ dense. Now we can apply Lemma 4.

There are two simple consequences of Theorem 5.

Corollary 1. *Let n, r, s be positive integers, P (resp. Q) a left L_n^r-manifold (resp. L_n^{r+s-1}-manifold). If a mapping $f^s: T_n^s P \to Q$ is a nearly-regular basis of $\pi_n^{r+s,r+s-1}$-invariants of $T_n^s P$, then for any $t > s$ the composition $f^t: T_n^t P \to T_n^{t-s} Q$ of the canonical embedding $\tau: T_n^t P \to T_n^{t-s} T_n^s P$ and the prolonged mapping $T_n^{t-s} f^s: T_n^{t-s} T_n^s P \to T_n^{t-s} Q$ is a nearly-regular basis of $\pi_n^{r+t,r+t-1}$-invariants of $T_n^t P$.*

Proof. This follows from Theorem 4, and Theorem 5.

Corollary 2. *Let $\pi_1: G_1 \to G_2$ (resp. $\pi_2: G_2 \to G_3$) be a surjective Lie group homomorphism and let $f_1: P_1 \to P_2$ (resp. $f_2: P_2 \to P_3$) be a nearly-regular basis of π_1-invariants (resp. regular basis of π_2-invariants) of P_1 (resp. P_2). Then the composition $f_2 \circ f_1$ is a nearly-regular basis of $(\pi_2 \circ \pi_1)$-invariants of P_1.*

Proof. This follows from Theorem 2, and Theorem 5.

3. Example: Invariants of velocities

In this section we find all natural differential operators of n-tuples of vector fields, using the methods of Section 2.

Let X be a smooth manifold, $\dim X = m$, n an integer. Sections of the bundle $T^1_n X$ of $(1, n)$-velocities of X are canonically identified with n-tuples of vector fields on X. The bundle $T^1_n X$ is associated with the principal L^1_m-bundle $F^1 X$. Type fibre of $T^1_n X$ is a manifold $P_0 = R^m \otimes R^{n*}$ with the canonical coordinates (x^σ_k), where $\sigma = 1, \ldots, m$, and $k = 1, \ldots, n$ (in this section greek indices will always run over $1, \ldots, m$, and latin indices over $1, \ldots, n$), and with a left action of L^1_m given by

$$x^\sigma_k(a \cdot p) = a^\sigma_\iota x^\iota_k(p)$$

or simply

$$\bar{p}^\sigma_k = a^\sigma_\iota p^\iota_k.$$

We start by computing a basis of $\pi^{2,1}_m$-invariants of $T^1_m P_0$. The induced action of L^2_m on $T^1_m P_0$ has the following equations:

$$\bar{p}^\sigma_k = a^\sigma_\iota p^\iota_k$$

$$\bar{p}^\sigma_{k,\nu} = b^\vartheta_\nu (a^\sigma_{\vartheta\iota} p^\iota_k + a^\sigma_\iota p^\iota_{k,\vartheta})$$

Further let $P_1 = (R^m \otimes R^{n*}) \times (R^m \otimes (R^{n*} \wedge R^{n*}))$. Denote the canonical coordinates on P_1 by (x^σ_k), (x^σ_{kl}) and define the left action of L^1_m on P_1 by

$$\bar{p}^\sigma_k = a^\sigma_\iota p^\iota_k$$

$$\bar{p}^\sigma_{kl} = a^\sigma_\iota p^\iota_{kl}.$$

Finally define a mapping $f: T^1_m P_0 \to P_1$ in components by

$$f^\sigma_k = x^\sigma_k$$

$$f^\sigma_{kl} = x^\sigma_{k,\iota} x^\iota_l - x^\sigma_{l,\iota} x^\iota_k.$$

From this definition it follows that the mapping f is a representation of a differential operator which each n-tuple $\xi_1, \ldots, \xi_n$ of vector fields on X assignes the vector fields $\xi_1, \ldots, \xi_n, [\xi_1, \xi_1], [\xi_1, \xi_2], \ldots, [\xi_{n-1}, \xi_n]$, $[\xi_n, \xi_n]$ (some of these vector fields are equal to zero, some depend on the others).

We shall need the canonical projections $\varrho: T^1_m P_0 \to P_0$, and $\tau: P_1 \to P_0$.

Let $\operatorname{reg} P_0 \subset P_0$ be defined by $\operatorname{rank} x^\sigma_k = \max$. $\operatorname{reg} P_0$ is an open dense subset of P_0. Denote the restriction $f|\operatorname{reg} P_0$ by $\operatorname{reg} f$.

Theorem 6. *The mapping f is a nearly-regular basis of $\pi^{2,1}_m$-invariants of $T^1_m P_0$. The mapping* $\operatorname{reg} f$ *is a regular basis of $\pi^{2,1}_m$-invariants of $T^1_m \operatorname{reg} P_0$.*

Proof. By Theorem 5 we prove only the second assertion. By a direct calculation we obtain that f is a $\pi_m^{2,1}$-invariant.

First suppose that $n \le m$. Let $p, \bar{p} \in T_m^1 \mathrm{reg}\, P_0$, $p = (p_k^\sigma, p_{k,\nu}^\sigma)$, $\bar{p} = (\bar{p}_k^\sigma, \bar{p}_{k,\nu}^\sigma)$. There evidently exist a neighbourhood $U \ni \rho(p)$, and a square regular matrix A defined on $T_m^1 U$ such that $A_k^\sigma = x_k^\sigma$. Denote by B its inverse matrix.

Suppose that $f(p) = f(\bar{p})$. Then

$$p_k^\sigma = \bar{p}_k^\sigma$$

$$p_{k,\iota}^\sigma p_l^\iota - p_{l,\iota}^\sigma p_k^\iota = \bar{p}_{k,\iota}^\sigma \bar{p}_l^\iota - \bar{p}_{l,\iota}^\sigma \bar{p}_k^\iota.$$

Further for $\iota > n$ set $p_{\iota,\nu}^\sigma = \bar{p}_{\iota,\nu}^\sigma = 0$. Hence

$$p_{\kappa,\iota}^\sigma A_\lambda^\iota(p) - p_{\lambda,\iota}^\sigma A_\kappa^\iota(p) = \bar{p}_{\kappa,\iota}^\sigma A_\lambda^\iota(p) - \bar{p}_{\lambda,\iota}^\sigma A_\kappa^\iota(p),$$

which implies

$$(\bar{p}_{\kappa,\iota}^\sigma - p_{\kappa,\iota}^\sigma) B_\vartheta^\kappa(p) = (\bar{p}_{\kappa,\vartheta}^\sigma - p_{\kappa,\vartheta}^\sigma) B_\iota^\kappa(p).$$

Then the expression

$$a_{\iota\vartheta}^\sigma = (\bar{p}_{\kappa,\iota}^\sigma - p_{\kappa,\iota}^\sigma) B_\vartheta^\kappa(p)$$

is symmetric in the indices ι, ϑ and represents coordinates of an element $a \in K_m^{2,1}$. It can be easily verified that

$$\bar{p}_{k,\nu}^\sigma = a_{\nu\iota}^\sigma p_k^\iota + p_{k,\nu}^\sigma.$$

Hence $\bar{p} = a \cdot p$.

Now define a mapping $s\colon \tau^{-1}(U) \cap \bar{P}_1 \to T_m^1 P_0$ by

$$s_k^\sigma = x_k^\sigma$$

$$s_{k,\nu}^\sigma = \frac{1}{2} x_{kh}^\sigma B_\nu^h.$$

Since $f_{kl}^\sigma \circ s = s_{k,\nu}^\sigma s_l^\nu - s_{l,\nu}^\sigma s_k^\nu = \frac{1}{2} x_{kh}^\sigma B_\nu^h A_l^\nu - \frac{1}{2} x_{lh}^\sigma B_\nu^h A_k^\nu = \frac{1}{2} x_{kl}^\sigma - \frac{1}{2} x_{lk}^\sigma = x_{kl}^\sigma$ then $f \circ s = \mathrm{id}_{\tau^{-1}(U)}$. As we have already proved there is an element $a \in K_m^{2,1}$ such that $p = (a \cdot s) \circ f(p)$. Then the mapping f is a surjective submersion.

In the case $n > m$ the proof will be analogous. Let $p, \bar{p} \in T_m^1 \mathrm{reg}\, P_0$, $p = (p_k^\sigma, p_{k,\nu}^\sigma)$, $\bar{p} = (\bar{p}_k^\sigma, \bar{p}_{k,\nu}^\sigma)$ and let $w\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ be a permutation such that there exists an open set $U \ni \varrho(p)$ such that the matrix A defined on $T_m^1 U$ by $A_\nu^\sigma = x_{w(\nu)}^\sigma$ is regular. Let $B = A^{-1}$. We have a cartesian projection defined on P_1 by $\mathrm{pr}\colon (p_k^\sigma, p_{kl}^\sigma) \to (p_k^\sigma, p_{kw(\lambda)}^\sigma)$.

Now suppose $f(p) = f(\bar{p})$. Then it holds

$$p_{k,\iota}^\sigma p_l^\iota - p_{l,\iota}^\sigma p_k^\iota = \bar{p}_{k,\iota}^\sigma \bar{p}_l^\iota - \bar{p}_{l,\iota}^\sigma \bar{p}_k^\iota.$$

It follows

$$(\bar{p}_{w(\kappa),\iota}^\sigma - p_{w(\kappa),\iota}^\sigma) B_\vartheta^\kappa(p) = (\bar{p}_{w(\kappa),\vartheta}^\sigma - p_{w(\kappa),\vartheta}^\sigma) B_\iota^\kappa(p).$$

Then the expression

$$a^{\sigma}_{\iota\vartheta} = (\bar{p}^{\sigma}_{w(\kappa),\iota} - p^{\sigma}_{w(\kappa),\iota})B^{\kappa}_{\vartheta}(p)$$

is symmetric in $\iota\vartheta$. Thus we have $a \in K^{2,1}_m$. Now

$$a^{\sigma}_{\nu\iota}p^{\iota}_{w(k)} + p^{\sigma}_{w(k),\nu} = (\bar{p}^{\sigma}_{w(\kappa),\iota} - p^{\sigma}_{w(\kappa),\iota})B^{\kappa}_{\nu}(p)p^{\iota}_{w(k)} + p^{\sigma}_{w(k),\nu}$$

$$= (\bar{p}^{\sigma}_{w(k),\iota} - p^{\sigma}_{w(k),\iota})B^{k}_{\nu}(p)p^{\iota}_{w(\kappa)} + p^{\sigma}_{w(k),\nu} = \bar{p}^{\sigma}_{w(k),\nu}.$$

Then $\bar{p} = a \cdot p$.

Now define a mapping $s\colon \mathrm{pr}(\tau^{-1}(U)) \to T^1_m P_1$ by

$$s^{\sigma}_k = x^{\sigma}_k$$

$$s^{\sigma}_{k,\nu} = \frac{1}{2}x^{\sigma}_{kw(\lambda)}B^{\lambda}_{\nu}.$$

Since $f^{\sigma}_{kw(\kappa)} \circ s = s^{\sigma}_{k,\nu}s^{\nu}_{w(\kappa)} - s^{\sigma}_{w(\kappa),\nu}s^{\nu}_k = \frac{1}{2}x^{\sigma}_{kw(\lambda)}B^{\lambda}_{\nu}x^{\nu}_{w(\kappa)} - \frac{1}{2}x^{\sigma}_{w(\kappa)w(\lambda)}B^{\lambda}_{\nu}x^{\nu}_k = \frac{1}{2}x^{\sigma}_{kw(\kappa)} + \frac{1}{2}x^{\sigma}_{kw(\lambda)}B^{\lambda}_{\nu}x^{\nu}_{w(\kappa)} = x^{\sigma}_{kw(\kappa)}$ then $\mathrm{pr} \circ f \circ s = \mathrm{id}_{\mathrm{pr}(\tau^{-1}(U))}$ and there exists $a \in K^{2,1}_m$ such that $(a \cdot s) \circ \mathrm{pr} \circ f(p) = p$. Then the set $f(T^1_m \mathrm{reg}\, P_0) \subset P_1$ is a submanifold whose intersection with $\tau^{-1}(U)$ is the image of the embedding $f \circ s$ and the mapping $f\colon T^1_m P_0 \to \mathrm{Im}\, f$ is a surjective submersion. Theorem 6 is proved.

Now we give a solution of our general problem. Denote $I_0 = \{1, 2, \ldots, n\}$. For every $r > 0$ put $I_r = J_r \cup I_0$, where J_r is the set of terms $[K, L]$, where $K, L \in I_{r-1}$. For $[k, l] \in J_1$ we write $[l, k] \in N([k, l])$. For $r > 1$, $[K_1, K_2] \in J_r$ we write $[L_1, L_2] \in N([K_1, K_2])$ if just one of the following conditions is satisfied:

$$L_1 = K_2, \quad \text{and } L_2 = K_1$$

$$L_1 = K_1, \quad \text{and } L_2 \in N(K_2)$$

$$L_1 \in N(K_1), \quad \text{and } L_2 = K_2.$$

Put $Q_r = R^m \otimes (R^{I_r})^*$, and denote by (x^{σ}_K), $\sigma = 1, \ldots, m$, $K \in I_r$, the canonical coordinates on Q_r. The associated coordinates on $T^s_m Q_r$ will be denoted by $(x^{\sigma}_K, x^{\sigma}_{K,\nu_1}, \ldots, x^{\sigma}_{K,\nu_1\ldots\nu_s})$.

Consider Q_r with the left action of L^1_m defined by

$$\bar{p}^{\sigma}_K = a^{\sigma}_{\nu}p^{\nu}_K,$$

and $T^s_m Q_r$ with the induced left action of L^{s+1}_m.

We have canonical projections $\varrho^s_{r_1,r_2}\colon T^s_m Q_{r_1} \to T^s_m Q_{r_2}$, $r_1 \geq r_2$, and $\varrho^{s_1,s_2}_r\colon T^{s_1}_m Q_r \to T^{s_2}_m Q_r$, $s_1 \geq s_2$.

Let $P_r \subset Q_r$ be defined by

$$x^{\sigma}_K = x^{\sigma}_L, \quad \text{where } L \in N(K).$$

Note that P_0 and P_1 defined now and before coincide.

Denote $f^r_{r,r} = \mathrm{id}_{P_r}$. Then define a mapping $f^{r+1}_{r,r+1}\colon T^1_m P_r \to P_{r+1}$ in components by

$$(f^{r+1}_{r,r+1})^\sigma_K = \begin{cases} p^\sigma_K, & \text{for } K \in I_0, \\ p^\sigma_{K_1,\nu} p^\nu_{K_2} - p^\sigma_{K_2,\nu} p^\nu_{K_1}, & \text{for } K \in J_{r+1},\ K = [K_1, K_2]. \end{cases}$$

Denote by $f^s_{r,r+1}\colon T^{s-r}_m P_r \to T^{s-r-1}_m P_{r+1}$, $s > r$, the composed mapping whose components are the canonical injection $T^{s-r}_m P_r \to T^{s-r-1}_m T^1_m P_r$, and the natural prolongation $T^{s-r-1}_m f^{r+1}_{r,r+1}\colon T^{s-r-1}_m T^1_m P_r \to T^{s-r-1}_m P_{r+1}$. Finally denote

$$f^s_{r_1,r_2} = f^s_{r_2-1,r_2} \circ f^s_{r_2-2,r_2-1} \circ \ldots \circ f^s_{r_1,r_1+1}, \qquad 0 \le r_1 \le r_2 \le s.$$

Hence $f^s_{r_1,r_2}$ maps $T^{s-r_1}_m P_{r_1}$ into $T^{s-r_2}_m P_{r_2}$.

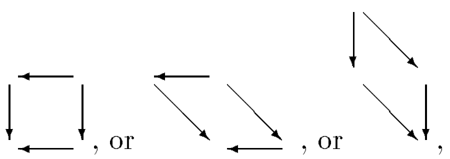
From this definition it follows that the mapping $f^s_{0,r}$ is a representation of a differential operator which assignes to each n-tuple $\xi_1, \ldots, \xi_n$ of vector fields on X all their Lie brackets up to r inclusions, and prolongations of these brackets up to $(s-r)$-th order.

The arising situation is described by the following diagram:

$$\begin{tikzcd}
{} \arrow[d, "\varrho_0^{4,3}"] & & & & \\
T^3_m P_0 \arrow[d, "\varrho_0^{3,2}"] \arrow[dr, "f^3_{0,1}"] & {} \arrow[l, "\varrho^3_{1,0}"'] \arrow[d, "\varrho_1^{3,2}"] & & & \\
T^2_m P_0 \arrow[d, "\varrho_0^{2,1}"] \arrow[dr, "f^2_{0,1}"] & T^2_m P_1 \arrow[l, "\varrho^2_{1,0}"'] \arrow[d, "\varrho_1^{2,1}"] \arrow[dr, "f^3_{1,2}"] & {} \arrow[l, "\varrho^2_{2,1}"'] \arrow[d, "\varrho_2^{2,1}"] & & \\
T^1_m P_0 \arrow[d, "\varrho_0^{1,0}"] \arrow[dr, "f^1_{0,1}"] & T^1_m P_1 \arrow[l, "\varrho^1_{1,0}"'] \arrow[d, "\varrho_1^{1,0}"] \arrow[dr, "f^2_{1,2}"] & T^1_m P_2 \arrow[l, "\varrho^1_{2,1}"'] \arrow[d, "\varrho_2^{1,0}"] \arrow[dr, "f^3_{2,3}"] & {} \arrow[l, "\varrho^1_{3,2}"'] \arrow[d, "\varrho_3^{1,0}"] & \\
P_0 & P_1 \arrow[l, "\varrho^0_{1,0}"'] & P_2 \arrow[l, "\varrho^0_{2,1}"'] & P_3 \arrow[l, "\varrho^0_{3,2}"'] & {} \arrow[l, "\varrho^0_{4,3}"']
\end{tikzcd}$$

Theorem 7. *The mapping $f^s_{0,r}$ is a nearly-regular basis of $\pi^{s+1,s+1-r}_m$-invariants of $T^s_m P_0$.*

Proof. Consider the above diagram. This diagram is non-commutative, but one can easily see that each of its parts of the form


, or , or ,

commutes. Moreover the diagram

$$\begin{tikzcd}
T^1_m P_0 \arrow[d, "\varrho_0^{1,0}"'] \arrow[dr, "f^1_{0,1}"] & \\
P_0 & P_1 \arrow[l, "\varrho^0_{1,0}"']
\end{tikzcd}$$

commutes. From these facts it easily follows that the mapping $f^s_{0,r}$, $s > r$, equalizes the mappings $\varrho_r^{s-r,s-r-1}$, and $\varrho^{s-r-1}_{r+1,r} \circ f^s_{r,r+1}$, i.e. it holds

$$\varrho_r^{s-r,s-r-1} \circ f^s_{0,r} = \varrho^{s-r-1}_{r+1,r} \circ f^s_{r,r+1} \circ f^s_{0,r} = \varrho^{s-r-1}_{r+1,r} \circ f^s_{0,r+1}.$$

Now we are able to prove the theorem. For arbitrary s, the mapping $f^s_{0,0}$ is obviously a regular basis of $\pi_m^{s+1,s+1}$-invariants of the manifold $T^s_m P_0$. Suppose that the mapping $f^s_{0,r}$, $r < s$, is a nearly-regular basis of $\pi_m^{s+1,s+1-r}$-invariants. Let $U \subset P_r$ be given by rank x^σ_k = max, rank x^σ_K = max $(K \in I_r)$. The set U is evidently an open dense L^1_n-invariant subset of P_r.

From Theorem 6, Corollary 1 it follows that the mapping $(\varrho_r^{s-r,s-r-1}, f^s_{r,r+1}) : T^{s-r}_m P_r \to T^{s-r-1}_m P_r \times T^{s-r-1}_m P_{r+1}$ is a nearly-regular basis of $\pi_m^{s+1-r,s-r}$-invariants of $T^{s-r}_m P_r$ such that its restriction to the set $T^{s-r}_m U$ is regular. The restriction of $f^s_{0,r}$ to the set $(f^s_{0,r})^{-1}(U)$ is a nearly-regular basis of $\pi_m^{s+1,s+1-r}$-invariants. Then by Corollary 1 and Theorem 5 the mapping $(\varrho_r^{s-r,s-r-1}, f^s_{r,r+1}) \circ f^s_{0,r}$ is a nearly-regular basis of $\pi_m^{s+1,s-r}$-invariants of $T^s_m P_0$. Since the mapping $f^s_{0,r+1}$ is one of its components, it is a $\pi_m^{s+1,s-r}$-invariant. Moreover, we have

$$(\varrho_r^{s-r,s-r-1}, f^s_{r,r+1}) \circ f^s_{0,r} = (\varrho_r^{s-r,s-r-1} \circ f^s_{0,r+1}, f^s_{0,r+1}).$$

Then the mapping $f^s_{0,r+1}$ is a nearly-regular basis of $\pi_m^{s+1,s-r}$-invariants of $T^s_m P_0$ and the theorem is proved.

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