Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 265–271

NATURAL CONNECTIONS OF 1–FORMS ON TANGENT BUNDLES

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Abstract. Ideas of the Lagrange formalism are used to constructions of connections from a scalar 1-form on TM.
Keywords. Semibasic scalar and vector forms, semispray, connections, symplectic forms.
MS classification. 58F05, 70H35.

In the Lagrange formalism of classical mechanics connections are a useful tool, see for example [1], [2], [6], [7], [8]. Let us recall that a time independent lagrangian $L : TM \to R$ of first order the mechanical system is a semispray S_L on TM such that the basic equation $i_{S_L} d\omega = dE$ is satisfied, where $\omega = J^* dL$, $E = L - CL$, J is the canonical morphism on TM and C is the Liouville field on TM, see Godbillon [4]. Griffone, [5], has shown that if S is a semispray on TM then the Lie derivative $L_S J$ is a $(1,1)$-form on TM determining a connection Γ_S on TM. In [3] we have proved that every connection constructed by natural way of order one from a semispray S is of the form $\Gamma_S + cJ$, $c \in R$. In this paper we show that the set of connection connected with every regular lagrangian L is larger that the one connected with S_L. We deal with this problem in a little more general case of 1-forms on TM.

Let M be a smooth manifold, $m = \dim M$. Let (x^i) be a local chart on M. Denote (x^i, x^i_1) the induced chart on the tangent bundle $p_M : TM \to M$. In this chart the Liouville field C or the canonical morphism is of the form $C = x^i_1 \partial/\partial x^i_1$ or $J = dx^i \otimes \partial/\partial x^i_1$. Recall that a semispray is a vector field S on TM such that $J(S) = C$ and that a spray is semispray such that $[C, S] = S$.

In this paper a connection Γ on TM is understood in a some of the following equivalent senses:

1. Γ is a global section $TM \to JTM$ of the first jet prolongation of TM.
2. Γ is a distribution of Γ-horizontal subspaces $H\Gamma \subset TTM$ such that $TTM = VTM \oplus H\Gamma$, where $VTM \to TM$ is the vector bundle of all vertical vectors on TM.
3. Γ is given by the Γ-horizontal projection h_Γ that is a vector 1-form on TM such that $h_\Gamma(Y) = 0$ for any $Y \in VTM$ and $Tp_M \cdot h_\Gamma = Tp_M$, i.e. $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma^i_j dx^j \otimes \partial/\partial x^i_1$ in the induced chart.
4. Γ is determined by the Griffone form on TM that is a vector 1-form φ on TM such that $\varphi J = -J$, $J\varphi = J$, i.e.,

$$\varphi = dx^i \otimes \partial/\partial x^i + (\varphi^i_j dx^j - dx^i_1) \otimes \partial/\partial x^i_1.$$

[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

The map $\varphi \mapsto \frac{1}{2}(\varphi + \mathrm{id}_{TTM}) = h_\Gamma$ is a bijection between the set of the Griffone forms φ and the set of the horizontal projections h_Γ.

Let us recall that a scalar or a vector p-form ω on a fibred manifold $\pi : Y \to M$ is semibasic if $\omega(X_1, \dots, X_p) = 0$ whenever at least one vector X_i is π-vertical. Then if h_Γ is a Γ-horizontal projection of a connection Γ and ψ is a semibasic VTM-valued 1-form on TM then $h_\Gamma = \psi$ is the horizontal projection of another connection on TM which will be denoted by $\Gamma + \psi$.

Properties of semibasic forms on TM

Let $\gamma = \gamma_i(x, x_1)dx^i$ be a semibasic 1-form on TM. It determines the morphism $\bar\gamma : TM \to T^*M$, $\gamma(X)(Z) = \gamma(Z_1)$, $Tp_M(Z_1) = Z$, $Z_1 \in T_X TM$, i.e. $z_i = \gamma_i(x, x_1)$ in the induced chart (x^i, z_i) on T^*M, see [4].

Denote $V\bar\gamma : T\bar\gamma \mid_{VTM}: VTM \to VT^*M$ the restriction of the tangent prolongation of $\bar\gamma$ to VTM Let $p_2 : VT^*M \equiv T^*M \times_M T^*M \to T^*M$ be the projection on the second factor. Using the identification $V^*TM := p_M^* T^*M \equiv TM \times_M T^*M$ we get the map $V\gamma : VTM \to V^*TM$ defined by the pull-back $p_M^*(p_2 \cdot V\bar\gamma)$, i.e. $V\gamma$ is a section $TM \to V^*TM \otimes V^*TM$, $V\gamma = \gamma_{ij_1} dx_1^i \otimes dx_1^j$, where we use the notations $\frac{\partial f}{\partial x^i} := f_i$, $\frac{\partial f}{\partial x_1^i} := f_{i_1}$.

A semibasic form γ is said to be V-regular (V-symmetric) if $V\gamma$ is a regular (symmetric) bilinear form. In the case of a V-regular form, $V\tilde\gamma = \tilde\gamma^{ij}\partial/\partial x_1^i \otimes \partial/\partial x_1^j$, $\tilde\gamma^{ij}\gamma_{jk_1} = \partial_k^i$ denotes the inverse map of $V\gamma$. Obviously γ is V-regular iff $\bar\gamma$ is regular.

Recall that a vector field $X = \xi^i\partial/\partial x^i + \eta^i\partial/\partial x_1^i$ on TM is projectable if $Tp_M X = \xi^i\partial/\partial x^i$ is a vector field on M. The field $TX = \xi^i\partial/\partial x^i + \frac{\partial\xi^i}{\partial x^j} x_1^j \partial/\partial x_1^i$ the flow of which is the tangent prolongation of the flow of a vector field $X = \xi^i\partial/\partial x^i$ on M is projectable.

A field $X : TM \to TTM$ on TM determines the map $Tp_M \cdot X : TM \to TM$, $x^{-i} = x^i$, $x_1^{-i} = \xi^i(x, x_1)$. Then the map $V(Tp_M \cdot X) : VTM \to VTM$ is a vector bundle morphism over $Tp_M \cdot X$. Using the projection $p_2 : VTM \equiv TM \times_M TM \to TM$ on the second factor we get the map

$$VX := p_2 \cdot V(Tp_M \cdot X) : VTM \to TM, \quad x^i = x^i, \quad x_1^i \frac{\partial \xi^i}{\partial x_l^j} dx_l^j$$

that is a section $TM \to T^*M \otimes_{TM} VTM \equiv V^*TM \otimes VTM \equiv T^*M \otimes_{TM} TM$. We say that a vector field X is a *b-regular* if VX is regular, (locally if $\det \xi^i_{j_1} \neq 0$). Every semispray on TM is regular.

Recall that a semibasic 1-form γ is closed iff $\gamma = p_M^*\varphi$, where φ is a closed 1-form on M. The Lie-derivative $L_X\gamma$ of a semibasic 1-form γ with respect to a vector field X on TM is of the coordinate form

$$L_X\gamma = (\gamma_{ij}\xi^j + \gamma_j\xi_i^i + \gamma_{ij_1}\eta^j)dx^i + \gamma_j\xi_{i_1}^j dx_1^i. \tag{1}$$

Denote by V_γ or V_X the space of all vector fields or all semibasic forms on TM such that $L_X\gamma$ is semibasic. Any projectable vector field belongs to V_γ. A semibasic form γ belongs to V_X iff $\gamma \in \mathrm{Ker}(VX)^*$ where $(VX)^* : V^*TM \to V^*TM$ is the join morphism to VX. It is clear that if X is b-regular then $V_X = \{0\}$. From (1) it follows

Lemma 1. *Let a semibasic form γ is V-regular. Then for every $X \in V\gamma$ and for any semibasic form ε there exists a unique field $\bar{X}$ on TM such that $Tp_M X = Tp_M \bar{X}$ and $L_{\bar{x}}\gamma = \varepsilon$.*

Let $Y = c^i \partial/\partial x^i$ be a vector field on M. Let $T^*Y = c^i\partial/\partial x^i - \frac{\partial c^k}{\partial x^i} z_k \partial/\partial z_i$ denote the vector field on T^*M induced by the flow $(T\phi)^*$, where ϕ is the flow of the field Y.

Lemma 2. *Let X be a projectable vector field on TM over a vector field Y on M. Then the vector field $\bar{X}$ described in Lemma 1 for $\varepsilon = 0$ satisfies $T\bar{\gamma}(X) = T^*Y$.*

Proof. If $Y = c^i\partial/\partial x^i$ then $\bar{X} = c^i\partial/\partial x^i + \eta^i \partial/\partial x_1^i$ where $\eta^i = \tilde{\gamma}$ is $(-\gamma_{sj}c^j - c_s^j \gamma_j)$. Then $T\bar{\gamma}(\bar{X}) = c^i\partial/\partial x^i + (\gamma_{ij}c^j - \gamma_{ij}c^j - c_i^j\gamma_j)\partial/\partial z_i = T^*Y$. $\square$

If γ is a V-regular then $\bar{\gamma} : TM \to T^*M$ is an isomorphism of the symplectic structures $(TM, d\gamma)$ and $(T^*M, d\lambda)$, where $\lambda = z_i dx^i$ is the Liouville form on T^*M. Let $I : T(TM) \to T^*M$ denote the map defined by the contraction $I(X) = i_X d\gamma = \alpha_i dx^i + \beta_i dx_1^i$ where

$$\alpha_i = (\gamma_{ij} - \gamma_{ji})\xi^j + \gamma_{ij_1}\eta^j, \quad \beta_j = -\gamma_{ij_1}\xi^i. \tag{2}$$

Recall that a vector field X on TM is a dynamical system of the structure $(TM, d\gamma)$ if the form $I(X) = i_X d\gamma$ is closed. Since the Lie derivative $L_X\gamma = (i_x d = di_X)^\gamma$ then X is a dynamical system of $(TM, d\gamma)$ iff $L_x\gamma$ is closed or iff $L_X d\gamma = 0$. If X is a symmetry of γ, $(L_X\gamma = 0)$, then is a dynamical system. Therefore by Lemma 1 to any vector field $X \in V_\gamma$ there is a unique dynamical system $\bar{X}$ of $(TM, d\gamma)$ such that $L_{\bar{X}}\gamma = 0$.

A vector field X on TM is called to be a quasidynamical system of $(TM, d\gamma)$ if $J^*(dL_X\gamma) = 0$. Every projectable vector field on TM is a quasidynamical systems of $(TM, d\gamma)$.

Lemma 3. *A semispray on TM is the quasidynamical system of $(TM, d\gamma)$ iff γ is V-symmetric.*

Proof follows from $J^*dL_X\gamma = \gamma_{it_1}\xi^i_{k_1} dx^t \wedge dx^k$, $\xi^i_{k_1} = \delta^i_k$.

According to (2) it is clear that if γ is V-regular then I is a bijection from the set of all vertical vectors onto the set of all semibasic forms on TM. Therefore the Poisson bracket (α, β) of semibasic forms α, β is semibasic.

Lemma 4. *The Poisson bracket (α, β) of two semibasic forms α, β satisfies $(\alpha, \beta) = \langle V\beta \otimes \alpha - V\alpha \otimes \beta,\ \widetilde{V\varepsilon}\rangle$ where the bracket $\langle\ \rangle$ denotes the contraction.*

Corollary. *Let $\alpha = p_M^*\alpha_0$, $\beta = p_M^*\beta_0$ be the p_M-pullback of forms α_0, β_0 on M. Then $(\alpha, \beta) = 0$.*

It is easy to show that the vector field $[C, X]$ is vertical iff $C \in \operatorname{Ker} VX$. If X_1, X_2 are semispray on TM then $[X, \bar{X}]$ is a semispray iff $X_2 = X_1 - C$.

Scalar 1-forms and connections on TM

Let $\varepsilon = a_i dx^i + b_i dx_1^i$ be a scalar 1-form on TM. Then $\gamma = J^*\varepsilon = \varepsilon \cdot J = b_i dx^i$ is semibasic on TM. We say that ε is V-symmetric or V-regular if $\gamma = J^*\varepsilon$ is V-symmetric or V-regular respectively. For the sections $VJ^*\varepsilon : TM \to V^*TM \times V^*TM$ $\widetilde{VJ^*\varepsilon} : TM \to VTM \otimes VTM$ we will use the shortened notations $V\varepsilon, \widetilde{V\varepsilon}$.

We are interested in the constructions of connections on TM from the form ε. First we will use the Lagrange formalism in the classical mechanics.

It is clear that the form $d\gamma = dJ^*\varepsilon = b_{ij}dx^j \wedge dx^i + b_{ij_1}dx_1^i \wedge dx^i$ is symplectic iff ε is V-regular. Denote $A := \varepsilon(C) = b_s x_1^s$.

Let $\lambda = \lambda_i(x, x_1)dx^i + \bar{\lambda}(x, x_1)dx_1^i$ be a 1-form on TM.

Lemma 5. *If ε is a V-regular scalar 1-form on TM then the vector field X on TM satisfying the equation*

$$i_X dJ^*\varepsilon = \varepsilon - dA + \lambda \tag{3}$$

is a semispray if and only if λ is semibasic.

Proof follows from the coordinate form

$$(b_{ij} - b_{ji})\xi^j + b_{ij_1}\eta^j = a_i - b_{si}x_1^s + \lambda_i, \quad b_{sj-1}\xi^s = b_{sj_1}x_1^s - \bar{\lambda}_j$$

of the equation (3).

We will suppose that λ is semibasic. The semispray X from the equation (3) will be denoted by $X = S_\varepsilon\lambda$ and called the Lagrange field of the structure (ε, λ). In the case $\lambda = 0$ we use the denotation S_ε and we say that S_ε is the Lagrange field of the V-regular 1-form ε.

Remark. If ε is semibasic and V-regular then $d\varepsilon$ is symplectic. We can use $A = \varepsilon(S) = b_s x_1^s$, where S is an arbitrary semispray but the solution X of the equation $i_X d\varepsilon = \varepsilon - dA$ is not a semispray.

We will suppose that ε is not semibasic and is V-regular. Then the Lie derivatives $L_{S_{\varepsilon\lambda}}J$, $L_{S_\varepsilon}J$ are the Griffone forms and determine the connections $\Gamma_{\varepsilon\lambda}$, Γ_ε the horizontal projection of which are of the coordinate form

$$h_\Gamma = dx^i \otimes \partial/\partial^i + \Gamma_j^i dx^j \otimes \partial\ \partial x_1^i, \quad \Gamma_j^i = \frac{1}{2}\frac{\partial B^i}{\partial x_1^j}$$

where B^i are the coordinates of the semisprays $S_{\varepsilon\lambda}$ or S_ε.

It is clear that the connections $\Gamma_{\varepsilon\lambda} + cJ$, $c \in R$, are not the only connections determined by the forms ε and λ. For example $\Gamma_{S_{\varepsilon\lambda}} + kJ^*\varepsilon \otimes C + f(L, V\varepsilon(C, C))J$, where $k \in R$ and f is a smooth function of two variables, are another connections on TM. We will construct some geometrical objects, that provide to construct large families of connections from ε.

a) We will prove that if ε is a V-symmetric 1-form on TM then $(a_{ij_1} - b_{ji})$ are the coordinates of a tensor field $\beta_\varepsilon : TM \to V^*TM \otimes V^*TM \equiv T^*M \otimes_{TM} T^*M$. Let $(x^i, x_{10}^i, x_{01}^i, x_{11}^i)$, $(x^i, x_{100}^i, x_{010}^i, x_{110}^i, x_{001}^i, x_{101}^i, x_{011}^i, x_{111}^i)$ be the induced charts on TTM and $TTTM$. Let us recall that the canonical involution i_3 on $TTTM \equiv T_2(TM)$ is of the coordinate form

$$i_3(x^i, x_{100}^i, \ldots, x_{111}^i) = (x^i, x_{100}^i, x_{001}^i, x_{101}^i, x_{010}^i, x_{110}^i, x_{011}^i, x_{111}^i).$$

Consider the form ε as a function $\varepsilon = a_i x_{01}^i + b_i x_{11}^i$ on TTM. Let $p_2 : TR \equiv R \times R \to R$ is the projection on the second factor. Then

$$p_2 T\varepsilon - p_2 \cdot T\varepsilon \cdot i_3 = (a_{ij} - a_{ji})x_{010}^i x_{001}^i + (a_{ij_1} - b_{ji})x_{010}^i x_{101}^j + (b_{ij} - a_{ji})x_{110}^i x_{001}^j$$

is a function on $TTTM$ and its restriction to $V(Tp_M)$ is of the coordinate form $\bar{\varepsilon} = (a_{ij_1} - b_{ji})x^i_{010}x^i_{101}$ where we use the denotation $V(\pi)$ for the space of all vertical vectors of a surjective submersion $\pi : Y \to M$. Let $X = c^i\partial/\partial x^i$, $Y = y^i = \partial/\partial x^i \in T_xY$. There exists $Z \in V_h(Tp_M)$, $h \in T_xM$ such that $Tp_M \cdot p_{TTM}(Z) = X \cdot \bar{p}_2 \cdot Tp_{TM}(Z) = Y$, where $\bar{p}_2 : VTM \equiv TM \times_M TM \to TM$ is the projection on the second factor. For example, $Z = (x^i, x^i_{100}, c^i, x^i_{100}, 0, y^i, 0, x^i_{111})$. Put $\beta_\varepsilon(X,Y) = \bar{\varepsilon}(Z) = (a_{ij_1} - b_{ji})c^iy^j$. Obviously $\beta_\varepsilon : TM \to T^*M \otimes_{TM} T^*M$.

b) We will show that if $\alpha : TM \to VTM \otimes VTM$, $(x^i, x^i_1) \mapsto (x^i, x^i_1, x^{ij} = \alpha^{ij})$ is a section then the functions $\alpha^{ij}_{k_1} = \partial\alpha^{ij}/\partial x^k_1$ determine a section $\bar{V}\alpha : TM \to T^*M \otimes (VTM \otimes VTM)$. Denote $\kappa_1 : VTM \otimes VTM \to TM$, $\kappa_2 = p_M \cdot \kappa_1$ the canonical submersions. Using the identification $(VTM \otimes_{TM} VTM)_x \equiv T_xM \times (T_xM \otimes T_xM)$ we get the other identification $T(VTM \otimes VTM)_x \equiv T(T_xM) \times T(T_xM \otimes T_xM) = (VTM)_x \times T(T_xM \otimes T_xM)$. Then we can extend the map α onto $\xi_\alpha : VTM \to T(VTM \otimes VTM)$, $(x^i, x^i_1, 0, dx^i_1) \mapsto (x^i, x^i_1, \alpha^{ij}, 0, dx^i_1, 0)$. Using the map $V\alpha = T\alpha\mid_{VTM}$ we obtain the map $\bar{V}\alpha := V\alpha - \xi_\alpha : (x^i, x^i_1, 0, dx^i_1) \mapsto (x^i, x^i_1, \alpha^{ij}, 0, 0, \alpha^{ij}_{k_1}dx^k_1) = (x^i, x^i_1, \alpha^{ij}_{k_1}dx^k_1)$ which defines the section $\overline{V\alpha} : TM \to T^*M \otimes (VTM \otimes VTM)$.

c) Remember that $[C,S]$ is a semispray and then $S^* = [C,S] - S_\varepsilon$ is a vertical vector field. In the case of a V-regular and V-nonsymmetric or of a V-regular and V-symmetric form ε denote by K the set $K = \{C, S_\varepsilon, \varepsilon, \omega = J^*\varepsilon, V\varepsilon, \alpha = \widetilde{V\varepsilon}, \overline{V\alpha}, J\}$ or $K = \{C, S_\varepsilon, J^*\varepsilon, V\varepsilon, \alpha = \widetilde{V\varepsilon}, \overline{V\alpha}, \beta_\varepsilon, J\}$. Using tensor products and contractions of objects of K we can construct a family of semibasic VTM-value 1-forms φ on TM and the connections $\Gamma_{\varepsilon\lambda} + \varphi$ joined to ε.

Let us construct connections for which the field S_ε is horizontal. Let Γ be an arbitrary connection on TM, $h_\Gamma = dx^i \otimes \partial/\partial x^i + \Gamma^i_j dx^j \otimes \partial/\partial x^i_1$. Then there exists a unique semispray S_Γ which is Γ-horizontal, $S_\Gamma = h_\Gamma(S) = x^i_1\partial/\partial x^i + \Gamma^i_j x^j_1 \partial/\partial x^i_1$, where S is an arbitrary semispray on TM. We say that S_Γ is associated to Γ. The values of the semisprays S_Γ on the zero-section $0 : M \to TM$, $0(x) = 0$, are equaled to zero-vectors.

Let $S_{\varepsilon\lambda}$ be the Liouville field of (ε, λ). The necessary condition for $S_{\varepsilon\lambda}$ to be globally associated to Γ is: $S_{\varepsilon\lambda}(0) = 0$. Every connection Γ on TM can be expressed in the form $\Gamma = \Gamma_\varepsilon + \varphi$, where φ is a semibasic 1-form on TM with values in VTM. Let S denote the semispray associated to the connection $\Gamma_\varepsilon + \varphi$. We have

$$S_\varphi = x^i_1\partial/\partial x^i + (\Gamma^i_j + \varphi^i_j)x^j_1\partial/\partial x^i_1, \quad \Gamma^i_j = \tfrac{1}{2}B^i_{j_1}, \quad B^i = \tilde{b}^{is}(a_s - b_{sk}x^k_1).$$

$$S_{\varepsilon\lambda} = x^i\partial/\partial x^i + \tilde{b}^{is}(a_s - b_{sk}x^k_1 + \lambda_s)\partial/\partial x^i_1 = x^i_1\partial/\partial x^i + (B^i + \tilde{b}^{is}\lambda_s)\partial/\partial x^i_1.$$

Then $S_{\varepsilon\lambda} = S_\varphi$ if and only if

$$\lambda_i = b_{it_1}(\Gamma^t_j + \varphi^t_j)x^j_1 - b_{it_1}B^t.$$

For any semispray S it holds $\varphi(S) = \varphi^i_t x^t_1$. Since $S_\varphi - S_\varepsilon = [(\Gamma^i_j + \varphi^i_j)x^j_1 - B^i]\partial/\partial x^i_1$ then $S_{\varepsilon\lambda} = S_\varphi$ if and only if $\lambda = \langle V\varepsilon \otimes (S_\varphi - S_\varepsilon)\rangle_2 \equiv V\varepsilon(\cdot, S_\varphi - S_\varepsilon)$. Also $S_\varphi = S_{\Gamma_\varepsilon} + \varphi(S)$ and then $S_\varphi - S_\varepsilon = S_{\Gamma_\varepsilon} - S_\varepsilon + \varphi(S)$, i.e. $\lambda = \langle V\varepsilon, S_{\Gamma_\varepsilon} - S_\varepsilon + \varphi(S)\rangle_2$. We have proved

Proposition 1. *Let ε be a V-regular 1-form on TM. Let $\Gamma = \Gamma_\varepsilon + \varphi$ be an arbitrary connection on TM. Then there is a unique semibasic 1-form $\lambda = V\varepsilon(\cdot, S_{\Gamma_\varepsilon} - S_\varepsilon + \varphi(S))$ such that $S_\varphi = S_{\varepsilon\lambda}$, i.e. such that the semispray associated to $\Gamma_\varepsilon + \varphi$ is the Lagrange field of the structure (ε, λ). Conversaly, if λ is a given semibasic 1-form on TM then the semibasic $(1,1)$-form φ with values in VTM (determining the connection $\Gamma = \Gamma_\varepsilon + \varphi$) such that $S_\varphi = S_{\varepsilon\lambda}$ satisfies $\varphi(S) = S_\varepsilon - S_{\Gamma_\varepsilon} + \widetilde{V}\varepsilon(\cdot, \lambda)$, where S is an arbitrary semispray on TM.*

Corollary. *If φ is a semibasic $(1,1)$-form with value in VTM such that $S_\varphi = S_\varepsilon$ then* $\varphi(S) = S_\varepsilon - S_{\Gamma_\varepsilon} = (B^i - \frac{1}{2}B^i_{j_1}x^j_1)\partial/\partial^i_1$.

We will find semibasic $(1,1)$-forms with values in VTM such that $S_\varphi = S_\varepsilon$. Consider, (compare with [1]),

$$S^*_\varepsilon := [C, S_\varepsilon] - S_\varepsilon = (B^j_{s_1}x^s_1 - 2B^j)\partial/\partial x^j_1.$$

Then $S_\varepsilon - S_{\Gamma_\varepsilon} = -\frac{1}{2}S^*_\varepsilon$. Let $i^1_c V\varepsilon$ be a 1-form such that $i^1_c V\varepsilon(Y) = V\varepsilon(C,Y)$. Then we get the section $i^1_c V\varepsilon \otimes S^*_\varepsilon : TM \to V^*TM \otimes VTM$. Put

$$\varphi_0 = -\frac{1}{2}\frac{1}{V\varepsilon(C,C)} i^1_c V\varepsilon \otimes S^*_\varepsilon = \frac{1}{b_{kt_1}x^k_1 x^t_1}\Big[B^i - \frac{1}{2}\frac{\partial B^i}{\partial x^j_i}x^j_1\Big] b_{kt_1}x^k_1 dx^t \otimes \partial/\partial x^i_1,$$

i.e. φ_0 is a semibasic $(1,1)$-form with values in VTM defined on $TM - 0^{V\varepsilon}$, where $0^{V\varepsilon}$ denotes the set of all vectors in TM such that $V\varepsilon(C,C) = 0$. When $V\varepsilon$ is positively definite then $0^{V\varepsilon}$ is the set of all zero-vectors on M. Obviously $\varphi_0(S) = S_\varepsilon - S_{\Gamma_\varepsilon}$.

Proposition 2. *The Lagrange field S_ε of a V-regular form ε is the semispray associated to the connection $\Gamma_\varepsilon + \varphi_0$.*

Proof. If S_{φ_0} is associated to the connection $\Gamma_\varepsilon + \varphi_0$ then

$$S_{\varphi_0} = x^i_l\partial/\partial x^i + \Big[\frac{1}{2}B^i_{j_1}x^j_1 + \frac{1}{b_{kt_1}x^k_1 x^t_1}(B^i - \tfrac{1}{2}B^i_{j_1}x^j_1)b_{kt_1}x^k_1 x^t_1\Big]\partial/\partial x^i_1 = S.$$

□

Remark 1. If S_ε is a spray then $S^*_\varepsilon = 0$ and therefore $S_\varepsilon = S_{\Gamma_\varepsilon}$. That is the well known result that if S is a spray then S is associated to Γ_ε.

Remark 2. Put

$$\varphi_1 = -\frac{1}{2}\frac{1}{\varepsilon(C)}J^*_1\varepsilon \otimes S^*_\varepsilon, \quad J^*\varepsilon = b_i dx^i,$$

i.e. $\varphi_1 : TM \to T^*M \otimes VTM$ is a section defined for $\varepsilon(C) \neq 0$. As $J^*\varepsilon(S) = \varepsilon(C)$ for any semispray S therefore $\varphi_1(S) = S_\varepsilon - S_{\Gamma_\varepsilon}$ if $\varepsilon(C) \neq 0$. Then the connection $\Gamma_\varepsilon + \varphi_1$ is such that S_ε is associated to $\Gamma_\varepsilon + \varphi_1$.

Remark 3. Denote $\varepsilon^V : \varepsilon \mid_{VTM}$. Put $\alpha = \widetilde{V\varepsilon}$, $h = \overline{V\alpha}(\varepsilon^V, \varepsilon^V, C) = b^{ij}_k b_i b_j x^k_1$, $\beta = \langle \overline{V\alpha} \otimes (\varepsilon^V \otimes \varepsilon^V)\rangle = \tilde{b}^{ij}_k b_i b_j dx^k$. Obviously $\beta(S) = h$. Then the semibasic $(1,1)$-form $\varphi_2 = -\frac{1}{2}\frac{1}{h}\beta \otimes S^*_\varepsilon$ is such that S_ε is associated to the connection $\Gamma_\varepsilon + \varphi_2$ for $h \neq 0$.

Remark 4. If $L : TM \to R$ is a lagrangian then $\varepsilon = dL$, $(a_i = L_i, b_i = L_{i_1})$ is V-symmetric and it is V-regular iff lagrangian is regular. We use denotation $S_L \equiv S_\varepsilon$, $\Gamma_L = \Gamma_\varepsilon$. Since $a_{ij_1} = b_{ji}$ therefore $\beta_\varepsilon = 0$. Now the form φ_0 coincides with the form considered in [1] where the connection $\Gamma_L + \varphi_0$ is called canonical. We see that it is not a unique connection on TM such that S_L is associated with it. There are another connections of this property which domains are in general open submanifolds of TM.

If $d(J^*\varepsilon)$ is symplectic then every connection Γ on TM determines the connection $\bar\Gamma = \operatorname{Orth}\Gamma$ which is orthogonal to Γ according to $d\omega$, i.e. $d\omega(X,Y) = 0$ for any Γ-horizontal vector X and any $\bar\Gamma$-horizontal vector Y. If Γ^i_j are the coordinate functions of Γ then the functions of $\bar\Gamma$ are of the form $\bar\Gamma^i_k = \tilde{b}^{is}(b_{ks} - b_{sk} + b_{ku_1}\Gamma^u_s)$. A connection Γ is called the Lagrange connection if $\bar\Gamma = \Gamma$. It is easy to deduce that if ε is V-symmetric and V-regular then the connection $\Gamma = \Gamma_{\varepsilon\lambda}$ is the Lagrange connection if and only if $a_{kp_1} - b_{pk} - a_{pk_1} + b_{kp} = \lambda_{pk_1} - \lambda_{kp_1}$. It gives

Proposition 3. *Let ε be V-symmetric and V-regular. Then the connection $\Gamma_{\varepsilon\lambda}$ is the Lagrange connection if $a(\beta_\varepsilon) = -a(V\lambda)$ where a denotes the antisymmetrization of forms.*

Corollary. *If $\varepsilon = dL$ then the connection $\Gamma_{\varepsilon\lambda}$ is the Lagrange connection if and only if λ is V-symmetric.*

Remark. We see that there is a large family of connections on TM constructed by the connection $\Gamma_{\varepsilon\lambda}$ and the set K. The mapping $\Gamma \mapsto \bar{\Gamma} = \operatorname{Orth}\Gamma$ gives another connections. Using the procedure of the theory of natural operators it is possible to solve the problem of the constructions of all connections deduced from a V-regular scalar 1-form ε on TM. But it is not such interesting as the question of deeper informations about the geometry of ε (and thus of the regular lagrangian) obtained by connections constructed from ε.

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Received 27 August 1992