Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 257–264 

# ON STRUCTURE EQUATIONS FOR SECOND ORDER CONNECTIONS

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**Abstract**

The basic facts concerning the structure equations of connections in the principal bundle of second order (holonomic) frames $F^2M$ on a differentiable manifold $M$ are presented here in a self contained and synthetic way. Advantage is taken from the embedding of $F^2M$ as a subset of $F(FM)$ and from viewing the group $G^2(n)$ as a semidirect product.

**Keywords.** Connections, frame bundles.

**MS classification.** 53C.

## 1 Introduction

There is a great paralellism between the linear (or first order) frame bundle, $LM \equiv F^1M$, and the second order (holonomic) frame bundle, $F^2M$, on a n-dimensional manifold $M$. As standard references see e.g. [1],[3]. On both principal bundles there exist canonical 1-forms: in the linear case it is the well known $\mathbb{R}^n$-valued 1-form, denoted by $\theta^L$; in the second order case, the canonical 1-form, denoted simply by $\theta$, is valued into the vector space $\mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R})$.

On both bundles we can define connection 1-forms, denoted by $\lambda$ or $\omega$, valued in the Lie algebras, $\mathfrak{gl}(n, \mathbb{R})$ or $\mathfrak{g}^2(n)$, of their respective structure groups, $GL(n, \mathbb{R})$ or $G^2(n)$. The so called first and second structure equations of a connection $\lambda$ in $LM$ are

$$D^\lambda \theta^L = d\theta^L + \lambda \dot{\wedge} \theta^L$$
$$D^\lambda \lambda = d\lambda + \frac{1}{2}[\lambda, \lambda]$$

where $D^\lambda$ means the covariant differentiation with respect to the connection $\lambda$.

We could expect to obtain similar expressions for the second order case. The second structure equation, giving an expression for the covariant derivative of $\omega$ with respect to $\omega$ itself, is well known as a general equation of a connection in any principal bundle:

$$D^\omega \omega = d\omega + \frac{1}{2}[\omega, \omega].$$

*This paper is in final form and no version of it will be submitted for publication elsewhere.

The first structure equation, giving an expression of $D^\omega\theta$ in terms of $\theta$ and $\omega$, was first obtained by Yuen [5]; it could also be deduced from the work of Kolář [4] in the more abstract framework of prolongations of general principal bundles.

We want to stress two previous steps, undoubtly well known, but having not been fully exploited before. The first one is to see $G^2(n)$ as a semidirect product of $GL(n,\mathbb{R})$ and $S^2(n)$, where $S^2(n)$ means the abelian Lie group of symmetric bilinear maps of $\mathbb{R}^n\times\mathbb{R}^n$ into $\mathbb{R}^n$, and also its abelian Lie algebra. The second step is to see $F^2M$ as the first prolongation $(LM)_1$ of $LM$, in the sense of [3] (thus, as a subset of $L(LM)$).

## 2 The Lie group $G^2(n)$ and its Lie algebra $\mathfrak{g}^2(n)$

The Lie group $G^2(n)$ is the set of all 2-jets $j^2(g)$, with source $0\in\mathbb{R}^n$, of diffeomorphisms between open neighbourhoods of 0 in $\mathbb{R}^n$, with $g(0)=0$, and with product law given by the 2-jet of the composition map.

The following two descriptions of $G^2(n)$ by global charts are interesting:

$$I\,:\,G^2(n)\xrightarrow{\approx}(GL(n,\mathbb{R})\times S^2(n),\star)$$

$$II\,:\,G^2(n)\xrightarrow{\approx}(GL(n,\mathbb{R})\times S^2(n),\bullet)$$

where $\left\{\begin{array}{l}(a,\tau)\star(a',\tau'):=(aa',a\tau'+\tau(a',a'))\\ (a,t)\bullet(a',t'):=(aa',t+at'(a^{-1},a^{-1}))\end{array}\right.$

and $\left\{\begin{array}{l}I:j^2(g)\longmapsto(a,\tau)\\ II:j^2(g)\longmapsto(a,t)\end{array}\right.$ with $\left\{\begin{array}{l}a:=Dg(0)\equiv\left(\left.\frac{\partial g^i}{\partial u^j}\right|_0\right)\\ \tau:=D^2g(0)\equiv\left(\left.\frac{\partial^2 g^i}{\partial u^j\partial u^k}\right|_0\right)\\ t:=D^2g(0)(Dg(0)^{-1},Dg(0)^{-1}),\end{array}\right.$

$(u^j)$ being the standard coordinates in $\mathbb{R}^n$.

Description $I$ is the standard one with partial derivatives, and the product law $\star$ expresses nothing else but the chain rule.

Description $II$ is very useful because it describes $G^2(n)$ as a semidirect product, $GL(n,\mathbb{R})\circledS S^2(n)$, associated to $Q\in\operatorname{Hom}(GL(n,\mathbb{R}),\operatorname{Aut}S^2(n))$ given by

$$Q(a)\,:\,t\longmapsto t_a\equiv a\,t(a^{-1},a^{-1}).$$

We will use only $II$.

The Lie algebra $\mathfrak{g}^2(n)$ of $G^2(n)$ can be identified –in both descriptions $I$ or $II$– with the semidirect sum $\mathfrak{gl}(n,\mathbb{R})\oplus S^2(n)$, associated to $q:=Q_*|_{Id}\in\operatorname{Hom}(\mathfrak{gl}(n,\mathbb{R}),\operatorname{End}S^2(n))$, explicitly

$$q(\alpha)\,:\,t\longmapsto t^\alpha\equiv\alpha t-t(\alpha,\cdot)-t(\cdot,\alpha).$$

The adjoint representations are given by

$$\begin{array}{llcl}Ad: & GL(n,\mathbb{R})\circledS S^2(n) & \longrightarrow & \operatorname{Aut}(\mathfrak{gl}(n,\mathbb{R})\oplus S^2(n))\\ & (a,t) & \longmapsto & ((\beta,s)\mapsto(a\beta a^{-1},s_a-t^{a\beta a^{-1}}))\\ ad: & \mathfrak{gl}(n,\mathbb{R})\oplus S^2(n) & \longrightarrow & \operatorname{End}(\mathfrak{gl}(n,\mathbb{R})\oplus S^2(n))\\ & (\alpha,t) & \longmapsto & ((\beta,s)\mapsto([\alpha,\beta],s^\alpha-t^\beta))\end{array}.$$

It is well known that there exists a representation of $G^2(n)$ into $\mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R})$. We can see it in the following way:

Let $g : U \longrightarrow V$ be a diffeomorphism between neighbourhoods of $0 \in \mathbb{R}^n$, with $g(0) = 0$. Let us define

$$\begin{array}{rccc} \mathcal{I}_g : & L\mathbb{R}^n|_U & \longrightarrow & L\mathbb{R}^n|_V \\ & j^1(\psi) & \longmapsto & j^1(g \circ \psi \circ g^{-1})\,; \end{array}$$

it thus follows that

$$\mathcal{I}_{g*}|_{j^1(id)} : T_{j^1(id)} L\mathbb{R}^n \longrightarrow T_{j^1(g)} L\mathbb{R}^n,$$

$id$ being the identity map in $\mathbb{R}^n$, depends only on $j^2(g)$.

Then, identifying $L\mathbb{R}^n$ with the product manifold $\mathbb{R}^n \times GL(n, \mathbb{R})$ and $T_{j^1(g)} L\mathbb{R}^n$ with $\mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R})$, we get a representation of $G^2(n)$ into this vector space. Explicitly:

$$\begin{array}{rccc} \mathcal{A} : & GL(n, \mathbb{R}) \circledS S^2(n) & \longrightarrow & \mathrm{Aut}(\mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R})) \\ & (a, t) & \longmapsto & ((v, \beta) \mapsto (av, t(av) + a\beta a^{-1})) \end{array}$$

and the corresponding representation of the Lie algebra

$$\begin{array}{rccc} \mathfrak{a} := \mathcal{A}_*|_{j^2(id)} : & \mathfrak{gl}(n, \mathbb{R}) \oplus S^2(n) & \longrightarrow & \mathrm{End}(\mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R})) \\ & (\alpha, t) & \longmapsto & ((v, \beta) \mapsto (\alpha v, t(v) + [\alpha, \beta])). \end{array}$$

This representation plays an important role, as we will see.

## 3 The Second Order Frame Bundle, $F^2M$

The second order frame bundle $F^2M$, is the set of all 2-frames on $M$, i.e. 2-jets, $j^2(\phi)$, with source $0 \in \mathbb{R}^n$, of local diffeomorphisms $\phi$ of open neighbourhoods of $0 \in \mathbb{R}^n$ into $M$.

It has a structure of $G^2(n)$-principal bundle, with right action $j^2(\phi) \cdot j^2(g) := j^2(\phi \circ g)$ and projection

$$\pi_2 : F^2M \longrightarrow M, \;\; j^2(\phi) \longmapsto \phi(0)\,.$$

There exists a canonical embedding of $F^2M$ into $L(LM)$, obtained by the following process:

Given a local diffeomorphism $\phi : U \subset \mathbb{R}^n \to M$ with $0 \in U$, we define

$$\begin{array}{rccc} \tilde{\phi} : & L\mathbb{R}^n|_U & \longrightarrow & LM|_{\phi(U)} \\ & j^1(\psi) & \longmapsto & j^1(\phi \circ \psi) \end{array}$$

then, it can be proved that its tangent map at $j^1(id)$

$$\tilde{\phi}_*\Big|_{j^1(id)} : \mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R}) \approx T_{j^1(id)} L\mathbb{R}^n \longrightarrow T_{j^1(\phi)} LM$$

depends only on $j^2(\phi)$ ; and so we obtain the embbeding

$$\begin{array}{rccc} \iota : & F^2M & \longrightarrow & L(LM) \\ & j^2(\phi) & \longmapsto & \tilde{\phi}_*\Big|_{j^1(id)} \equiv z \end{array} .$$

The isomorphisms $z : \mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R}) \to T_{j^1(\phi)} LM$ corresponding to 2-frames $j^2(\phi)$ are characterized as follows:

**Lemma 1** *$z \in L(LM)$ belongs to $\iota(F^2M)$ if and only if:*

**(a)** *The vertical part is determined by the condition $z(0,\alpha) = \alpha^*|_l, \alpha^*$ being the corresponding fundamental vectorfield on $LM$, $\forall \alpha \in \mathfrak{gl}(n, \mathbb{R})$.*

**(b)** *Defining $z(\mathbb{R}^n, 0) =: H^z$, the isomorphism $\mathbb{R}^n \to H^z$, $v \mapsto z(v, 0)$, is the inverse of the canonical form acting on $H^z$; i.e. $\theta^L(z(v,0) = v$, $\forall v \in \mathbb{R}^n$.*

**(c)** *The subspaces $H^z$ have no torsion; i.e. $d\theta^L|_{H^z \times H^z} = 0$.*

Then we have a one-to-one correspondence between 2-frames at points of $M$ and horizontal torsionless n-subspaces in $TLM$.

On the other hand, it can be proved that, given $z \equiv \iota(j^2(\phi) \in L(LM)$, every connection $\lambda$ in $LM$ having $H^z$ as horizontal plane at $l \equiv j^1(\phi)$ yields a normal chart $x_\lambda$, centered at $m \equiv \phi(0)$ and corresponding to the linear frame $l$, such that $j^2(\phi) = j^2(x_\lambda^{-1})$. We denote by $\mathcal{C}_z$ the set of linear connections having $H^z$ as horizontal plane. The isomorphism $z$ satisfies $z(v,\alpha) = B^L_\lambda(v)|_l + \alpha^*|_l$, $B^L_\lambda(v)$ being the standard $\lambda$-horizontal vectorfield corresponding to $v$.

It turns out that the properties a, b, and c define what Kobayashi calls (see [3]) first prolongation $(LM)_1$ of the linear frame bundle $LM$ . Initially, $(LM)_1$ is a $S^2(n)$-principal bundle over $LM$ with projection $\Pi : (LM)_1 \to LM$ given by the restriction of the projection $L(LM) \to LM$, and right action

$$zt := z \circ \mathcal{A}_{(I,t)}\,.$$

But there is also a natural $GL(n, \mathbb{R})$-action on $(LM)_1$ given by

$$za := R^L_{a\,*} \circ z \circ \mathcal{A}_{(a,0)}$$

($R^L$ is the right action on $LM$), which may be characterized by the condition $R^L_{a\,*}(H^z) = H^{za}$, or $\mathcal{C}_{za} = \mathcal{C}_z$.

Mixing in appropriate way the last two actions we obtain:

**Lemma 2**

**(a)** *The bundle $\pi^L \circ \Pi : (LM)_1 \longrightarrow M$ is a $GL(n, \mathbb{R}) \otimes S^2(n)$-principal bundle over $M$ (denoting $\pi^L : LM \to M$), with action*

$$\begin{array}{rcl} \mathcal{R} : (LM)_1 \times (GL(n,\mathbb{R}) \otimes S^2(n)) & \longrightarrow & LM_1 \\ (z,(a,t)) & \longmapsto & z(a,t) := (zt)a = \mathcal{R}^L_{a\,*} \circ z \circ \mathcal{A}_{(a,t)}. \end{array}$$

**(b)** *The bijection $\iota : F^2M \longrightarrow (LM)_1$, becomes an isomorphism of principal bundles over $M$ corresponding to the identity map in $GL(n, \mathbb{R}) \otimes S^2(n)$.*

We resume the situation in the following diagram:

$$
\begin{tikzcd}
F^2M \arrow[r, "\iota"] \arrow[dr, "\pi_{2,1}"] \arrow[ddr, "\pi_2"'] & LM_1 \arrow[r, "\subset"] \arrow[d, "\Pi"] & L(LM) \arrow[dl] \\
& LM \arrow[d, "\pi^L"] & \\
& M &
\end{tikzcd}
$$

In what follows we shall identify $F^2M \equiv LM_1$, $\pi_{2,1} \equiv \Pi$, $\pi_2 \equiv \pi^L \circ \Pi$.

Let us now fix our conventions concerning exterior products of forms on $F^2M$ : Let $\Psi \in \Lambda^j(F^2M, \mathfrak{gl}(n,\mathbb{R}) \dotplus S^2(n))$ and $\Phi \in \Lambda^k(F^2M, \mathbb{R}^n + \mathfrak{gl}(n,\mathbb{R}))$ be two pseudotensorial forms of type $Ad$ and $\mathcal{A}$, respectively. We define $\Psi \dot\wedge \Phi \in \Lambda^{j+k}(F^2M, \mathbb{R}^n + \mathfrak{gl}(n,\mathbb{R}))$ by

$$\Psi \dot\wedge \Phi(X_1, \ldots, X_{j+k}) := \frac{1}{j!k!} \sum_\sigma (-1)^\sigma \mathfrak{a}_{\Psi(X_{\sigma(1)},\ldots,X_{\sigma(j)})} \Phi(X_{\sigma(j+1)}, \ldots, X_{\sigma(j+k)}),$$

where the sum is taken over the set of all permutations $\sigma$ of $\{1, \ldots, j+k\}$ and $(-1)^\sigma = \pm 1$ is the sign of $\sigma$.

Also, if $\Psi \in \Lambda^j(F^2M, \mathfrak{gl}(n,\mathbb{R}) \dotplus S^2(n))$ and $\Psi' \in \Lambda^k(F^2M, \mathfrak{gl}(n,\mathbb{R}) \dotplus S^2(n))$, we define $[\Psi, \Psi'] \in \Lambda^{j+k}(F^2M, \mathfrak{gl}(n,\mathbb{R}) \dotplus S^2(n))$ by

$$[\Psi, \Psi'](X_1, \ldots, X_{j+k}) := \frac{1}{j!k!} \sum_\sigma (-1)^\sigma [\Psi(X_{\sigma(1)}, \ldots, X_{\sigma(j)}), \Psi'(X_{\sigma(j+1)}, \ldots, X_{\sigma(j+k)})].$$

# 4 Structure Equations of a Connection in $F^2M$

We proceed to define the main objects appearing in the structure equations, and to give their most important properties.

The *fundamental vectorfield* $(\alpha, s)^*$ on $F^2M$ is defined as usual in any principal bundle, $\forall (\alpha, s) \in \mathfrak{gl}(n,\mathbb{R}) \dotplus S^2(n)$.

The *canonical 1-form* $\theta \equiv (\theta_{-1}, \theta_0) \in \Lambda^1(F^2M, \mathbb{R}^n + \mathfrak{gl}(n,\mathbb{R}))$ is defined by $\theta(X) := z^{-1}(\Pi_* X)$, $\forall z \in F^2M$, $\forall X \in T_z F^2M$.

**Lemma 3** *It holds:*

**(a)** *$\theta$ is pseudotensorial of $\mathcal{A}$ type: i.e. $\mathcal{R}^*_{(a,t)}\theta = \mathcal{A}_{(a,t)^{-1}}\theta$. It is not tensorial (contrarily to the linear case), because $\theta((\alpha, s)^*|_z) = (0, \alpha)$.*

**(b)** *The $\mathbb{R}^n$-and $\mathfrak{gl}(n,\mathbb{R})$-valued components of $\theta$ verify:*
*$\theta_{-1} = \Pi^* \theta^L$ and $\theta_0|_z = (\Pi^* \lambda)|_z$, $\forall \lambda \in \mathcal{C}_z$.*

**(c)** $d\theta_{-1} = -\theta_0 \dot{\wedge} \theta_{-1}$ *(as a direct application of the first structure equation of a symmetric linear connection* $d\theta_{-1} = -\lambda \dot{\wedge} \theta^L$ *).*

A *second order connection* $\omega \equiv (\omega_0, \omega_1) \in \Lambda^1(F^2M, \mathfrak{gl}(n, \mathbb{R}) \oplus S^2(n))$ is a connection form in $F^2M$ defined as usual. From $\omega$ we obtain a linear connection $\lambda^\omega$ by the condition $\Pi^* \lambda^\omega = \omega_0$ (because $\Pi$ is a morphism of principal fiber bundles).

We define the *standard $\omega$-horizontal vectorfield corresponding to* $v \in \mathbb{R}^n$ as the unique vectorfield $B_\omega(v)$ on $F^2M$ satisfying $\omega(B_\omega(v)) = 0$ and $\theta_{-1}(B_\omega(v)) = v$.

**Lemma 4**

**(a)** $\Pi_*(B_\omega(v)|_z) = B^L_{\lambda^\omega}(v)|_l$ *with* $B^L_{\lambda^\omega}(v)$ *the corresponding standard* $\lambda^\omega$*-horizontal vectorfield on* $LM$.

**(b)** $[(\alpha, t)^*, B_\omega(v)] = B_\omega(\alpha v)$.

**Lemma 5** *Given a second order connection* $\omega$ *and* $v \in \mathbb{R}^n$*, the* $C^\infty$*-function* $\Upsilon_\omega(v) := \theta_0(B_\omega(v)) : F^2M \to \mathfrak{gl}(n, \mathbb{R}),\ z \mapsto \theta_0(B_\omega(v)|_z)$ *satisfies* $(\alpha, t)^* \Upsilon_\omega(v) = \Upsilon_\omega(\alpha v) - [\alpha, \Upsilon_\omega(v)] - t(v)$.

**Proof:** Let $exp$ and $Exp$ be the exponential maps in the Lie groups $GL(n, \mathbb{R})$ and $GL(n, \mathbb{R}) \otimes S^2(n)$, respectively. Note that the $GL(n, \mathbb{R})$-component of $Exp(\alpha, t)$ is $exp\alpha$. Then we have ($\forall z \in F^2M$ and $\forall (\alpha, t) \in \mathfrak{gl}(n, \mathbb{R}) \oplus S^2(n)$ ) :

$$(\alpha, t)^*_z \Upsilon_\omega(v) = \tfrac{d}{d\epsilon}|_{\epsilon=0}\ \theta_0(B_\omega(v)|_{z \cdot Exp\, \epsilon(\alpha,t)}) = \text{(Lemma 4b)}$$

$$\tfrac{d}{d\epsilon}|_{\epsilon=0}\ \theta_0(\mathcal{R}_{Exp\, \epsilon(\alpha,t)\, *} B_\omega(exp\epsilon\alpha\, v)|_z) = \text{(Lemma 3a)}$$

$$\tfrac{d}{d\epsilon}|_{\epsilon=0}\ (\mathcal{A}_{Exp\, -\epsilon(\alpha,t)}\theta)_0\ (B_\omega(exp\epsilon\alpha\, v)|_z) = (\mathfrak{a}_{-(\alpha,t)}\theta)_0\ (B_\omega(v)|_z) + \theta_0(B_\omega(\alpha\, v)|_z) =$$

$$(-[\alpha, \theta_0] - t \circ \theta_{-1})\ (B_\omega(v)|_z) + \theta_0(B_\omega(\alpha\, v)|_z) = -[\alpha, \Upsilon_\omega(v)\,(z)] - t(v) + \Upsilon_\omega(\alpha\, v)\,(z)$$

and the result follows. □

We remark that, given $X \equiv B_\omega(v)|_z + (\alpha, t)^*|_z \in T_z(F^2M)$, it holds: $(\theta_0 - \omega_0)(X) = \theta_0(B_\omega(v)|_z)$.

Finally, we could define, by analogy to the linear case, the *'torsion' of second order* (see [5]) as the two form

$$\Theta = (\Theta_{-1}, \Theta_0) := D^\omega \theta \in \Lambda^2(F^2M, \mathbb{R}^n + \mathfrak{gl}(n, \mathbb{R})),$$

which is a tensorial 2-form of $\mathcal{A}$ type, and the *curvature of second order* as

$$\Omega = (\Omega_0, \Omega_1) := D^\omega \omega \in \Lambda^2(F^2M, \mathfrak{gl}(n, \mathbb{R}) \oplus S^2(n)),$$

which is a tensorial 2-form of $Ad$-type.

**Theorem** *(Structure equations). Let* $\omega$ *be a second order connection. It holds:*

**(a)** $d\theta = -\omega \dot{\wedge} \theta + \frac{1}{2}\,[\omega_0, \omega_0] + \Theta$

**(b)** $d\omega = -\frac{1}{2}[\omega, \omega] + \Omega$

**Proof:** (a) By linearity, it suffices to compute both members on pairs $X, Y \in T_z(F^2M)$ in the following three cases:

- Both $X$ and $Y$ are $\omega$-horizontal. The equation follows trivially.
- Both $X$ and $Y$ are vertical, thus $X = (\alpha, t)^*|_z$ and $Y = (\beta, s)^*|_z$ for some $\alpha, \beta \in \mathfrak{gl}(n, \mathbb{R})$ and $t, s \in S^2(n)$. Then:

$$\begin{cases} d\theta(X,Y) = X(\theta(\beta,s)^*) - Y(\theta(\alpha,t)^*) - \theta|_z\,[(\alpha,t)^*, (\beta,s)^*] = \quad (\text{Lem. 3a}) \\ \qquad -(0,\ [\alpha,\beta]) \\ \omega\dot{\wedge}\theta\ (X,Y) = \mathfrak{a}_{(\alpha,t)}(\theta|_z(\beta,s)^*) - \mathfrak{a}_{(\beta,s)}(\theta|_z(\alpha,t)^*) = 2(0,\ [\alpha,\beta]) \\ [\omega_0,\omega_0]\ (X,Y) = 2\,[\omega_0(\alpha,t)^*|_z,\ \omega_0(\beta,s)^*|_z] = 2(0,\ [\alpha,\beta]) \\ \Theta(X,Y) = 0 \end{cases}$$

  and the result follows.
- $X$ is vertical and $Y$ is $\omega$-horizontal, thus $X = (\alpha, t)^*|_z$ and $Y = B_\omega(v)|_z$ for some $\alpha \in \mathfrak{gl}(n, \mathbb{R})$, $t \in S^2(n)$ and $v \in \mathbb{R}^n$. Then:

$$\begin{cases} d\theta(X,Y) = X(\theta B_\omega(v)) - Y(\theta(\alpha,t)^*) - \theta|_z\,[(\alpha,t)^*, B_\omega(v)] = (\text{Lem. 5 and 4b}) \\ \qquad (0, \Upsilon_\omega(\alpha\, v) - [\alpha, \Upsilon_\omega(v)] - t(v)) - \theta|_z\ B_\omega(\alpha\, v) = (-\alpha\, v,\ -[\alpha, \Upsilon_\omega(v)] - t(v)) \\ \omega\dot{\wedge}\theta\ (X,Y) = \mathfrak{a}_{(\alpha,t)}(\theta|_z\ B_\omega(v)) = (\alpha\, v,\ t(v) + [\alpha, \Upsilon_\omega(v)]) \\ [\omega_0,\omega_0]\ (X,Y) = 0 \\ \Theta(X,Y) = 0 \end{cases}$$

  and the result follows.

(b) This is a general result for connections in any principal bundle. □

We remark that the equation (a) decomposes into:

- $d\theta_{-1} = -\omega_0\dot{\wedge}\theta_{-1} + \Theta_{-1}$ ($\mathbb{R}^n$-valued component)
- $d\theta_0 = -\omega_1\dot{\wedge}\theta_{-1} - [\omega_0, \theta_0] + \frac{1}{2}[\omega_0, \omega_0] + \Theta_0$ ($\mathfrak{gl}(n, \mathbb{R})$-valued component)

Taking covariant derivatives in the equations of the Theorem, we obtain the following (Bianchi) identities:

**Corollary**

**(a)** $D^\omega\Theta = \Omega\dot{\wedge}\theta - [\Omega_0, \omega_0]$

**(b)** $D^\omega\Omega = 0$.

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*Received 5 November 1992*