Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 185–192

ON NONLINEAR QUANTUM MECHANICS

Pavel Bóna

Abstract. There is a "natural" formal extension of quantum mechanics (QM) containing, besides the traditional (linear) QM of systems with finite number of degrees of freedom, its nonlinear generalization, as well as classical mechanics (CM), and also some other ("quasiclassical") theories. A formal expression of this extension has the form of a Poisson system on an infinite-dimensional manifold. The essentially statistical nature of interpretation of this theory requires introduction of description of physical quantities ("observables") by projection measure valued functions on the state space and their interpretation. A brief review of parts of this theory is given. Some open problems are mentioned.
Keywords. Projective Hilbert space, Kähler structure, Poisson bracket, Frèchet differential, spectral measure, symmetry, coadjoint action, momentum map.
MS classification scheme. 53C55, 58B20, 81B99, 81C25.

1. Some general aspects of physical theories

Physical theories consist of their formal (essentially mathematical) description – their formalism, as well as of their interpretation connecting this formalism with empirical procedures and data. Our forthcoming generalization of the quantum theory consists in an extension of these general features. Formalisms of "basic" contemporary theories contain three main mutually related classes of mathematical objects describing a considered "physical system":

I. *States* $\mathcal{S}$; these objects correspond intuitively to different "situations" controlled by specific empirical procedures identifying and preparing the *physical system.* The fact that *different* elements of $\mathcal{S}$ correspond to the *same physical system* in different situations is reflected in the theory by groups of transformations of $\mathcal{S}$ onto $\mathcal{S}$ – by symmetries.

II. *Quantities* $\mathcal{C}$; elements $\mathfrak{f}, \mathfrak{h}$ of this set are interpreted as empirical procedures (representing "measuring apparatuses") applicable to the considered physical system in various states, and defining functions of the state with values in a chosen set (usually $\mathbb{R}$, or $\mathbb{R}^n$) interpreted as "the measured (expectation, e.g.) values of the quantity $\mathfrak{q}$", or "the pointer positions of the apparatus $\mathfrak{q}$". The set $\mathcal{C}$ is considered as a characteristics of "the considered physical system", and by the symmetries are elements of $\mathcal{C}$ transformed leaving the set $\mathcal{C}$ invariant. Some elements of $\mathcal{C}$ could be connected with subgroups of symmetries leaving (e.g.) the values of themselves in the whole orbit of the states unchanged; in this sense symmetries could endow some of the "observables" by a physical meaning.

This paper is in final form and no version of it will be submitted for publication elsewhere.

III. *Symmetries* $\mathcal{G}$; this set consists of transformations of the couple $(\mathcal{S};\mathcal{C})$ leaving its "general structure" (if there is some determined) invariant. We consider, e.g., the one-parameter group of time translations – "the dynamical group" – to be a subset of $\mathcal{G}$. It is usual to consider $\mathcal{G}$ to be a group of automorphisms (w.r.t. the given structure) of $(\mathcal{S};\mathcal{C})$. Symmetries are useful also in formulation of interpretation of the theory, cf. the preceding paragraph.

Let us give examples of these mathematical objects in the cases of Hamiltonian CM and traditional QM of systems with finite number of degrees of freedom:

CM : Classical mechanics.

I. States are probability measures $\mu \in \mathcal{M}_1^+(M)$ on a symplectic manifold $(M;\Omega)$ (Ω is the symplectic form, [1]), and the "pure states" (i.e. the extremal points of the convex set $\mathcal{M}_1^+(M)$) containing theoretically maximal possible information on the system are described by the Dirac measures $\delta_{\boldsymbol{x}} \in \mathcal{M}_1^+(M)(\boldsymbol{x} \in M)$. The set M is in many important cases (which we are mainly interested in) a homogeneous space of a Lie group G, and can be identified as the symplectic manifold with a coadjoint orbit of G, or of its central extension by $\mathbb{R}$, [8].

II. Quantities consist of real functions on M, e.g. $f, h \in C^\infty(M)$. The interpretation of the formalism says, that the integral $\mu(f) := \int f(\boldsymbol{x})\mu(d\boldsymbol{x})$ is the expectation value of f in the state μ.

III. Symmetries are represented by symplectomorphisms of $(M;\Omega)$. One-parameter groups of symmetries are generated by $f, h \in C^\infty(M)$. The corresponding "dynamical equation" corresponding to the generator h (the "Hamilton function") is of the form

$$\dot{f}_t = \{h, f_t\} \tag{1}$$

for all $f \in C^\infty(M)$, where $\{h, f\}$ is the Poisson bracket corresponding to Ω. It describes implicitly the Poisson flow φ^h on M corresponding to the Hamiltonian h; $f_t(\boldsymbol{x}) \equiv f(\varphi_t^h \boldsymbol{x})$. The states are transformed by a symplectomorphism φ in such a way that $\mu \mapsto \mu \circ \varphi^{-1}$.

QM : Quantum mechanics.

I. States are described by "density matrices" $\rho \in \mathcal{S}$, i.e. positive trace class operators on a separable complex Hilbert space $\mathcal{H}$ with the trace equal to unity, $\mathrm{Tr}\,\rho = 1$. The extremal points of the set $\mathcal{S}$ are one-dimensional orthogonal projections $P_j := P_{x_j}$ onto the subspaces of $\mathcal{H}$ spanned by the vectors $x_j \in \mathcal{H}$. Each ρ is an at most countable convex combination of $P_j^! s$. Let us note that the set of all the $P_j^! s$ can be identified with the projective Hilbert space $P(\mathcal{H})$ which is a Kähler manifold in a canonical way, [5].

II. Quantities are described in QM by selfadjoint operators $F, H, \dots$ on $\mathcal{H}$ or, equivalently, by their (spectral) projection-valued measures $E_X, E_H, \dots$ on $\mathbb{R}$. The interpretation of the theory says that the probability of finding the measured value of X in the Borel set $B \subset \mathbb{R}$ if X is measured in the state ρ equals to $\mathrm{Tr}(\rho E_X(B))$, hence the expectation of X in the state ρ equals $\mathrm{Tr}(\rho X)$.

III. Symmetries are described here by transformations φ of $P(\mathcal{H})$ conserving the Kähler metrics, hence they can be represented by unitary or antiunitary transformations $U \equiv U_\varphi$ of $\mathcal{H}$ so that $\varphi(P_x) = P_{Ux}(= U_\varphi P_x U_\varphi^{-1}$ in the unitary case). Unitary flows correspond to one-parameter symmetry groups $U_t \equiv \exp(-\mathrm{it}\,H)$, $H = \int \lambda\; E_H(d\lambda)$, and these are generated as flows $\tilde{\varphi}^H$ on $P(\mathcal{H})$ (or even on $\mathcal{S}$) by an equation formally similar to (1):

$$\dot{\rho}_t = \frac{1}{i}[H, \rho_t] \tag{2}$$

with the solution $\rho_t = \exp(-\,\mathrm{it}\, H)\rho\exp(\mathrm{it}\, H)$, for $\rho_0 = \rho$.

It is seen from the said that in the both cases of CM and QM the set of generators of one parameter groups of symmetries can be identified with a subset of "observables". In the extension of quantum mechanics presented in the following sections the set $D\mathcal{G}$ of generators of the symmetry groups in $\mathcal{G}$ will be much larger then that of QM and CM, and the set $\mathcal{C}$ of generalized "observable" quantities will be constructed as a different set "containing", however, in a certain sense, the set $D\mathcal{G}$: Each generator can be obtained from many different "observables".

It can be shown that a wide scale of existing physical theories can be found in our [2] general scheme for dynamical theories. Its specifications to various "subtheories" mainly consists in specific choices of classes of functions (defined on subsets of, e.g., $\mathcal{E} := P(\mathcal{H})$, resp. on $\mathcal{E}\times\mathcal{E}$) describing "the generators of symmetries", resp. "the set of observables" in each of the cases; also an appropriate choice of a set of "physical states" of the system is connected with such a choice of observables and generators.

We can obtain in this way, e.g., formalisms of the following general theories as well as some of their "caricatures" (or approximations):

(i) Hamiltonian classical mechanics (CM) of systems with finite number of degrees of freedom on homogeneous phase spaces.

(ii) Traditional quantum mechanics (QM).

(iii) Nonlinear (symplectic) quantum mechanics (NLQM).

(iv) Specific "quasiclassical" or "selfconsistent" approximations for QM described as dynamics on manifolds of generalized coherent states – the "classical projections of QM"; this class includes the time-dependent variational principle including in turn the time-dependent Hartree-Fock approximations, cf. also [10].

2. Generalized quantum mechanics as a symplectic dynamics on $P(\mathcal{H})$

It would be useful first to describe the structure of $P(\mathcal{H})$ in some details. Note, that also density matrices $\rho \neq \rho^2$ could be described on a projective Hilbert space, e.g. as elements of $P(\mathfrak{H})$ corresponding to (normalized) vectors $\rho^{1/2} \in \mathfrak{H}$ ($\equiv \mathcal{H}\otimes\mathcal{H}^* \equiv$ the space of Hilbert Schmidt operators on $\mathcal{H}$) or by taking the Hilbert space of some composed system containing the considered system as a subsystem. Elements x of $\mathcal{H}$ will be naturally associated with the corresponding elements x^* of the topological dual space $\mathcal{H}^*$ of $\mathcal{H}$ via the Riesz theorem, i.e. $x^*(y) := (x,y)(y\in\mathcal{H})$; the mapping $x \to x^*$ is an antilinear isometry of $\mathcal{H}$ onto $\mathcal{H}$. The space $\mathfrak{H} \subset \mathcal{L}(\mathcal{H})$ ($\equiv$ bounded operators on $\mathcal{H}$) will be (linearly and isometrically) identified with the tensor product $\mathcal{H}\otimes\mathcal{H}^*$ in such a way that the operator (cf. the Dirac notation, [6]) $x\otimes y \equiv |x\rangle\langle y| \in \mathcal{L}(\mathcal{H})$ acts as follows: $|x\rangle\langle y|z := |x\rangle\langle y|z\rangle := x(y,z), z\in\mathcal{H}$. The scalar product in $\mathfrak{H}$ is then $(x\otimes y^*, z\otimes u^*)_2 = \mathrm{Tr}(|y\rangle\langle x||z\rangle\langle u|) = (x,z)(u,y) := \langle x|z\rangle\langle u|y\rangle$.

The projective Hilbert space $P(\mathcal{H})$ will be considered as a complex analytic manifold with the charts mapping the neighbourhoods $\mathcal{V}_{\boldsymbol{y}} := \{P_x \in P(\mathcal{H}) : \mathrm{Tr}(P_xP_y)\neq 0\}$ of $P_y \in P(\mathcal{H})$ onto the orthogonal complements $[y]^\perp$ of $y \in \boldsymbol{y}$ in $\mathcal{H}$ (considered as complex Hilbert subspaces of $\mathcal{H}$, see also [5]) via the y-dependent mappings θ_y,

$$\theta_y : P_x \to \|y\|^2(y,x)^{-1}(I-P_y)x =: \theta_y(x); \tag{3}$$

the elements of $P(\mathcal{H})$ will be denoted also by the lower case boldface letters: $\boldsymbol{y} \equiv P_y \in P(\mathcal{H})$ (for $y\in\boldsymbol{y}$, if $\boldsymbol{y}$ is considered as the equivalence class in $\mathcal{H}$ of the factor-space; if $y\in\mathcal{H}$ is expressed by a *formula* written in any type of letters, then we shall use the boldface brackets instead of $\boldsymbol{y}$, i.e. $\boldsymbol{(}formula\boldsymbol{)} := P_y$). The tangent space $T_{\boldsymbol{y}}P(\mathcal{H})$ of

$P(\mathcal{H})$ at $\boldsymbol{y} \in P(\mathcal{H})$ is identified in the usual way, [1], with the linear space of classes of mutually tangent differentiable curves at $\boldsymbol{y}$. For any differentiable mapping θ of a neighbourhood of $\boldsymbol{y}$ onto a neighbourhood of $\theta(\boldsymbol{y})$ in another differentiable manifold the corresponding tangent mapping $T_{\boldsymbol{y}}\theta$ maps the vector $v \in T_{\boldsymbol{y}}P(\mathcal{H})$ represented by a curve $t \to c_{\boldsymbol{v}}(t)(c_{\boldsymbol{v}}(0) = \boldsymbol{y})$ onto the vector tangent at $\theta(\boldsymbol{y})$ represented by the curve $t \to \theta(c_{\boldsymbol{v}}(t))$. If $\boldsymbol{x} \in \mathcal{V}_{\boldsymbol{y}}$ (hence $\boldsymbol{y} \in \mathcal{V}_{\boldsymbol{x}}$), $T_{\boldsymbol{y}}\theta_x$ maps $T_{\boldsymbol{y}}P(\mathcal{H})$ onto $[x]^{\perp}$ and the choice of $x \equiv y \in \boldsymbol{y}$ in θ_x leads to a natural (y-dependent) identification of $T_{\boldsymbol{y}}P(\mathcal{H})$ with $[y]^{\perp}$. For $\boldsymbol{v} \in T_{\boldsymbol{y}}P(\mathcal{H})$ (let $\boldsymbol{y}$ be fixed), one has $\boldsymbol{v}_x := T_{\boldsymbol{y}}\theta_x(\boldsymbol{v}) \in [x]^{\perp}$,

$$\boldsymbol{v}_x := \frac{d}{dt}\,|_{t=0}\ \ \theta_x(c_{\boldsymbol{v}}(t)). \tag{4}$$

One can choose, e.g., $c_{\boldsymbol{v}}(t) := (\exp(-\,\mathrm{it}\, B(\boldsymbol{v}))y) = \mathrm{Ad}(\exp(-\,\mathrm{it}\, B(\boldsymbol{v})))P_y$, where $B(\boldsymbol{v})$ is a selfadjoint element of $\mathcal{L}(\mathcal{H})$ representing the vector $\boldsymbol{v}$ in this way. With such a choice of $\boldsymbol{v}$, one has the expression

$$\boldsymbol{v}_x = -i(x,y)^{-1}\|x\|^2(I - P_yP_x/\operatorname{Tr}(P_yP_x))B(\boldsymbol{v})y. \tag{5}$$

For $x = y$, this leads to

$$\boldsymbol{v}_y = -i(I - P_y)B(\boldsymbol{v})y, \tag{6}$$

and the choice of any $v \in [y]^{\perp}$, and $B(\boldsymbol{v}) := i\|y\|^{-2}(|v\rangle\langle y| - |y\rangle\langle v|)$ gives $\boldsymbol{v}_y = v$. With a chosen $y \in \boldsymbol{y}$, and with the corresponding "identification θ_y" of $T_{\boldsymbol{y}}P(\mathcal{H})$ with $[y]^{\perp}$, one can identify $\boldsymbol{v} \equiv v \equiv \boldsymbol{v}_y$. For different choices of x in (4), on the other hand, one obtains expressions of $\boldsymbol{v}$ in different charts θ_x and θ_z mutually related by

$$\boldsymbol{v}_z = \|x\|^{-2}(z,y)^{-1}\|z\|^2(x,y)(I - (z,y)^{-1}|y\rangle\langle z|)\boldsymbol{v}_x. \tag{7}$$

Let us note that $(z,y)^{-1}|y\rangle\langle z| = P_yP_z/\operatorname{Tr}(P_yP_z)(y, z \in \mathcal{H})$. One can check easily validity of the following two mutually inverse relations:

$$\boldsymbol{v}_x = (x,y)^{-1}\|x\|^2(I - P_yP_x/\operatorname{Tr}(P_yP_x))\boldsymbol{v}_y,$$
$$\boldsymbol{v}_y = \|x\|^{-2}(x,y)(I - P_y)\boldsymbol{v}_x.$$

We shall consider $P(\mathcal{H})$ as a real analytic manifold with the complex structure (integrable) $J \in \mathcal{T}_1^1(P(\mathcal{H}))(\mathcal{T}_s^r(P(\mathcal{H}))$ will denote the vector space of all smooth r times contravariant and s times covariant tensor fields on the manifold M) defined as the section $\boldsymbol{y} \to J_{\boldsymbol{y}} \in \mathcal{L}([y]^{\perp})$ by $(J_{\boldsymbol{y}}\boldsymbol{v})_y := i\boldsymbol{v}_y$, i.e. with a help of the given multiplication by the imaginary unit "i" in $\mathcal{H}$. The Kähler metrics $\Gamma \in \mathcal{T}_2^0(P(\mathcal{H}))$ is expressed in the following form:

$$\Gamma_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) := 2\|y\|^{-2}\operatorname{Re}(\boldsymbol{v}_y,\boldsymbol{w}_y), \boldsymbol{v},\boldsymbol{w} \in T_{\boldsymbol{y}}P(\mathcal{H}). \tag{8}$$

The corresponding symplectic form $\Omega \in \Omega^2(P(\mathcal{H}))$ (:= the space of two-forms) $\subset \mathcal{T}_2^0(P(\mathcal{H})))$ on $P(\mathcal{H})$ then is:

$$\Omega_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) := \Gamma_{\boldsymbol{y}}(\boldsymbol{v},J\boldsymbol{w}) = -2\|y\|^{-2}\operatorname{Im}(\boldsymbol{v}_y,\boldsymbol{w}_y). \tag{9}$$

Expressed in the chart θ_x, the Kähler structure Ψ on $P(\mathcal{H})$ has the form:

$$\Gamma_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) - i\Omega_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) = 2\|y\|^{-2}\operatorname{Tr}(P_xP_y)(\boldsymbol{v}_x,(I - P_y)\boldsymbol{w}_x). \tag{10}$$

Inserting into (8) and (9) from (6), one obtains an expression of the Kähler structure in terms of the selfadjoint operators $B(\boldsymbol{v}(\boldsymbol{x}))$ and $B(\boldsymbol{w}(\boldsymbol{x}))$ representing the vector fields $\boldsymbol{v}$ and $\boldsymbol{w}$ in any point $\boldsymbol{x} \in P(\mathcal{H})$:

$$\Psi_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) = \Gamma_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) - i\Omega_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) = 2\|y\|^{-2}(\boldsymbol{v}_y,\boldsymbol{w}_y) =$$

$$2\,\mathrm{Tr}(P_y B(\boldsymbol{v}(\boldsymbol{y}))B(\boldsymbol{w}(\boldsymbol{y}))) - 2\,\mathrm{Tr}(P_y B(\boldsymbol{v}(\boldsymbol{y})))\,\mathrm{Tr}(P_y B(\boldsymbol{w}(\boldsymbol{y}))). \tag{11}$$

It can be shown that the distance function $d(\boldsymbol{x},\boldsymbol{y})$ on $P(\mathcal{H})$ corresponding to the Riemannian metrics Γ is expressed by

$$d(\boldsymbol{x},\boldsymbol{y}) = \sqrt{2}\arccos\sqrt{\mathrm{Tr}(P_x P_y)}, \tag{12}$$

with values in the interval $[0; 2^{-1/2}\pi]$. Let us rephrase the fundamental Wigner theorem in the following form:

Proposition 1. *Let φ be any bijection of $P(\mathcal{H})$ onto itself conserving d from (12). Then there is a linear or antilinear isometry $\boldsymbol{u}_\varphi$ of $\mathcal{H}$ onto itself representing φ in the sense that $\varphi(\boldsymbol{y}) = (\boldsymbol{u}_\varphi y)$ for all $\boldsymbol{y} \in P(\mathcal{H})(y \in \boldsymbol{y})$. If $\boldsymbol{u}_\varphi$ is linear, it conserves also Ω:*

$$(\varphi^*\Omega)_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}) := \Omega_{\varphi(\boldsymbol{y})}(\varphi_*\boldsymbol{v},\varphi_*\boldsymbol{w}) = \Omega_{\boldsymbol{y}}(\boldsymbol{v},\boldsymbol{w}), \tag{13}$$

i.e. φ is a symplectomorphism of $P(\mathcal{H})$. For antilinear $\boldsymbol{u}_\varphi$ the mapping φ changes sign at $\Omega : \varphi^*\Omega = -\Omega$.

A general symplectomorphism of $P(\mathcal{H})$, on the other hand, does not conserve Γ (equivalently: the distance function d).

Now we can describe Hamiltonian dynamics on $P(\mathcal{H})$ generated by arbitrary differentiable Hamiltonians $f, h \in C^\infty(P(\mathcal{H}),\mathbb{R})$. These considerations will show us differences between the dynamics of standard (linear) QM generated by $f := h_a$ (cf. (15) for definition, $a = a^* \in \mathcal{L}(\mathcal{H})$, functions f of this form will be called "affine"; all others f are "nonlinear"), and the nonlinear one (cf. Proposition 2). Let us denote by $\mathcal{F}$ the set of infinitely (Frèchet) differentiable real-valued functions on $P(\mathcal{H})$. The differential $df \in \mathcal{T}_1^0(P(\mathcal{H}))$ of $f \in \mathcal{F}$ can be determined by the formula

$$d_{\boldsymbol{x}} f(\boldsymbol{w}(\boldsymbol{x})) := \frac{d}{dt}\,|_{t=0}\, f((\exp(-\,\mathrm{it}\,B(\boldsymbol{w}(\boldsymbol{x}))x)),$$

with $B(\boldsymbol{w})$ specified below (4), for any $\boldsymbol{w} \in \mathcal{T}_0^1(P(\mathcal{H}))$. The symplectic form Ω is (strongly) nondegenerate on $P(\mathcal{H})$, [5], hence it associates with each $f \in \mathcal{F}$ a unique Hamiltonian vector field $\boldsymbol{v}_f$ on $P(\mathcal{H})$ such that

$$\Omega(\boldsymbol{v}_f,\boldsymbol{w}) = -df(\boldsymbol{w}) \tag{14}$$

for all $\boldsymbol{w} \in \mathcal{T}_0^1(P(\mathcal{H}))$. The Lie derivative $L(\boldsymbol{v}_f)$ (defined on the graded algebra $\mathcal{T}(P(\mathcal{H}))$ $= \bigcup \mathcal{T}_s^r(P(\mathcal{H}))$ of tensor fields in the standard way, [4]) leaves Ω invariant, $L(\boldsymbol{v}_f)\Omega \equiv 0$.

The Poisson bracket $\{f,h\} := \Omega(\boldsymbol{v}_f,\boldsymbol{v}_h) \in \mathcal{F}$ determines the differential equation (which is equivalent to the Schrödinger equation in the case of affine f) for the Hamiltonian (local) flow $\tilde{\varphi}^f$, which can be written also in the form: $dh(\boldsymbol{v}_f) = \{f,h\}(\forall h \in \mathcal{F})$. According to [5], Propositions 3.2 and 3.3 one has:

Proposition 2. *Let $f \in \mathcal{F}$, and let $\boldsymbol{v}_f$ be the corresponding Hamiltonian vector field on $P(\mathcal{H})$. Then $L(\boldsymbol{v}_f)\Gamma \equiv 0$ iff there is a bounded selfadjoint operator a, $a = a^* \in \mathcal{L}(\mathcal{H})$, such that*

$$f(\boldsymbol{x}) \equiv h_a(\boldsymbol{x}) := \mathrm{Tr}(P_x a), (x \in \boldsymbol{x}). \tag{15}$$

In the case of $f \equiv h_a$, v_f is complete, and one has

$$\tilde{\varphi}_t^f(\boldsymbol{x}) = (\exp(-\,\mathrm{it}\,a)x),\ t \in \mathbb{R},\ \boldsymbol{x} \in P(\mathcal{H}),\ 0 \neq x \in \boldsymbol{x}.$$

From (9), (6) and (14) one can see that

$$\{h_a, h_b\}(\rho) = i\,\mathrm{Tr}(\rho[a,b]).$$

For general $f, h \in \mathcal{F}$ one has

$$\{h, f\}(\rho) = i\,\mathrm{Tr}(\rho[d_\rho h, d_\rho f]),$$

where $d_\rho f$ – the differential of f in the point $\rho \in P(\mathcal{H})$ – is considered as an element of $\mathcal{L}(\mathcal{H})$ in a canonical way, and the commutator $[d_\rho h, d_\rho f]$ is then defined.

Let us introduce also the *Riemann bracket* in accordance with Ref.[5]:

$$[\![f, h]\!] := \Gamma(\boldsymbol{v}_f, \boldsymbol{v}_h). \tag{16}$$

An immediate consequence of (11) and of the last proposition is

Lemma 3. *Let $h_a \in \mathcal{F}$ be defined for any $a \in \mathcal{L}(\mathcal{H})$ as in (15). For any selfadjoint $a, b \in \mathcal{L}(\mathcal{H})$, the following formula holds:*

$$2h_{ab} = [\![h_a, h_b]\!] - i\{h_a, h_b\} + 2h_a h_b. \tag{17}$$

The mappings $a \mapsto h_a(\boldsymbol{x})(\boldsymbol{x} \in P(\mathcal{H}))$ are continuous in the weak-operator topology on $\mathcal{L}(\mathcal{H})$.

Due to (17), we can calculate $\mathrm{Tr}(P_x a^n)(n \in \mathbb{N})$ in terms of the function h_a, hence we can recover the probabilistic interpretation of QM in differential geometrical terms on $P(\mathcal{H})$: The formula (17) leads to a prescription for calculation of $h_{f(a)}(a^* = a)$ for any bounded real Borel function f on the spectrum $\mathrm{sp}(a)$ of a. Then $h_{f(a)}(\boldsymbol{x})$ can be interpreted as the expectation value of the "observable" (represented by the operator) $f(a)$ in the pure state $\boldsymbol{x}$. Calculating expectations of any $b = b^* \in \mathcal{L}(\mathcal{H})$ in an "elementary mixture", i.e. the state ρ described by a density matrix $\rho \in \mathcal{S}$, we obtain the number $\mathrm{Tr}(\rho b)$ according to QM. To obtain this in our "phase space formalism" of QM on $P(\mathcal{H})$, we can decompose ρ (in many different ways) into a countable convex combination of pure states $P(j) = P(j)^2 \in P(\mathcal{H}), \rho = \Sigma \lambda_j P(j)$, i.e. to obtain a (discrete – in this case) measure μ on $P(\mathcal{H})$ with the barycentre $\mathfrak{b}(\mu) = \rho$ concentrated at points $P(j) \in P(\mathcal{H})$, with $\mu(P(j)) := \lambda_j$:

$$\mathrm{Tr}(\rho b) = \Sigma_j \lambda_j \,\mathrm{Tr}(P(j)b) = \Sigma_j \lambda_j h_b(P(j)) = \int h_b(\boldsymbol{x})\mu(d\boldsymbol{x}). \tag{18}$$

This expression for the expectation of "the observable $f = h_b$" does not depend on μ with the same barycentre ρ, and this independence is conserved by time evolution corresponding to linear generators $Q \equiv h_H$. This is not the case of a nonlinear generator Q (i.e. $Q \not\equiv h_X$ for any $X^* = X$) in our theory. Then the flow $\tilde{\varphi}^Q$ does not conserve the metrics Γ according to Proposition 2, hence it does not conserve the "transition probabilities" $|\langle x|y\rangle|^2 = \mathrm{Tr}(P_x P_y)$ according to (12). This shows, however, that the validity of (18) will generally depend on time: various measures $\mu \neq \mu'$ with the same barycentres $\mathfrak{b}(\mu) = \mathfrak{b}(\mu')$ could have different barycentres after some time $t \neq 0 : \mathfrak{b}(\mu \circ \tilde{\varphi}^Q_{-t}) \neq \mathfrak{b}(\mu' \circ \tilde{\varphi}^Q_{-t})$. If different decompositions of ρ would be interpreted empirically as resulting from the "reductions of wave packets" corresponding to different measurements performed on a correlated but noninteracting system as in the Einstein–Podolsky–Rosen type experiments, [7], [9], then the measurable distinction of these decompositions (obtained after some time) would lead to possibility of superluminal communication.

3. Notes on "observables" and their interpretation

In the traditional interpretation of QM, the expectation value of the numerical results of measurement of an observable f in an arbitrary state $\boldsymbol{y} \in P(\mathcal{H})$ is its value $f(\boldsymbol{y})$, e.g.

$h_X(\boldsymbol{y}) = \langle y|X|y\rangle$, if $\|y\| = 1$. The "stationary states" corresponding to the observable described by the function f are naturally defined as points $\boldsymbol{x}(i) \in P(\mathcal{H})$ in which $d_{\boldsymbol{x}(i)}f = 0$, which correspond to the eigenvectors $x(i)$ of an operator X, if $f := h_X$: $X|x(i)\rangle = \kappa_i|x(i)\rangle$. In the last case the probability of "finding the value $h_X(\boldsymbol{x}(i)) = \langle x(i)|X|x(i)\rangle = \kappa_i$ if X is measured in an arbitrary state $\boldsymbol{y}$" equals to $\mathrm{Tr}(P_{x(i)}P_y)$. Hence, the expectation of X in an arbitrary $\boldsymbol{y} \in P(\mathcal{H})$ is

$$h_X(\boldsymbol{y}) = \Sigma_i \,\mathrm{Tr}(P_{x(i)}P_y)h_X(\boldsymbol{x}(i)) \equiv \mathrm{Tr}(\Sigma_i h_X(\boldsymbol{x}(i))P_{x(i)}P_y)$$

if X has pure point spectrum: $X \equiv \Sigma_i h_X(\boldsymbol{x}(i))P_{x(i)}$, and what can be also modified (with a help of the spectral measure of X) for an arbitrary spectrum of X. Such an expression of $f(\boldsymbol{y})$ for a nonlinear f corresponding to the described interpretation would lead, however, to linearity of f (i.e. f would be equal to h_Y with the operator $Y := \Sigma_i f(\boldsymbol{x}(i))P_{x(i)}$, with $d_{x(i)}f = 0$), hence to a contradiction.

Axiomatic analysis of QM seems to show that probabilistic properties of observables are described by linear operators. "Natural" algebraic properties of the used sets of operators lead to sets containing (spectral projections of) selfadjoint operators of universal enveloping algebras of generators of symmetry group representations in $\mathcal{H}$. Moreover, it is possible to incorporate to quantum theory also classical quantities with vanishing dispersion of their values measured in pure states, if we shall consider them as the operator-valued functions on the "classical" (generalized) phase space $P(\mathcal{H})$ with values proportional to the identity operator $I \in \mathcal{L}(\mathcal{H})$. General "observables" are introduced in our generalized QM as selfadjoint-operator valued functions on $P(\mathcal{H})$, $\nu(\in P(\mathcal{H})) \mapsto X(\nu) = X(\nu)^*$. One can consider also (for $X(\nu) \in \mathcal{L}(\mathcal{H})$) as the "observable" the function $(\rho;\nu)(\in P(\mathcal{H}) \times P(\mathcal{H})) \to \mathrm{Tr}(\rho X(\nu))(\in \mathbb{R})$. For sufficiently differentiable $\nu \to X(\nu)$ one obtains a bounded generator by setting $\rho \equiv \nu$. The interpretation of these generalized observables cannot be obtained (for $X(\nu) \not\equiv$ const.) as mentioned in notes following Lemma 3, if one starts from the function $\nu \to \mathrm{Tr}(\nu X(\nu))$ of one variable. Those technics can be used, however, for the function $\nu \to \mathrm{Tr}(\nu X(\rho))$ at each fixed ρ. In the case of unbounded operators $X(\nu) = \int \lambda E_{X(\nu)}(d\lambda)$ it would be better to start directly from the spectral measures $E_{X(\nu)}$ instead of real-valued densely defined functions on $P(\mathcal{H}) \times P(\mathcal{H})$ to obtain probabilistic interpretation of the "observables". Generally, the probability of finding in the Borel set $B \subset \mathbb{R}$ the value of the quantity determined by the operator-valued function $X : \nu \mapsto X(\nu)$ measured in the state given by a probability measure μ on the Borel space $P(\mathcal{H})$ equals to

$$\mathrm{prob}(X,\mu;B) := \int \mathrm{Tr}(\nu E_{X(\nu)}(B))\mu(d\nu). \tag{19}$$

Hence, classical quantities are obtained as operator-valued functions $\nu \to f(\nu)I$, with $f(\nu) \in \mathbb{R}$, and I is the identity of $\mathcal{L}(\mathcal{H})$.

4. Some problems

As a technical problem of the theory is a general definition of the "Poisson flow" $\tilde{\varphi}$ corresponding to the densely defined functions $h_H : \nu \to \mathrm{Tr}(\nu H)$ corresponding to constant unbounded selfadjoint H and to those unbounded nonlinear real functions h on $P(\mathcal{H})$ that are considered as their generalizations.

For specific functions connected with continuous representations $U(G)$ of Lie groups G the problem is (essentially) solved in [2] (with some unpublished completions and corrections) with a help of a momentum mapping F of its dense domain of definition $\mathcal{D}(F)$

containing projections P_x corresponding to the analytic vectors x of the representation $U(G)$. The mapping $F\colon \mathcal{D}(F) \to \mathrm{Lie}(G)^*$ ($\equiv$ the dual of the Lie algebra $\mathrm{Lie}(G)$ of G) is defined by

$$F : \rho \to F_\rho \in \mathrm{Lie}(G)^*, \text{ with } F_\rho(\xi) := (\mathrm{Tr}(\rho X(\xi))(\xi \in \mathrm{Lie}(G)),$$

and $X(\xi)$ is the selfadjoint generator (in general unbounded) of the one parameter unitary group $t \to U(\exp(t\xi))$. Let a function $f \in C^\infty(\mathrm{Lie}(G)^*, \mathbb{R})$ be given. Then $\boldsymbol{f} : \rho \to \boldsymbol{f}(\rho) := f(F_\rho)$ is a (unbounded real) function on the $U(G)$-invariant domain $\mathcal{D}(F)$. These functions can be used as generators of a "Poisson flow" (Poisson brackets of such $\boldsymbol{f's}$ with differentiable functions can be defined) on $\mathcal{D}(F)$ which is projected by F onto the Poisson flow of f on $F(\mathcal{D}(F)) \subset \mathrm{Lie}(G)^*$ generated via the canonical Poisson structure on $\mathrm{Lie}(G)$ and can be reproduced by the coadjoint action of G, cf. [2], and also [3].

The another open problem of the formal theory is its physical justification and interpretation to specific systems and phenomena.

References

1. Abraham R., J.E. Marsden, Foundations of Mechanics, Benjamin Cummings, Reading Mass.
2. Bóna P., *Quantum mechanics with mean-field background*, (Preprint Ph10-91), Comenius University, Bratislava, Czechoslovakia; cf. also P. Bóna; *"Generalized Quantum Mechanics and Classical Observables"*, Classical and Quantum Systems – Foundation and Symmetries, World Scientific, 1992.
3. Bóna P., J. Math. Phys. **29** (1988), 2223–2235.
4. Bourbaki N. Varétés différentielles et analytiques. Fascicule de résultats. Hermann, Paris 1967 and 1971; russ.ed. Mir, Moscow 1975.
5. Cirelli R., A. Mania, L. Pizzocchero, J. Math. Phys. **31** (1990), 2891–2897; P. Bóna, Czech. J. Phys. **B33** (1983), 837.
6. Dirac P.A.M., *The Principles of Quantum Mechanics*, Claredon Press, Oxford, 1958.
7. Gisin N., Phys Lett A **143** (1990), 1.
8. Kirillov A.A., *Elements of Theory of Representations*, russ. ed. Nauka, Moscow, 1978.
9. Polchinski J., Phys. Rev. Lett. **66** (1991), 397.
10. Spohn H., Rev. Mod. Phys. **52** (1980), 569.

Pavel Bóna
Department of Theoretical Physics
Faculty of Mathematics and Physics
Comenius University
Mlynská dolina F2
CS-842 15 Bratislava, Czechoslovakia
E-mail: bona@mff.uniba.cs

Received 29 October 1992