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GENERATION METHOD FOR SOLUTIONS OF EINSTEIN-MAXWELL EQUATIONS AND ITS GEOMETRICAL BACKGROUND

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Abstract. Geometrical facts are presented supporting the conjecture on the connection between gravity and electromagnetism, leading to a new method finding classes of solutions of Einstein-Maxwell equations. On this basis, the conjecture in question is more precisely formulated.

Keywords. Generation method, Einstein-Maxwell equations, Killing vectors.

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1. INTRODUCTION

Many attempts were made to unify gravity and electromagnetism. However, some interesting and not completely explored connections between them exist also in the standard general relativity [4]. In 1988, a new conjecture concerning this connection was formulated by Horský and Mitskiévic [2]. Our considerations are based on this conjecture, which lead to the construction of new classes of Einstein-Maxwell fields. We start from an exact Einstein equations seed solution having some symmetry expressed by Killing vector field (with related 1-form). We multiply this 1-form by a parameter q and identify it with an electromagnetic four-potential (the metric will be "charged"). Then we replace the original seed metric by a metric of similar analytic form, but containing some unknown functions. Solving the Einstein-Maxwell equations for this new metric with the electromagnetic potential from the seed metric obtained, we find a new class of solutions having the seed metric as a limit for $q \to 0$. Perhaps it is possible to express the essence of conjecture saying that every Killing field ξ in vacuo solution (with exception of the case $d\xi = 0$) could be treated as a relic of electromagnetic potential in some class of solutions with an electromagnetic field.

[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

This method was succesfully applied in several cases [1], [3]. Nevertheless, it is desirable to give an exact proof of the conjecture, eventually to find limits of its validity. For this reason, a more concrete specification of relations between the "seed metric" and "new metric" is necessary. In order to find it, we formulate some geometric theorems supporting the Horský-Mitskiévic conjecture and, on this basis, we propose its more precised variant.

2. GEOMETRIC THEOREMS AND THEIR INTERPRETATION

Let us have a metric field g with some Killing field ξ and, eventually, with other Killing field ψ . Let us denote $\Xi = g\xi$, $\Psi = g\psi$ related 1-forms and let $\mathcal{L}_\xi$, $\mathcal{L}_\psi$ be the Lie derivatives with respect to the fields ξ, ψ respectively. Then the following theorems are valid:

1a) Let $R^i_k \xi^k = 0$ (especially $R^i_k = 0$ in vacuo). Then $\left(\xi^{i;k} - \xi^{k;i}\right)_{;k} = 0$.
Proof:

$$\left(\xi^{i;k} - \xi^{k;i}\right)_{;k} = \xi^{i;k}_{;k} + \xi^{k;i}_{;k} - 2\xi^{k;i}_{;k} = \left(\xi^{i;k} + \xi^{k;i}\right)_{;k} - 2\xi^{k}_{;k;i} + 2R^i_k \xi^k,$$

first two terms vanish as a consequence of Killing equations and the third one is equal to zero with respect to our assumption.

2a) It holds $\mathcal{L}_\xi d\Xi = 0$.
Proof:

$$\mathcal{L}_\xi d\Xi = d\mathcal{L}_\xi \Xi,$$
$$\mathcal{L}_\xi \Xi = \mathcal{L}_\xi g\xi = g\mathcal{L}_\xi \xi,$$
$$\mathcal{L}_\xi \xi = [\xi, \xi] = 0.$$

3a) It holds $\mathcal{L}_\psi d\Xi = dg\,[\psi, \xi]$.
Proof:

$$\mathcal{L}_\psi d\Xi = d\mathcal{L}_\psi \Xi,$$
$$\mathcal{L}_\psi \Xi = \mathcal{L}_\psi g\xi = g\mathcal{L}_\psi \xi,$$
$$\mathcal{L}_\psi \xi = [\psi, \xi]\,.$$

Following commentaries have to be added to our theorems:

The first theorem means that the expression $\xi^{i;k} - \xi^{k;i}$ (or the related form $d\Xi$) fulfills the Maxwell equations. Thus it can be interpreted as a testing electromagnetic field. This well-known fact served as an initial impulse to the idea that the form $qd\Xi$ can be treated as an electromagnetic field with a non negletable influence on metrics.

The second theorem informs us that the electromagnetic field $d\Xi$ is invariant with respect to the related Killing field ξ. It supports the assumption that the original symmetry given by the vector field ξ will be also preserved in new metrics.

The third theorem means that the field $d\Xi$ is invariant with respect to the other Killing field ψ, if the relation $dg[\xi, \psi] = 0$ holds. It leads us to the assumption of preservation of symmetry connected with the ψ also in new metrics.

4a) Let us also discuss the reflective symmetries in the seed metrics. It is natural to suppose that such a symmetry $\mathcal{R}$ will be preserved when

$$\mathcal{R}d\Xi = \pm d\Xi$$

where the upper/lower sign is related to the space/time reflection. (Our demand reflects the hybrid properties of electromagnetic four potential A_i with respect to space/time reflections.)

Now, we can formulate the conjecture for vacuo case in a more concrete form:

Let g be a solution of Einstein equations in vacuo with one or more Killing fields and eventually with reflection symmetries. Then there exists a one-parameter class of metrics $\widetilde{g}(q)$ on the same manifold, representing a solution of Einstein-Maxwell equations in vacuum with the following properties:

1b) The expression $A = q\Xi$ ($\Xi = g\xi$) represents the electromagnetic potential of $\widetilde{g}(q)$, where $\lim_{q\to 0} \widetilde{g}(q) = g$.

2b) The Killing field ξ of g is also Killing field of $\widetilde{g}$.

3b) Another Killing field ψ of g represents also a Killing field of $\widetilde{g}$, if $dg[\xi, \psi] = 0$ holds.

4b) Reflection symmetries of g are conserved also in $\widetilde{g}$, if the condition expressed in (4a) holds.

3. EXAMPLES

Let us apply our considerations to the Taub solution in the form

$$ds^2 = \frac{1}{x}dt^2 - x dx^2 - x^2\left(dy^2 + dz^2\right) \tag{1}$$

as the seed metric.

There are the following Killing fields in this solution:

a) $\xi = \frac{\partial}{\partial t}$
time translation,

b) $\eta = a\frac{\partial}{\partial y} + b\frac{\partial}{\partial z}$
plane translation,

c) $\vartheta = -(z+m)\frac{\partial}{\partial y} + (y+n)\frac{\partial}{\partial z}$
plane rotation

(a, b, m, n are arbitrary parameters). Naturally, also all linear combinations of a), b), c) are Killing fields of the metrics (1). Further, the metrics (1) contains plane reflections and time reflection.

A) According to 1b) we can introduce the electromagnetic potential

$$A = \frac{q}{x} dt, \qquad dA = -\frac{q}{x^2} dx \wedge dt = F$$

connected with the vector field ξ of the time translation. Because of the fact

$$[\xi, \eta] = [\xi, \vartheta] = 0,$$

all Killing isometries of the seed metric (1) will be preserved (in accordance with the conditions 2b), 3b)). The Killing equations are

$$\xi^{i;k} + \xi^{k;i} = \widetilde{g}^{ks} \xi^{i}_{,s} + \widetilde{g}^{is} \xi^{k}_{,s} - \widetilde{g}^{ik}_{,s} \xi^{s} = 0, \tag{2}$$

analogically for η, ϑ. Applying them on the field ξ we obtain

$$\widetilde{g}^{ik}_{,t} = 0, \tag{3}$$

all components of the new metric will be independent of time.

Let us apply, further, (2) on η. Using (3) one gets

$$a \widetilde{g}^{ik}_{,y} + b \widetilde{g}^{ik}_{,z} = 0.$$

Because a, b are arbitrary parameters, the components of the new metric are independent of y, z

$$\widetilde{g}^{ik}_{,y} = \widetilde{g}^{ik}_{,z} = 0. \tag{4}$$

Finally, we can apply (2) on the field ϑ. Using (3) and (4), we get

$$\widetilde{g}^{yy} = \widetilde{g}^{zz}, \qquad \widetilde{g}^{yz} = \widetilde{g}^{tz} = \widetilde{g}^{xz} = \widetilde{g}^{ty} = \widetilde{g}^{xy} = 0. \tag{5}$$

The symmetry of metric with respect to the time reflection means

$$\widetilde{g}^{tx} = 0. \tag{6}$$

Consequently, according to (3) - (6), the form of metrics, fulfilling the requirements of our conjecture, is

$$ds^2 = T(x) dt^2 - P(x) dx^2 - Q(x) \left(dy^2 + dz^2 \right). \tag{7}$$

Functions $T(x)$, $P(x)$, $Q(x)$ are given in [2].

B) Let us connect the electromagnetic potential with the plane translation η. Without loss of generality we can choice this translation in the y-direction. Then we have

$$A = -q x^2 dy, \qquad dA = -2qx dx \wedge dy = F.$$

It holds

$$\left[\eta_{(1)}, \eta_{(2)} \right] = 0$$

for different plane translations, but

$$dg \left[\eta, \vartheta \right] \neq 0$$

for the plane translation and rotation, respectivelly. Consequently, we can apply (2) only on ξ and η, then we obtain again (3), (4), i.e. independency of $\widetilde{g}$ on t, y, z. It is easily to seen, that the form dA fulfills the condition formulated in 4a) for the case of $z \to -z$ reflection but not in the case of time reflection. It means that

$$\widetilde{g}^{zy} = \widetilde{g}^{zx} = 0. \tag{8}$$

in the case of $z \to -z$ reflection.

Thus the supposed form of new class of metrics is (see (3), (4), (8))

$$ds^2 = T(x)dt^2 - P(x)dx^2 - Q(x)dy^2 - R(x)dz^2 - 2U(x)dxdy - 2V_\alpha(x)dq^\alpha dt \tag{9}$$

where q^α means the space coordinates x, y, z.

c) The last possibility is to connect the electromagnetic potential with rotation. In this case it is suitable to introduce polar coordinates r, φ and replace

$$dy^2 + dz^2 = dr^2 + r^2 d\varphi^2.$$

We can choice the rotation Killing field as $\theta = \frac{\partial}{\partial\varphi}$. The electromagnetic potential is

$$A = -qx^2r^2d\varphi, \qquad dA = -2q\left(r^2xdx + x^2rdr\right) \wedge d\varphi = F.$$

Because

$$dg\left[\theta_{(1)}, \theta_{(2)}\right] \neq 0,$$

there can be time translation and one plane rotation as the Killing fields of a new class of metrics. It means that the metric coefficients are independent of t, φ,

$$\widetilde{g}^{ik}_{,t} = 0, \qquad \widetilde{g}^{ik}_{,\varphi} = 0. \tag{10}$$

Comparing dA with the condition 4a) we can see, that the time reflection and the $\varphi \to -\varphi$ reflection do not fulfill it. Consequently, we can suppose the new class of metrics in the form (see (10))

$$ds^2 = g_{ik}(r,t)dq^idq^k \tag{11}$$

where q^i means the space-time coordinates (t, x, r, φ).

Comparing (7), (9) and (11) we can see that decreasing symmetry leads to a much less restrictive form of metrics.

Let us remember that the metric coefficients depend also on a parameter q. One could expect that the limit of T is $\frac{1}{x}$, the limit of P is x and the limits of Q, R are x^2. Remaining coefficients have limit equal to 0 for $q \to 0$.

4. CONCLUSION

In our contribution we have connected the conjecture on the existence of classes of solutions of Einstein-Maxwell equations with some natural requirements concerning form of metrics of these classes. The requirements in question are expressed by Killing equations and by some symmetries with respect to the reflections. The concrete example presented here shows that the Killing equations could be solved and we obtained an analytical form of metrics with some unknown functions. These functions could be determined from the Einstein-Maxwell vacuo equations. We intend to realize more general calculations later.

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