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ON THE THEORY OF HOLOMORPHICALLY–PROJECTIVE MAPPINGS OF PARABOLICALLY–KÄHLERIAN SPACES

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Abstract. General aspects of the theory of the holomorphically-projective mappings of parabolically-Kählerian spaces are considered. Spaces that do not admit these mappings are found.
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1. Introduction

In the paper holomorphically-projective mappings of parabolically-Kählerian spaces $K_n^{o(m)}$ are considered. A basis for these investigations was laid by V.V. Vishnevsky [3]. Some facts from the theory of holomorphically-projective mapping $K_n^{o(m)}$ were published in [1],[2].

A Riemannian space $K_n^{o(m)}$ is said to be *m-parabolically-Kählerian spaces* if together with a metric tensor $g_{ij}(x)$ it possesses a structure $F_i^h(x)$ of a rank $m \geq 2$ satisfying the following relations

$$a) \quad F_\alpha^h F_i^\alpha = 0; \quad b) \quad F_{(ij)} = 0; \quad c) \quad F_{i,j}^h = 0, \tag{1}$$

where comma denotes covariant differentiation, (i,j) denotes symmetrization with respect to i and j, $F_{ij} \equiv g_{i\alpha} F_j^\alpha$.

By an *analytically planar curve* in $K_n^{o(m)}$ we shall mean a curve defined by the equations $x^h = x^h(t)$ such that the following condition holds

$$d\lambda^h/dt + \Gamma_{\alpha\beta}^h \lambda^\alpha \lambda^\beta = \rho_1(t)\lambda^h + \rho_2(t) F_\alpha^h \lambda^\alpha$$

where $\lambda^h \equiv dx^h/dt$; ρ_1, ρ_2 are function of the argument t, Γ_{ij}^h is the Christoffel symbol.

A diffeomorphism f: $K_n^{o(m)} \to \bar{K}_n^{o(m)}$ is said to be *holomorphically-projective mapping*, if all analytically planar curves in $K_n^{o(m)}$ are mapped into analytically planar curves in $\bar{K}_n^{o(m)}$.

This paper is in final form and no version of it will be submitted for publication elsewhere.

It is proved in [2] that $K_n^{o(m)}$ admits a holomorphically-projective mapping onto some $\bar{K}_n^{o(m)}$ if and only if there exists a solution of the following equations

$$a_{ij,k} = \bar{\lambda}_i g_{jk} + \bar{\lambda}_j g_{ik} + \lambda_i F_{jk} + \lambda_j F_{ik} \tag{2}$$

on an unknown tensor a_{ij} and a vector λ_i, $\bar{\lambda}_i \equiv \lambda_\alpha F_i^\alpha$, satisfying

$$a_{ij} = a_{ji};\ a_{\alpha(i} F_{j)}^\alpha = 0;\ det\|a_{ij}\| \neq 0. \tag{3}$$

If $\lambda_i \not\equiv 0$, then the holomorphically-projective mapping will be called *non-trivial.* Contracting (2) with g^{ij} ($\|g^{ij}\| = \|g_{ij}\|^{-1}$) we see that the vector $\bar{\lambda}_i$ is a gradient: $\bar{\lambda}_i = \frac{1}{4}\partial_i(a_{\alpha\beta} g^{\alpha\beta}) \equiv \bar{\lambda}_{,i}$. Thus, the conditions $\bar{\lambda}_{i,j} = \bar{\lambda}_{j,i}$ hold. It follows that $\bar{\lambda}_{\alpha,\beta} F_i^\alpha F_j^\beta = 0$.

2. Main equation

Theorem 1. *A diffeomorphism f: $K_n^{o(m)} \to \bar{K}_n^{o(m)}$ is a holomorphically-projective mapping if and only if there exists a solution of the following linear Cauchy-like system*

$$\begin{array}{lrl} a) & a_{ij,k} &= \bar{\lambda}_{(i} g_{j)k} + \lambda_{(i} F_{j)k}; \\ b) & \lambda_{i,j} &= \tau F_{ij} + a_{\alpha\beta} M_{ij}^{\alpha\beta}; \\ c) & \tau_{,i} &= \lambda_\alpha M_i^\alpha + a_{\alpha\beta} M_i^{\alpha\beta} \end{array} \tag{4}$$

on an unknown tensor a_{ij}, a vector λ_i, and a function τ, satisfying conditions (3). Here $M_{ij}^{\alpha\beta}$, M_i^α, $M_i^{\alpha\beta}$ are tensors determined by formulas (8)-(12) below and dependent on g_{ij} and F_i^h of the space $K_n^{o(m)}$.

Proof. Consider the integrability conditions of the equation (2):

$$a_{\alpha(i} R_{j)kl}^\alpha = g_{k(i} \bar{\lambda}_{j),l} - g_{l(i} \bar{\lambda}_{j),k} - F_{k(i} \lambda_{j),l} + F_{l(i} \lambda_{j),k}. \tag{5}$$

Contracting (5) with $F_{k'}^k$ and also with $F_{l'}^l$ we obtain two expressions. After removing primes we sum them up. Since in $K_n^{o(m)}$ it holds $R_{i\alpha k}^h F_j^\alpha + R_{ij\alpha}^h F_k^\alpha = 0$, we get $F_{k(i} \Lambda_{j)l} - F_{l(i} \Lambda_{j)k} = 0$ where $\Lambda_{ij} \equiv \bar{\lambda}_{i,j} + \lambda_{i,\alpha} F_j^\alpha$.

It follows that $\Lambda_{ij} = \Lambda F_{ij}$, i.e. $\bar{\lambda}_{i,j} + \lambda_{i,\alpha} F_j^\alpha = \Lambda F_{ij}$ where Λ is a function. Contracting (5) with g^{jk} we then get

$$n\bar{\lambda}_{i,j} = \mu g_{ij} + \nu F_{ij} + a_{\alpha i} R_j^\alpha - a_{\alpha\beta} R_{.ij.}^{\alpha..\beta} \tag{6}$$

where R_{ijk}^h and R_{ij} are Riemann and Ricci tensors, respectively, the operations of lifting and lowering of indices are induced by the metric tensor, and μ, ν are certain functions.

After symmetrizing (6) we get

$$n\bar{\lambda}_{i,j} = \mu g_{ij} + \tfrac{1}{2} a_{\alpha(i} R_{j)}^\alpha - a_{\alpha\beta} R_{.ij.}^{\alpha..\beta}. \tag{7}$$

Substituting (7) into (5) we obtain

$$a_{\alpha\beta} M_{ijkl}^{\alpha\beta} = F_{li}\lambda_{j,k} + F_{lj}\lambda_{i,k} - F_{ki}\lambda_{j,l} + F_{kj}\lambda_{i,l}. \tag{8}$$

Here and in what follows M are tensors determined by g_{ij} and F_i^h on $K_n^{o(m)}$. More concretely

$$M^{\alpha\beta}_{ijkl} \equiv \delta^{\alpha}_{(i}R^{\beta}_{j)kl} + M^{\alpha\beta}_{1|k(i}g_{j)l} - M^{\alpha\beta}_{1|l(i}g_{j)k}; \quad nM^{\alpha\beta}_{1|li} \equiv \tfrac{1}{2}\delta^{\alpha}_{(i}R^{\beta}_{j)} - R^{\alpha..\beta}_{.ij.}$$

where δ_i^h is the Kronecker symbol.

Let ε^j and ν^k be vectors such that $\varepsilon^j\nu^k F_{jk} = 1$. Denote $M_i \equiv \varepsilon^\alpha F_{\alpha i}$.

Contracting (8) with $\varepsilon^i\varepsilon^j\nu^k$ we get

$$\varepsilon^\alpha \lambda_{\alpha,l} = \tau M_l + a_{\alpha\beta}M^{\alpha\beta}_{2|l} \tag{9}$$

where $\tau \equiv \lambda_{\alpha,\beta}\varepsilon^\alpha\nu^\beta$. Contracting (8) with $\varepsilon^j\nu^k$ and using (9) we arrive at

$$\lambda_{i,j} = \tau F_{ij} + \Lambda_i M_j + a_{\alpha\beta}M^{\alpha\beta}_{3|ij} \tag{10}$$

where Λ_i is a vector. Substituting (10) into (8) we obtain

$$\Lambda_i(M_kF_{lj} - M_lF_{kj}) + \Lambda_j(M_kF_{li} - M_lF_{ki}) = a_{\alpha\beta}M^{\alpha\beta}_{4|ijkl}. \tag{11}$$

Contracting (11) with $\nu^i\nu^j\nu^k\varepsilon^l$ we get $\nu^\alpha\Lambda_\alpha = a_{\alpha\beta}M^{\alpha\beta}_{5|}$, and contracting (11) with $\nu^j\nu^k\varepsilon^l$ we get $\Lambda_i = a_{\alpha\beta}M^{\alpha\beta}_{6|i}$. Then (10) assumes the form (4b). Integrability conditions of (4b) are

$$F_{ij}\tau_{,k} - F_{ik}\tau_{,j} = a_{\alpha\beta}M^{\alpha\beta}_{7|ijk} + \lambda_\alpha M^{\alpha\beta}_{8|ijk}. \tag{12}$$

Contracting (11) with $\varepsilon^i\varepsilon^j\nu^k$ we get $\tau_{,a}\varepsilon^\alpha = a_{\alpha\beta}M^{\alpha\beta}_{9|} + \lambda_\alpha M^{\alpha\beta}_{10|}$. Then contracting (11) with $\varepsilon^j\nu^i$ we get finally (4c). The theorem is proved.

The system (4) has at most one solution for the initial values in a point x_o: $a_{ij}(x_o)$, $\lambda_i(x_o)$ and $\tau(x_o)$. Hence, the general solution of this system depends on no more than $(n+2)(n+1)/2 - m(n-m+1)$ essential parameters.

The integrability conditions of the system (4) and their differential prolongations are linear algebraic equations on the components of the unknown tensors a_{ij}, λ_i and τ with coefficients from $K_n^{o(m)}$.

Thus, one can, in principle, solve the problem whether a given m-parabolically-Kählerian space $K_n^{o(m)}$ admits or not a holomorphically-projective mapping and its uniqueness.

3. Holomorphically-projective mappings of Ricci-recurrent spaces

As in the theory of the holomorphically-projective mappings of the elliptical and hyperbolical Kählerian spaces, there exist generally-recurrent m-parabolical Kählerian spaces, which do not admit these mappings.

Recall that a space is *Ricci-recurrent* if its Ricci tensor satisfies the conditions

$$R_{ij,k} = \varphi_k R_{ij}$$

where φ_k is a vector.

Theorem 2. *A Ricci-recurrent m-parabolically Kählerian space $K_n^{o(m)}$ whose Ricci tensor satisfies the condition*

$$R_{i\alpha}F_j^\alpha \neq \rho F_{ij},$$

does not admit non-trivial holomorphically-projective mappings.

Proof. Let a Ricci-recurrent spaces $K_n^{o(m)}$ admits non-trivial holomorphically-projective mappings. Hence in $K_n^{o(m)}$ there exists a solution of equation (2) satisfying $\lambda_i \not\equiv 0$, and formulas (6) hold. Alternating (6) we obtain

$$2\nu F_{ij} + a_{\alpha i}R_j^\alpha - a_{\alpha j}R_i^\alpha = 0. \tag{13}$$

Differentiate covariantly (13) with respect to x^k. On the basis if (1), (2) and (13) we get

$$\begin{aligned} v_k F_{ij} + \bar\lambda_i R_{jk} + \bar\lambda_\alpha R_j^\alpha g_{ik} - \bar\lambda_j R_{ik} - \bar\lambda_\alpha R_i^\alpha g_{jk} + \\ + \lambda_i R_j^\alpha F_{\alpha k} + \lambda_\alpha R_j^\alpha F_{ik} - \lambda_j R_i^\alpha F_{\alpha k} - \lambda_\alpha R_i^\alpha F_{jk} = 0 \end{aligned} \tag{14}$$

where v_k is a vector. Contracting (14) with F_l^i we get

$$\bar\lambda_\alpha R_j^\alpha F_{lk} + \bar\lambda_j F_l^\alpha R_{\alpha k} + \bar\lambda_\alpha R_l^\alpha F_{jk} + \bar\lambda_l F_j^\alpha R_{\alpha k} = 0. \tag{15}$$

It follows that $\bar\lambda_\alpha R_i^\alpha = \alpha\bar\lambda_i$ or $F_i^\alpha R_{\alpha j} = \rho F_{ij}$. But then it follows from (15) that $\bar\lambda_i = 0$ and (14) reduces to

$$v_k F_{ij} + \lambda_i R_j^\alpha F_{\alpha k} + \lambda_\alpha R_j^\alpha F_{ik} - \lambda_j R_i^\alpha F_{\alpha k} - \lambda_\alpha R_i^\alpha F_{jk} = 0. \tag{16}$$

Symmetrizing (16) with respect to j and k we get

$$F_{ij}a_k + F_{ik}a_j - R_i^\alpha F_{\alpha j}\lambda_k - R_i^\alpha F_{\alpha k}\lambda_j = 0$$

where $a_k \equiv 2v_k + \lambda_\alpha R_k^\alpha$. Analogously, $a_i = \lambda_i = 0$.

Similarly, the following theorem is proved.

Theorem 3. *If a Ricci-recurrent space $K_n^{o(m)}$ $(m > 2)$ satisfying $Rg\|R_{ij} - R/n\, g_{ij}\| \geq 1$ admits nontrivial holomorphically-projective mappings, then the following conditions hold*

$$R_{i\alpha}F_j^\alpha = \rho F_{ij} \quad \text{and} \quad \bar\lambda_i = 0.$$

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