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LIOUVILLE AND JACOBI THEOREMS FOR VECTOR DISTRIBUTIONS[1]

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Abstract. The paper contains a generalization of the Liouville and Jacobi integration methods, known from the classical calculus of variations, to the case of characteristic distributions of closed 2-forms of constant rank on smooth manifolds. It is shown that the results can be used to find integral manifolds of completely integrable distributions on smooth manifolds. In this way, the theorems of Liouville and Jacobi are generalized to a wide class of generally higher order ordinary and partial differential equations, possibly degenerate, which are not supposed to come from a Lagrangian.

Keywords: completely integrable distribution, closed 2-form, characteristic distribution of a closed 2-form, Liouville theorem, Jacobi theorem, symmetries of differential equations, integration of differential equations, degenerate equations, non-variational equations, connections on fibered manifolds.

MS classification: 58A30, 34A05, 34A08, 70H10, 70H20.

1. Introduction

The classical calculus of variations provides us with powerful integration methods for regular second order ordinary differential equations which are variational, i.e. are the equations for extremals of a certain regular Lagrangian. It is, in particular, the *Liouville theorem*, saying that once one knows "a half" of first integrals "in involution", remaining ones can be computed by "quadratures". Another one is the *Jacobi theorem*, according to which one can find first integrals providing that a "complete integral" of the Hamilton-Jacobi equation is known.

A generalized version of the Liouville theorem, which applies to semiregular variational problems of higher-order mechanics has been proved in [4]. A more general version, Liouville theorem for closed 2-forms of constant rank, which applies to any distribution which can be represented as the characteristic distribution of a closed 2-form of constant rank, has been proved in [9]. Some possible generalizations of the Jacobi integration method for the case of closed 2-forms of maximal rank have been studied in [1, 11]. For the case of (possibly degenerate) higher order ordinary variational equations the Jacobi theorem has been generalized in [9].

The purpose of this paper is to propose integration techniques for characteristic distributions of closed 2-forms of constant rank on smooth manifolds, based on a general idea of the Liouville integration method: to use a knowledge of a part of *first integrals* of the given equations (or equivalently, a knowledge of a part of *symmetries* of the associated closed 2-form) for finding adapted charts to the distribution in question. Another aim is to understand properly the geometry "behind" these integration techniques.

The plan of the paper is as follows: in Sections 2 and 3 it is shown that the Liouville theorem and the Liouville and Jacobi integration methods can be generalized in such a

[1]This paper is in final form and no version of it will be submitted for publication elsewhere.

form which does not need the assumption of variationality (i.e. the existence of a Lagrangian). In Section 2 we study the geometric background of the Liouville integration method. Among others, the geometric nature of such notions, as e.g. that of a "complete integral", or "first integrals in involution" (commonly used in classical mechanics), becomes transparent. This part of the paper is a more detailed exposition of ideas and results contained in [9, Sect.4.1]. Section 3 contains a geometric look on the Jacobi integration method, and a generalization of the classical Jacobi theorem to an arbitrary distribution which is a characteristic distribution of a closed 2-form. It also provides us with a new understanding of the Jacobi theorem as a special case of the Liouville theorem. In comparison to [9], we present a general theory without any reference to the calculus of variations.

The integration methods introduced in this paper can be applied to any completely integrable distribution on a manifold, or on a jet-prolongation of a fibered manifold, resp. to any integrable system of ordinary or partial differential equations (possibly of higher order), which arises as a system for integral mappings of a vector distribution. These applications are briefly discussed in Sect. 4. In particular, the family of equations covered by this theory contains (1) semisprays on jet prolongations of fibered manifolds ("higher-order vector fields"), i.e. higher-order ordinary and partial differential equations "explicitly solved with respect to the highest derivatives", and not necessarily variational, or (2) degenerate equations of variational higher-order mechanics (cf. [9]).

We work in the category of smooth finite dimensional manifolds. We denote by d, i, ∂, and * the exterior derivative, the contraction, the Lie derivative, and the pullback, respectively. The summation convention is used as far as possible.

Throughout the paper, by a *distribution* of rank r on a manifold M we mean a mapping which to each point x of M assigns an r-dimensional subspace of T_xM. If a distribution Δ is spanned by r (smooth, linearly independent) vector fields $\xi_1, \cdots, \xi_r$, or equivalently by $k = \dim M - r$ (smooth, linearly independent) 1-forms $\omega_1, \cdots, \omega_k$, we write

$$\Delta = \mathrm{span}\{\xi_1, \cdots, \xi_r\} = \mathrm{span}\{\omega_1, \cdots, \omega_k\}.$$

For a vector field ξ (resp. a 1-form ω) the notation $\xi \in \Delta$ (resp. $\omega \in \Delta$) means that ξ (resp. ω) belongs to Δ, i.e. it can be expressed by means of generators of Δ.

2. Liouville theorem for closed 2-forms

In what follows, M is a smooth manifold of dimension n, and α is a *closed* 2*-form of constant rank* $= 2p$ on M.

Denote by $\mathcal{D}$ the *characteristic distribution* of α. Recall that this distribution is spanned by the kernel of *alpha* (consisting of all vector fields ζ on M such that $i_\zeta \alpha = 0$), or equivalently, by the associated system of α (consisting of the 1-forms $i_\xi \alpha$ where ξ runs over the set of all vector fields on M). It holds rank α = corank $\mathcal{D}$, and by a well-known Cartan theorem, $\mathcal{D}$ is completely integrable.

Let Δ be a distribution on M. Recall that a function f defined on an open set $U \subset M$ is called a *first integral* of Δ if the form df belongs to the set of the generators of Δ on U. Suppose Δ be completely integrable and of constant rank on M. A set of functions $\{f_i, 1 \leq i \leq \text{corank } \Delta\}$ on U is called a *complete set of independent first integrals* of Δ if the 1-forms df_i are linearly independent and span Δ on U. The existence of a complete set of independent first integrals of Δ in a neighbourhood of each point of M is ensured by Frobenius theorem (existence of adapted charts to the foliation defined by Δ). Conversely, every given complete set of independent first integrals of Δ can be completed in an obvious way to an adapted chart to Δ.

If, in particular, we take for Δ the characteristic distribution $\mathcal{D}$ of a closed 2-form α on M, we get important relations between first integrals of $\mathcal{D}$ and *symmetries* of α (i.e. vector fields ξ such that $\partial_\xi \alpha = 0$), which can be used to simplify the integration procedure. (In this context it should be noted that in classical mechanics, where a particular case of a closed 2-form arises, namely, α is the differential of a Cartan form, or more precisely, a Lepagean 2-form, the symmetries of α are called *Noetherian symmetries*).

Proposition 1. *Let α be a closed 2-form on M.*

(1) Let ξ be a symmetry of α. Then $di_\xi \alpha = 0$. Hence, locally, $i_\xi \alpha = df$ where f is a first integral of the characteristic distribution $\mathcal{D}$ of α.

(2) Let ρ be a 1-form such that $\alpha = d\rho$. Let ξ be a symmetry of ρ. Then ξ is a symmetry of α, and $i_\xi \rho$ is a first integral of the characteristic distribution $\mathcal{D}$ of α.

(3) Let f be a first integral of the characteristic distribution of α. Then there exists a class $\{\xi\}$ of symmetries of α such that $i_\xi \alpha = df$ for each $\xi \in \{\xi\}$. This class is of the form $\xi_0 + \zeta$ where ξ_0 is a symmetry of α such that $i_{\xi_0} \alpha = df$, and ζ runs over the set of the generators of the characteristic distribution of α.

(4) Let ξ be a symmetry of α, let ρ be defined on an open set $U \subset M$, and such that $\alpha = d\rho$ on U. Then in a neighbourhood of each point $x \in U$ there exists a symmetry $\overline{\xi}$ of ρ (and hence of α) such that $i_{\overline{\xi}} \rho = 0$. If $k \geq 1$ is a fixed integer and $\zeta_1, \ldots, \zeta_k$ are generators of the characteristic distribution $\mathcal{D}$ of α, defined in a neighbourhood of x, and such that $i_{\zeta_j} \rho \neq 0, 1 \leq j \leq k$, then

$$\overline{\xi} =]\xi - \frac{1}{k} \sum_{j=1}^{k} \frac{i_\xi \rho}{i_{\zeta_j} \rho} \zeta_j \tag{2.1}$$

is a required symmetry.

(5) The set of all symmetries of α is a Lie algebra. If f_1, f_2 are first integrals of the characteristic distribution $\mathcal{D}$ of α, and ξ_1, ξ_2 are some corresponding symmetries of α, then

$$g \equiv \{f_1, f_2\} = i_{\xi_1} i_{\xi_2} \alpha = i_{\xi_1} df_2 \tag{2.2}$$

is another first integral of $\mathcal{D}$, corresponding to the symmetry $[\xi_1, \xi_2]$ of α.

Proof is an easy exercise and is left to the reader.

The first integral $\{f_1, f_2\}$ defined by (2.2) is called the *Poisson bracket* of the first integrals f_1, f_2.

Note that if ξ is a symmetry of α and ζ is a generator of the characteristic distribution $\mathcal{D}$ of α then $[\xi, \zeta] = 0$. This means that knowing $\mathcal{D}$ and an involutive set of symmetries $\xi_1, \ldots, \xi_k$ of α we get a completely integrable distribution $\Delta = \operatorname{span}\{\xi_K, \zeta_L, 1 \leq K \leq k, 1 \leq L \leq \operatorname{rank} \mathcal{D}\}$, whose subdistribution is $\mathcal{D}$. There is a very important particular case when the number k of involutive symmetries of α is equal to $p = (1/2) \operatorname{rank} \alpha$. In this case, as we shall see below, Δ can be obtained directly (without the explicit knowledge of $\mathcal{D}$), and $\mathcal{D}$ can be solved easily; this will be subject of the generalized Liouville theorem.

Let Δ be a distribution defined on an open subset U of M, let α be a closed 2-form of constant rank $= 2p$ on M. Δ will be called a *complete integral* of α on U if (1) Δ is completely integrable and corank $\Delta = p$, and (2) α belongs to the differential ideal generated by Δ [9] (cf. also [2]). Note that the second condition means that $\alpha = 0$ on the maximal integral manifolds of Δ.

The existence of local complete integrals is ensured by Darboux theorem.

It is clear that the characteristic distribution $\mathcal{D}$ of α is a subdistribution of every complete integral of α. Hence, first integrals of Δ are first integrals of $\mathcal{D}$, and by Proposition 1 they are in correspondence with symmetries of α. Namely, if Δ is a complete integral and $(x^J, a^K), 1 \leq J \leq \operatorname{rank} \Delta, 1 \leq K \leq p = \operatorname{corank}\Delta$, is an adapted chart to Δ at a point $x \in M$, i.e. $\Delta = \operatorname{span}\{da^K, 1 \leq K \leq p\}$ in a neighbourhood of x, then the functions $a^K, 1 \leq K \leq p$, are independent first integrals of $\mathcal{D}$, and the vector fields $\partial/\partial a^k, 1 \leq K \leq p$, are symmetries of α.

There arises a question about the converse: under what conditions a family of symmetries of α, resp. a family of first integrals of α defines a complete integral of α? This question is solved by the following proposition.

Proposition 2. *Let α be a closed 2-form of rank $2p$ on M. Let $a^K, 1 \leq K \leq p$, be independent first integrals of the characteristic distribution $\mathcal{D}$ of α. Let $\xi_K, 1 \leq K \leq p$, be symmetries of α such that $i_{\xi_K} = da^K, 1 \leq K \leq p$. The following conditions are equivalent:*

(1) *The distribution $\Delta = \operatorname{span}\{da^k, 1 \leq K \leq p\} = \operatorname{span}\{\xi_K, \zeta_L, 1 \leq K \leq p, 1 \leq L \leq \operatorname{rank}\mathcal{D}\}$, where ζ_L are independent generators of $\mathcal{D}$, is a complete integral of α.*

(2) *$i_{\xi_K} da^L = i_{\xi_K} i_{\xi_L} \alpha = 0$ for all $1 \leq K, L \leq p$.*

(3) *The vector fields $\xi_K, 1 \leq K \leq p$, are involutive, i.e. $[\xi_K, \xi_L] = 0$, for all $1 \leq K$, $L \leq p$.*

Proposition 2 can be obviously reformulated in the following way:

Let α be a closed 2-form of rank $2p$ on M. Let $\xi_K, 1 \leq K \leq p$, be symmetries of α, defined on an open subset of M and independent at each point of their domain of definition. The following conditions are equivalent:

(1) *The distribution $\Delta = \operatorname{span}\{i_{\xi_K}\alpha, 1 \leq K \leq p\}$ is a complete integral of α,*

(2) *$i_{\xi_K} i_{\xi_L} \alpha = 0$ for all $1 \leq K, L \leq p$,*

(3) *$[\xi_K, \xi_L] = 0$ for all $1 \leq K, L \leq p$.*

Proof of Proposition 2. The equivalence of (1) and (2) has been proved in [9]. (1)$\Rightarrow$(3) is trivial. (2) follows from (3) by the assertion (4) of Proposition 1, if we use the formula $i_{[\bar\xi_K, \bar\xi_L]}\rho = \partial_{\bar\xi_K} i_{\bar\xi_L}\rho - i_{\bar\xi_L}\partial_{\bar\xi_K}\rho = i_{\bar\xi_K} i_{\bar\xi_L}\alpha = i_{\xi_K} i_{\xi_L}\alpha$ (in the notation of Proposition 1(4)).

If any of the equivalent conditions of Proposition 2 is satisfied we shall say that the independent first integrals $a^K, 1 \leq K \leq p$, are *in involution* (cf. [9],[4]).

The following theorem, which is a generalization of the classical Liouville theorem from the calculus of variations to closed 2-forms of constant rank, is of fundamental importance for integration of characteristic distributions of closed 2-forms [9].

Theorem 1. *Let Δ be a complete integral of a closed 2-form α of rank $2p$ on M. Then at each point of* $\operatorname{dom}\Delta$ *there exists a chart $(U, \varphi), \varphi = (a^K, b_K, y^S), 1 \leq K \leq p, 1 \leq S \leq \dim M - 2p$, such that*

(1) $\Delta = \operatorname{span}\{da^K, 1 \le K \le p\}$,

(2) $\alpha = da^K \wedge db_K$,

(3) *the set of functions* $\{a^K, b_K, 1 \le K \le p\}$ *is a complete set of independent first integrals of the characteristic distribution of* α.

Proof [9]. Let $(W, \chi), \chi = (x^J, a^K), 1 \le J \le \dim M - p, 1 \le K \le p$, be a chart adapted to Δ at a point of M. Then in this chart

$$\alpha = (g_{JK} dx^J + h_{LK} da^L) \wedge da^K, \tag{2.3}$$

where we can set $h_{LK} = -h_{KL}$. Let x be a point in W. Since α is closed, there exist a neighbourhood of x, and a form ρ defined on this neighbourhood, such that $\alpha = d\rho$. Denote

$$\rho = \rho_J dx^J + \hat{\rho}_K da^K. \tag{2.4}$$

Then it holds

$$\frac{\partial \rho_I}{\partial x^J} - \frac{\partial \rho_J}{\partial x^I} = 0, \; \frac{\partial \hat{\rho}_K}{\partial x^J} - \frac{\partial \rho_J}{\partial a^K} = g_{JK}, \; \frac{\partial \hat{\rho}_K}{\partial a^L} - \frac{\partial \hat{\rho}_L}{\partial a^K} = 2h_{KL}, \tag{2.5}$$

where $1 \le I, J \le \dim M - p, 1 \le K, L \le p$, and we conclude that

$$\rho_J = \frac{\partial f}{\partial x^J} \tag{2.6}$$

for a function $f(x^J, a^K)$ defined in a neighbourhood of x. Hence, we have shown that to each point of W there exist a neighbourhood $U \subset W$ and functions $\hat{\rho}_K, 1 \le K \le p$, and f on U such that $\alpha = d\rho$, where

$$\rho = \frac{\partial f}{\partial x^J} dx^J + \hat{\rho}_K da^K = df + \left(\hat{\rho}_K - \frac{\partial f}{\partial a^K} \right) da^K. \tag{2.7}$$

Put

$$b_K = \frac{\partial f}{\partial a^K} - \hat{\rho}_K. \tag{2.8}$$

Since $\partial b_K / \partial x^J = -g_{JK}$ and the matrix (g_{JK}) has maximal rank, there can be found functions $y^S, 1 \le S \le$ *dim* $M - 2p$, on U such that the mapping $(x^J, a^K) \to (a^K, b_K, y^S)$ is a coordinate transformation. Finally, from the expression of α in the chart $(U, \varphi), \varphi = (a^K, b_K, y^S)$ we see that the characteristic distribution of α is spanned by the forms $da^K, db_K, 1 \le K \le p$, which are linearly independent at each point of U. This proves that $\{a^K, b_K, 1 \le K \le p\}$ is a complete set of independent first integrals of the characteristic distribution of α.

The following corollary is a reformulation of Theorem 1 in a way similar to the classical Liouville theorem, and could be referred to as "*generalized Liouville integration method*":

Corollary 1. *Let* α *be a closed* 2*-form of constant rank* $2p$ *on* M. *Let* $a^K, 1 \le K \le p$, *be independent first integrals in involution of the characteristic distribution* $\mathcal{D}$ *of* α, *defined on an open subset of* M. *Then the set of functions* $\{a^K, b_K, 1 \le K \le p\}$, *where* b_K *are defined by* (2.8) *is a complete set of independent first integrals of* $\mathcal{D}$.

Note that the explicit formulas for ρ_J, $\hat{\rho}_K$, and f can be easily found with help of the Poincaré lemma. Hence the first integrals b_K can be found explicitly by means of differentiation and integration ("by quadratures"). For example, if we have found a form ρ such that $\alpha = d\rho$, we can take f in the form

$$f = x^J \int_0^1 (\rho_J \circ \mu) dv + h(a^K), \tag{2.9}$$

where ρ_J are defined by $\rho = \rho_J dx^J + \hat{\rho}_K da^K$, the mapping $\mu : [0,1] \times U \to U$ is defined by

$$\mu(v, (x^J, a^L)) = (vx^J, a^L), \tag{2.10}$$

and $h(a^K)$ is an arbitrary (but fixed) function on U (note that $\partial h / \partial x^J = 0$, i.e. h is a first integral of the distribution Δ).

Another possible reformulation of Theorem 1 is the following "coordinate-free Liouville theorem", providing a geometric look on the integration procedure described by Theorem 1.

Corollary 2. *Let Δ_1 be a complete integral of a closed 2-form α of constant rank $2p$ on M. Then to each point of* dom Δ_1 *there exists a neighbourhood U and a complete integral Δ_2 of α on U such that $\Delta_1 \cap \Delta_2$ is the characteristic distribution of α on U.*

There is another possibility than that described by Corollary 1 how to use Theorem 1 to integrate the characteristic distribution of α. Consulting the proof of Theorem 1 we get

Corollary 3. *Let Δ be a complete integral of a closed 2-form α of constant rank $2p$ on M. Then to each point x of* dom Δ *there exists a neighbourhood W and a 1-form $\bar{\rho}$ on W such that $\alpha = d\bar{\rho}$, and $\bar{\rho}$ belongs to Δ (i.e. $i_\xi \bar{\rho} = 0$ for all vector fields spanning Δ on U).*

Hence, once we know ρ (defined on an open subset $U \subset M$) such that $\alpha = d\rho$ on U, we are able to find a form $\bar{\rho}$ (on a certain open subset of U) such that $\alpha = d\bar{\rho}$ and $\bar{\rho} \in \Delta$. This is equivalent to the problem of finding a function f satisfying

$$\rho - df \in \Delta\,. \tag{2.11}$$

In every chart $(x^J, a^K), 1 \le J \le \operatorname{rank} \Delta, 1 \le K \le \operatorname{corank} \Delta$, adapted to Δ (i.e. such that $\Delta = \operatorname{span}\{da^K, 1 \le K \le \operatorname{corank} \Delta\}$) the condition (2.11) reads

$$\bar{\rho} = \rho - df = b_K da^K, \tag{2.12}$$

i.e., expressing ρ in the form $\rho = \rho_J dx^J + \hat{\rho}_K da^K$, we get the equations (2.6) and (2.8). Now, obviously, the functions $b_K = i_{\partial/\partial a^K} \bar{\rho}, 1 \le K \le p$, are independent first integrals of the characteristic distribution $\mathcal{D}$ of α. The above described integration method could be referred to as "*generalized Jacobi integration method*".

Remarks. (1) The problem of finding a one-form ρ such that $\alpha = d\bar{\rho}$ and $\bar{\rho}$ vanishes on the maximal integral submanifolds of Δ has been pointed out in [1] where the condition (2.11) has been introduced as a modification of the Hamilton-Jacobi equation.

(2) The Jacobi integration method for characteristic distributions of closed 2-forms can be explained in terms of a pair of complementary distributions. To this purpose notice that if Δ is a complete integral of α and Δ^c is a distribution complementary to Δ, then at each point x of the common domain of definition of Δ and Δ^c there exists a chart (x^J, a^K) adapted to both Δ and Δ^c, i.e. such that

$$\Delta = \text{span}\{da^K\} = \text{span}\{\frac{\partial}{\partial x^J}\}, \quad \Delta^c = \text{span}\{dx^J\} = \text{span}\{\frac{\partial}{\partial a^K}\}, \tag{2.13}$$

where $1 \leq K \leq \text{corank}\Delta, 1 \leq J \leq \text{rank}\Delta$. Conversely, if Δ is a complete integral of α and (U, φ), $\varphi = (x^J, a^K)$ is a chart adapted to Δ such that $\Delta = \text{span}\{da^K\}$, then by $\Delta^c = \text{span}\{\partial/\partial a^K\} = \text{span}\{dx^J\}$ a complementary distribution Δ^c to Δ is defined on U. Now, suppose we are given two complementary distributions Δ and Δ^c, defined on an open set $U \subset M$, and such that Δ is a complete integral of α. Since any 1-form ω on U can be uniquely decomposed into a sum $\omega = P\omega + P^c\omega$ where $P\omega$ belongs to Δ and $P^c\omega$ belongs to Δ^c, we get Corollary 3 in the following form:

Let Δ be a complete integral of α, and Δ^c be a distribution complementary to Δ, both Δ and Δ^c defined on an open set $U \subset M$. Then in a neighbourhood of each point of U there exists a 1-form $\bar{\rho}$ such that $\alpha = d\bar{\rho}$ and $P^c\bar{\rho} = 0$.

Hence, if ρ is a 1-form such that $\alpha = d\rho$, then "locally" there exists a function f such that

$$P^c\rho = P^c df; \tag{2.14}$$

$\bar{\rho}$ is then given by $\bar{\rho} = P\bar{\rho} = P\rho - Pdf$. Equation (2.14) covers two special cases discussed for the case rank α = maximal in [11]; namely (1) Δ and Δ^c are such that to a given 1-form ρ there exists f satisfying $P^c\rho = df$ (cf. the "Noether splitting" in [11]), or (2) Δ and Δ^c are such that to a given 1-form ρ there exists f satisfying $\rho = P^c df$ (cf. the "Jacobi splitting" in [11]).

We shall use a slightly different description of geometric structures connected with complete integrals than that mentioned in the above remark; this will be subject of the following section.

3. Jacobi theorem for closed 2-forms

In this section we shall show that Corollary 3 to Theorem 1 can be proved directly if one makes use of fibered-manifolds techniques. This approach will also provide us with a possibility to simplify (in a certain situation) the Jacobi integration method.

To this purpose we shall need some basic concepts from the theory of jets and connections on fibered manifolds (see e.g. [12] and [10]). For the reader's convenience we recall briefly some definitions and assertions from the calculus of horizontal and contact forms on fibered manifolds. For more details we refer to [5], [6].

Let $\pi : Y \to X$ be a fibered manifold, $\dim X = n$, $\dim Y = n + m$. Denote by $\pi_1 : J^1Y \to X$ the first jet prolongation of π (π_1, as well as the canonical projection $\pi_{1,0} : J^1Y \to Y$, is a fibered manifold). Points of J^1Y (i.e. 1-jets of local sections γ of π) will be denoted by $J^1_x\gamma$.

A vector field ξ on Y is called *π-vertical* if $T\pi.\xi = 0$.

A q-form ρ on Y is called *π-horizontal* if $i_\xi \rho = 0$ for each π-vertical vector field ξ on Y. ρ is called *π-projectable* if there exists a form ρ_0 on X such that $\pi^* \rho_0 = \rho$; (if ρ_0 exists it is unique, and is called the *π-projection* of ρ).

Let ρ be a q-form on Y, $q \geq 1$. For every $J^1_x \gamma \in J^1 Y$ and any vectors $\xi_1, ..., \xi_q \in TJ^1Y$ at the point $J^1_x \gamma$ we set

$$h\rho(J^1_x\gamma)(\xi_1, ..., \xi_q) = \rho(\gamma(x))(T_x\gamma.T\pi_1.\xi_1, ..., T_x\gamma.T\pi_1.\xi_p). \tag{3.1}$$

Evidently, $h\rho$ is a π_1-horizontal q-form on J^1Y. The mapping $h : \rho \to h\rho$ is called the *π-horizontalization.* It has the following properties:

(1) *for any q-forms ρ, ω and any p-form η*

$$h(\rho + \omega) = h\rho + h\omega,\ h(\rho \wedge \eta) = h\rho \wedge h\eta;$$

(2) *if $q > n$ then $h\rho = 0$;*

(3) *if ρ is π-horizontal then $h\rho$ is $\pi_{1,0}$-projectable* (if there is no danger of confusion we usually write $\rho = h\rho$ in this case);

(4) $hh\rho = h\rho$.

A q-form ρ on Y (or, possibly on some r-jet prolongation J^rY of π) is called *π-contact* if $h\rho = 0$. For any q-form ρ on Y the form $\pi^*_{1,0}\rho$ on J^1Y has the unique decomposition

$$\pi^*_{1,0}\rho = h\rho + p\rho. \tag{3.2}$$

Clearly, $p\rho$ is a π-contact form. The mapping p assigning to a q-form ρ on Y the π-contact q-form $p\rho$ on J^1Y is called the *contactization* (with respect to π). Note that p does not preserve the wedge product.

Let ρ be a π-contact form on Y (i.e. let $\pi^*_{1,0}\rho = p\rho$). We say that ρ is *1-contact* if for each π_1-vertical vector field ξ on J^1Y the form $i_\xi(\pi^*_{1,0}\rho)$ is π_1-horizontal. We say that ρ is *k-contact* $(1 < k \leq q)$ if $i_\xi(\pi^*_{1,0}\rho)$ is $(k-1)$-contact.

We shall recall two assertions which are of fundamental importance in the calculus on fibered manifolds:

Decomposition theorem [5], [6]. *Let ρ be a q-form on Y. Then $\pi^*_{1,0}\rho$ has the unique decomposition*

$$\pi^*_{1,0}\rho = h\rho + p_1\rho + \ldots + p_q\rho, \tag{3.3}$$

where $p_i\rho, 1 \leq i \leq q$, is a i-contact q-form on J^1Y.

For each i, $1 \leq i \leq q$, the mapping $p_i : \rho \to p_i\rho$ defined by (3.3) is called the *i-contactization* (with respect to the projection π).

Let ρ be a π-contact q-form on Y. Let k be a fixed integer, $1 \leq k \leq q$. We say that *the order of contactness of ρ is equal to k* if $p_i\rho = 0$ for all $i > k$.

Poincaré lemma for contact forms [5], [6]. *Let ρ be a q-form on Y, suppose that $d\rho = 0$ and the order of contactness of ρ is equal to $k \geq 1$. Then to each point of Y there exists its neighbourhood W and a $(q-1)$-form η on W such that*

(1) *the order of contactness of η is equal to $k-1$,*

(2) *$d\eta = 0$ on W.*

Now, let us return to closed 2-forms.

Let α be a closed 2-form of rank $2p$ on M, let Δ be a complete integral of α, defined on an open set $U \subset M$. Denote by $\mathcal{F}$ the foliation of U defined by Δ. Lowering U if necessary, we have a (global) submersion $\pi^{\mathcal{F}} : U \to U/\mathcal{F}$ of U onto the quotient-manifold $U/\mathcal{F}$ of U by the foliation $\mathcal{F}$ (recall that the equivalence relation in U is the following: $x_1, x_2 \in U$ are equivalent iff both they belong to the same leaf of $\mathcal{F}$).

In correspondence with the above notations let us denote by $\pi_1^{\mathcal{F}}$ the first jet prolongation of $\pi^{\mathcal{F}}$, and by $h^{\mathcal{F}}, p_1^{\mathcal{F}}$ and $p_2^{\mathcal{F}}$ the horizontalization, 1-contactization and 2-contactization with respect to $\pi^{\mathcal{F}}$, respectively. Just from the definition of $\pi^{\mathcal{F}}$ we recognize that the order of contactness of α is equal to 1, i.e. it holds $p_2^{\mathcal{F}}\alpha = 0$. Since α is closed, we can apply the Poincaré lemma for contact forms to get

Proposition 3. *To each point $x \in U$ there exist a neighbourhood W and $\pi^{\mathcal{F}}$-horizontal 1-form $\bar{\rho}$ on W such that $\alpha = d\bar{\rho}$ on W.*

Note that the condition that $\bar{\rho}$ is $\pi^{\mathcal{F}}$-horizontal means $h^{\mathcal{F}}\bar{\rho} = \bar{\rho}$ and $p_1^{\mathcal{F}}\bar{\rho} = 0$, i.e. Proposition 3 is nothing but a "coordinate free" reformulation of Corollary 3 to Theorem 1 in terms of the theory of fibered manifolds. Expressing the condition

$$\bar{\rho} = \rho - df = h^{\mathcal{F}}\bar{\rho} \tag{3.4}$$

in a fiber chart on $\pi^{\mathcal{F}}$ we get (2.6) and (2.8).

The generalized Jacobi integration method is then expressed in the following way:

Corollary. *Let α and $\bar{\rho}$ be as above, let $\mathcal{D}$ be the characteristic distribution of α,* $\operatorname{corank} \mathcal{D} = 2p$. *Then*

(1) *$\bar{\rho}$ is invariant with respect to $\mathcal{D}$, i.e. for each vector field ζ belonging to $\mathcal{D}$, $\partial_\zeta\bar{\rho} = 0$.*

(2) *In each fiber chart $(V, \psi), \psi = (a^K, x^J)$ on $\pi^{\mathcal{F}}$ it holds $\bar{\rho} = b_K da^K$ where b_K are first integrals of $\mathcal{D}$. The rank of the matrix $(\partial b_K/\partial x^J)$ is constant on V and equal to p (i.e. it is maximal).*

Proof. If ζ belongs to $\mathcal{D}$, it is $\pi^{\mathcal{F}}$-vertical (since $\mathcal{D} \subset \Delta$), and $i_\zeta\alpha = 0$. But $\bar{\rho}$ is $\pi^{\mathcal{F}}$-horizontal, i.e. $i_\zeta\bar{\rho} = 0$. Hence, $\partial_\zeta\bar{\rho} = i_\zeta d\bar{\rho} + di_\zeta\bar{\rho} = 0$, proving (1).

Let $(V, \psi), \psi = (x^J, a^K)$ be a fiber chart on U, associated with a chart $(W, \varphi), \varphi = (a^K)$ on $U/\mathcal{F}$. Since $\bar{\rho} = h^{\mathcal{F}}\bar{\rho}$, we have $\bar{\rho} = b_K da^K$ where b_K are functions on V. Let ζ belong to $\mathcal{D}$, and let $\zeta = \zeta^J\partial/\partial x^J$ be the chart expression of ζ (note that ζ is $\pi^{\mathcal{F}}$-vertical). Since $i_\zeta\alpha = 0$, we get $i_\zeta(db_K \wedge da^K) = (\partial_\zeta b_K)da^K = 0$, proving that $b_K, 1 \leq K \leq \dim U/\mathcal{F}$, are first integrals of $\mathcal{D}$. Since $\operatorname{rank}\alpha = 2p$ and $\alpha = db_K \wedge da^K$ on V, we get rank $(\partial b_K/\partial x^J) = \text{const.} = p$ on V.

Let Γ be a connection on $\pi^{\mathcal{F}}$, i.e. a section of the fibered manifold $\pi_{1,0}^{\mathcal{F}} : J^1\pi^{\mathcal{F}} \to U$. Suppose Γ be flat (which means that the associated $\pi^{\mathcal{F}}$-horizontal distribution on U, denoted also by Γ, is completely integrable). Denote by $\mathcal{G}$ the foliation defined by Γ. Lowering U if necessary, we can consider the quotient manifold $U/\mathcal{G}$ and the fibered manifold $\pi^{\mathcal{G}} : U \to U/\mathcal{G}$. (Notice that, on $\pi^{\mathcal{G}}, \mathcal{G}$ is the vertical bundle while $\mathcal{F}$ is a connection). Evidently, in general, the order of contactness of α with respect to the projection $\pi^{\mathcal{G}}$ is equal to 2.

Denote by $h^{\mathcal{G}}, p_1^{\mathcal{G}}$ and $p_2^{\mathcal{G}}$ the horizontalization, 1-contactization and 2-contactization with respect to $\pi^{\mathcal{G}}$, respectively.

A complete integral Δ of α on U will be called *canonically associated with a (flat) connection* Γ on $\pi^{\mathcal{F}}$ if $p_2^{\mathcal{G}}\alpha = 0$. Again, as a direct consequence of the Poincaré lemma for contact forms we obtain:

Proposition 4. *Let α, Δ and $\pi^{\mathcal{F}}$ be as above. If Δ is canonically associated with a flat connection Γ on $\pi^{\mathcal{F}}$, then to each point $x \in U$ there exist a neighbourhood W and a $\pi^{\mathcal{G}}$-horizontal form $\tilde{\rho}$ on W such that $\alpha = d\tilde{\rho}$ on W.*

Note that if Δ is a complete integral of α on U, and Γ is a completely integrable distribution on U, complementary to Δ, then Γ is a flat connection on $\pi^{\mathcal{F}}$. Further note that if $(V, \psi), \psi = (x^J, a^K)$ is a chart at a point $x \in U$, adapted to the complementary distributions Γ and Δ (i.e. such that $\Delta = \operatorname{span}\{da^K\}$ and $\Gamma = \operatorname{span}\{dx^J\}$, where $1 \leq K \leq \operatorname{corank} \Delta, 1 \leq J \leq \operatorname{rank} \Delta$), then (V, ψ) can be viewed as a fiber chart on $\pi^{\mathcal{F}}$ associated with a chart $(W_1, \varphi_1), \varphi_1 = (a^K)$ on $U/\mathcal{F}$, or as a fiber chart on $\pi^{\mathcal{G}}$ associated with a chart $(W_2, \varphi_2), \varphi_2 = (x^J)$ on $U/\mathcal{G}$, respectively.

Now, using Proposition 3 and 4 we get immediately a general version of the classical *Jacobi theorem.*

Theorem 4. *Let α be a closed 2-form of* rank $2p$ *on M. Let Δ be a complete integral of α, defined on an open set $U \subset M$, and denote by $\mathcal{F}$ the foliation of U by Δ. Let Γ be a distribution on U, completely integrable, and complementary to Δ; denote by $\mathcal{G}$ the foliation defined by Γ. Suppose that $\pi^{\mathcal{F}} : U \to U/\mathcal{F}$ and $\pi^{\mathcal{G}} : U \to U/\mathcal{G}$ are fibered manifolds, and that Δ is canonically associated with Γ. Then to each point $x \in U$ there exists a neighbourhood W and, on W*

(1) *a $\pi^{\mathcal{G}}$-horizontal 1-form $\tilde{\rho}$ such that $\alpha = d\tilde{\rho}$,*

(2) *a function f such that the 1-form $\bar{\rho} = h^{\mathcal{G}}\tilde{\rho} - df$ is $\pi^{\mathcal{F}}$-horizontal.*

If $(V, \psi), \psi = (x^J, a^K)$ is a chart at x adapted to Δ and Γ (i.e. $\Delta = \operatorname{span}\{da^K, 1 \leq K \leq \operatorname{corank}\Delta\}$, $\Gamma = \operatorname{span}\{dx^J, 1 \leq J \leq \operatorname{rank}\Delta\}$), then it holds $\tilde{\rho} = \tilde{\rho}_J dx^J$ where

$$\tilde{\rho}_J = \frac{\partial f}{\partial x^J}, \quad 1 \leq J \leq \operatorname{rank} \Delta \tag{3.5}$$

and the functions

$$b_K = \frac{\partial f}{\partial a^K}, \quad 1 \leq K \leq \operatorname{corank} \Delta \tag{3.6}$$

are first integrals of $\mathcal{D}$. The rank of the matrix

$$\left(\frac{\partial b_K}{\partial x^J}\right) = \left(\frac{\partial^2 f}{\partial x^J \partial a^K}\right) \tag{3.7}$$

is constant on V and equal to p.

Theorem 4 can be used to find local solutions of the characteristic distribution of a closed 2-form of constant rank, especially in case we know $\tilde{\rho}$. Then a function f satisfying (3.5) can be easily constructed and the "remaining" first integrals b_K can be found according to (3.6).

4. Applications to integration of differential equations

The Liouville and Jacobi integration methods can be used to find local solutions of characteristic distributions of closed 2-forms. The purpose of this section is to give a few examples of systems of ordinary and partial differential equations of order $r \geq 1$ which can be interpreted as equations for integral mappings of characteristic distributions of closed 2-forms, and hence integrated by means of the Liouville or Jacobi theorem.

4.1. Integration of vector distributions

Let M be a manifold, let $\mathcal{D}$ be a *completely integrable* distribution on M.

Proposition 5. (1) *Let* $\operatorname{corank} \mathcal{D}$ *be even and equal to* $2p$. *Then to each point* $x \in M$ *there exists a neighbourhood* W *and a closed* 2*-form* α *of rank* $2p$ *on* W *such that* $\mathcal{D}$ *is the characteristic distribution of* α.

(2) *Let* $\operatorname{corank} \mathcal{D}$ *be odd and equal to* $2p+1$. *Then to each point* $x \in M$ *there exists a neighbourhood* W *and a closed* 2*-form* α *of rank* $2p$ *on* W *such that* $\mathcal{D}$ *is a subdistribution of the characteristic distribution of* α.

Proof. (1) Frobenius theorem ensures the existence of an adapted chart (W, φ), $\varphi = (y^J, f_K)$, $1 \leq J \leq \operatorname{rank}\mathcal{D}$, $1 \leq K \leq 2p$, at x, such that f_K are first integrals of $\mathcal{D}$. Putting $\alpha = df_1 \wedge df_2 + \ldots + df_{2p-1} \wedge df_{2p}$ we get a desired closed 2-form on W.

(2) Let $f_1, \ldots, f_{2p+1}$ be independent first integrals of $\mathcal{D}$, defined in a neighbourhood W of x. Putting $\alpha = df_1 \wedge df_2 + \ldots + df_{2p-1} \wedge df_{2p}$ the proof is finished.

By the above proposition every involutive vector distribution $\mathcal{D}$ of constant rank on a manifold M can be associated with a (generally non-unique, locally defined) closed 2-form, i.e. it can be solved by means of either the Liouville or Jacobi integration method. If corank $\mathcal{D}$ is *even* then these methods provide us directly with a complete set of independent first integrals of $\mathcal{D}$. In case that corank $\mathcal{D}$ is *odd* we get in this way a foliation of dimension equal to rank $\mathcal{D} + 1$, whose subfoliation is $\mathcal{D}$; hence the integration problem for $\mathcal{D}$ is reduced to the problem of finding one additional first integral. In concrete practical situations, the problem of finding an appropriate 2-form α can lead to rather unpleasant computations. However, one has several possibilities how to proceed. For example, (1) one can solve the equations $i_{\zeta_K}\alpha = 0, 1 \leq K \leq \operatorname{rank} \mathcal{D}$, where ζ_K are independent generators of $\mathcal{D}$, with respect to a closed 2-form α of rank $2p$, or (2) one can solve the conditions $\partial_{\zeta_K}\rho = 0$, $di_{\zeta_K}\rho = 0$, $1 \leq K \leq \operatorname{rank} \mathcal{D}$, with respect to a 1-form ρ such that $\operatorname{rank} d\rho = 2p$, and put $\alpha = d\rho$, or (3) if Δ is spanned by the 1-forms $\omega_1, \cdots, \omega_{2p}$ (resp. $\omega_1, \cdots, \omega_{2p+1}$), to find functions a^{JK} such that the form $\alpha = a^{JK}\omega_J \wedge \omega_K$ is closed and of rank $2p$; in this case it is convenient to use the fact that the forms ω_K satisfy the Frobenius condition $d\omega_K = \eta_K^L \wedge \omega_L$, and moreover, that they can be completed to a basis of linear forms on M, and to perform the computations in this basis.

4.2. Distributions on fibered manifolds

All above applies to the case when $\mathcal{D}$ is a distribution defined on an r-jet prolongation $J^rY, r \geq 0$, of a fibered manifold $\pi : Y \to X, \dim X = n, \dim Y = m + n$. In this case obviously $M = J^r Y$. In most applications the distributions in question are *connections* [10], *semispray connections* [3], [13], or *generalized connections* [9] on fibered manifolds, and the problem is to search for integral sections of these distributions. Consequently, the Liouville and Jacobi integration methods can be used to find solutions of a wide class of systems of higher-order ordinary and partial differential equations.

4.3. Variational ordinary differential equations

It has been shown in [8] (see also [9], [7]) that a system of generally higher-order variational ordinary differential equations (i.e. arising from a possibly higher-order Lagrangian as its Euler-Lagrange equations) can be interpreted as equations for integral sections of a distribution $\mathcal{D}$ on a certain jet-prolongation of a fibered manifold $\pi : Y \to X$ (with the base X of dimension 1); in general, rank $\mathcal{D}$ is nonconstant and can be greater than one. The most important cases are those when $\mathcal{D}$ is an integrable distribution of constant rank, complementary to a subbundle Γ of the vertical bundle (i.e. when it is a generalized connection) [9]. Then obviously the integration problem for $\mathcal{D}$ can be solved by means of the Liouville or Jacobi integration method. However, it should be stressed that *due to the existence of Lagrangians the situation simplifies essentially*:

(1) the problem of finding an appropriate closed 2-form to $\mathcal{D}$ is solved *canonically* by the existence (and easy construction) of the so called *Lepagean 2-form* [9],

(2) for even-order equations, $\mathcal{D}$ is canonically associated with a subbundle of the vertical bundle, hence the Jacobi theorem can be applied easily to find the extremals.

For details on the integration methods arising from the Liouville and Jacobi theorems for (generally degenerate) higher-order variational equations we refer to [9].

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