Differential Geometry and Its Applications
Proc. Conf. Opava (Czechoslovakia), August 24–28, 1992
Silesian University, Opava, 1993, 47–50

ON COMPACT HOMOGENEOUS SPACES WITH NON–VANISHING MASSEY PRODUCTS

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Abstract. We construct explicitly examples of homogeneous spaces of compact simple Lie groups with non-vanishing Massey products.
Keywords. Homogeneous space, Massey product.
MS classification. 53C, 55P.

It is well-known that formality or non-formality (in the sense of Sullivan) of a manifold implies strong geometric consequences. For example, if a manifold carries a Kählerian structure, it must be formal [3]. In fact it is not easy to determine whether the given manifold is formal or not (see, e.g. [4]). It is known that if a topological space has non-zero Massey products, it is not formal. Therefore, it is an interesting question to describe various classes of manifolds with non-vanishing Massey products. The first example of that kind in the case of compact homogeneous nilmanifold was obtained by L. Cordero, M. Fernandez and A. Gray in [2]. Here we consider the "opposite" case of homogeneous spaces of compact simple Lie group.

Theorem. *There exist compact homogeneous spaces G/H of simple compact Lie groups G with non-vanishing Massey products. In particular, homogeneous space*

$$SU(n+1)/SU(3)\times SU(3), \quad n\geq 5$$

possesses non-vanishing Massey products.

Introduce the appropriate notations. As far as the author would like this note to be short, the reader is supposed to be familiar with the standard facts of cohomology theory of homogeneous spaces and rational homotopy theory. We shall use all the notations and facts in accordance with [1, 6] without additional explanations. Everywhere the symbol $W(G,T)$ denotes the Weyl group of a compact Lie group G with respect to its maximal torus T. For a subgroup $K\subset G$ we choose its maximal torus T' in such a way that $T'\subset T$. For any vector space V the symbol $k[V]$ denotes the polynomial algebra over V and $k[V]^\Gamma$ denotes the ring of Γ-invariants of the discrete group $\Gamma\leq GL(V)$. A free graded commutative algebra over a graded vector space V is denoted by $\Lambda(V)$ and the same notation is used for the exterior algebra over V (but the sense of the notation will be always clear from the context). Everywhere Lie algebras of Lie groups $G, H, \ldots$

[1]This paper is in final form and no version of it will be submitted for publication elsewhere

will be denoted by the corresponding Gothic letters $\mathfrak{G}, \mathfrak{H} \ldots$. We use also the notation $k[X_1, \ldots, X_n]$ for the polynomial ring of variables $X_1, \ldots, X_n$ over the field k.

Proof of the theorem. Consider, at first, the minimal case

$$M = SU(6)/SU(3) \times SU(3)$$

and calculate its Cartan algebra (C, δ). In our case (see [1] for details)

$$(C, \delta) = (\mathbb{R}[\mathfrak{t}']^{W(A_2 \times A_2)} \times \Lambda(y_1, \ldots, y_5), \delta), \quad \delta y_j = \tilde{f}_j = f_j \mid_{\mathfrak{t}'}, \ j = 1, \ldots, 5$$

where $\delta = 0$ on the first term of the tensor product, f_j and $\tilde{f}_j$ are defined by Chevalley isomorphisms

$$\mathbb{R}[\mathfrak{t}]^{W(A_5)} \cong \mathbb{R}[f_1, \ldots, f_5],$$
$$\mathbb{R}[\mathfrak{t}']^{W(A_2 \times A_2)} \cong \mathbb{R}[u_1, u_2, v_1, v_2].$$

Using the explicit expressions for f_j and u_i, v_i [5] one obtains

$$f_j(x_1, \ldots, x_6) = x_1^{j+1} + \cdots + x_6^{j+1}, \quad x_1 + \cdots + x_6 = 0, \quad j = 1, \ldots, 6,$$

$$u_i = x_1^{i+1} + x_2^{i+1} + x_3^{i+1}, \quad x_1 + x_2 + x_3 = 0, \quad i = 1, 2,$$

$$v_i = x_4^{i+1} + x_5^{i+1} + x_6^{i+1}, \quad x_4 + x_5 + x_6 = 0, \quad i = 1, 2.$$

Then, up to scalar multiplies, which are not important in our considerations

$$\tilde{f}_1 = u_1 + v_1, \quad \tilde{f}_2 = u_2 + v_2, \quad \tilde{f}_3 = u_1^2 + v_1^2,$$

$$\tilde{f}_4 = u_1 u_2 + v_1 v_2, \quad \tilde{f}_5 = u_1^3 + v_1^3 + u_2^2 + v_2^2$$

and finally

$$(C, \delta) = (\mathbb{R}[u_1, u_2, v_1, v_2] \otimes \Lambda(y_1, \ldots, y_5), \delta), \quad \delta u_i = \delta v_i = 0, \ i = 1, 2,$$

$$\delta y_1 = u_1 + v_1, \quad \delta y_2 = u_2 + v_2, \quad \delta y_3 = u_1^2 + v_1^2,$$
$$\delta y_4 = u_1 u_2 + v_1 v_2, \quad \delta y_5 = u_1^3 + v_1^3 + u_2^2 + v_2^2,$$

where the degrees of variables have values: $\deg(u_1) = 4$, $\deg(v_1) = 4$, $\deg(u_2) = \deg(v_2) = 6$, $\deg(y_1) = 3, \ldots, \deg(y_5) = 11$.

Observe that (C, δ) is not a minimal algebra (see [6] for the appropriate notions). Nevertheless, (C, δ) is a free graded commutative algebra and we can apply Sullivan's method for constructing the minimal model of (C, δ). In what follows we apply the calculations of [6] (Proposition V.7). Denote by V the linear span of $\{u_1, v_1, u_2, v_2, y_1, \ldots, y_5\}$:

$$V = L(u_1, u_2, v_1, v_2, y_1, \ldots, y_5)$$

and introduce a new derivation d', defined on V by the formula

$$d' : V^n \to V^{n+1}, \quad d' = \pi \circ \delta, \quad \pi : V^{n+1} \oplus L^{++}(V)^{n+1} \to V^{n+1},$$

($L^{++}(V)$ being the ideal of decomposable elements in $\Lambda(V)$, and π being the projection on the first summand). Then

$$d'u_i = d'v_i = 0 \ (i=1,2), \quad d'y_1 = u_1 + v_1, \quad d'y_2 = u_2 + v_2, \quad d'y_j = 0 \ (j>2).$$

Consider the direct sums

$$V = \operatorname{Im} d' \oplus V' \oplus W, \quad \operatorname{Im} d' \oplus V' = \operatorname{Ker} d'.$$

Then, obviously,

$$\operatorname{Ker} d' = L(u_1, u_2, v_1, v_2, y_3, y_4, y_5),$$
$$\operatorname{Im} d' = L(u_1 + v_1, u_2 + v_2).$$

Thus

$$L(u_1, u_2, v_1, v_2, y_3, y_4, y_5) = L(u_1+v_1, u_2+v_2) + L(v_1, v_2) + L(y_3, y_4, y_5),$$

which implies

$$V' = L(v_1, v_2) \oplus L(y_3, y_4, y_5), \quad W = L(y_1, y_2),$$
$$W' = d(W) = L(\delta y_1, \delta y_2) = L(u_1+v_1, u_2+v_2).$$

Consider the algebra

$$\bar{C} = \mathbb{R}[u_1+v_1, u_2+v_2] \otimes \Lambda(y_1, y_2) = \Lambda(W' \oplus W)$$

and generate the ideal $\langle \bar{C}^+ \rangle$ in $\Lambda(V)$ by $\bar{C}^+$ (it means by the elements of positive degrees). It is easy to calculate that

$$(\Lambda(V'), \delta) \cong (\Lambda(V)/\langle \bar{C}^+ \rangle, \delta) \cong (\mathbb{R}[u_1, u_2] \otimes \Lambda(y_3, y_4, y_6), D) = (\mathfrak{M}, D),$$
$$Du_i = 0, \quad i = 1,2, \quad Dy_3 = u_1^2, \quad Dy_4 = u_1 \cdot u_2, \quad Dy_5 = u_2^2$$

(up to scalars, of course). By Sullivan's theorem [6], $(\mathfrak{M}, D)$ is a minimal model for M.

In what follows we need to calculate certain Massey products.

Let $\bar{a}, \bar{b}, \bar{c}$ be the cohomology classes in $H^*(\mathfrak{M}, D)$ corresponding to the elements $a = u_1, b = u_1, c = u_2$. As far as $ab = u_1^2 = \delta y_3$, $bc = u_1 u_2 = \delta y_4$ we obtain $\bar{a}\bar{b} = 0$, $\bar{b}\bar{c} = 0$. Then

$$ay_4 - y_3 c = u_1 y_4 - u_2 y_3 \tag{1}$$

is a cocycle, representing the Massey triple product $\langle \bar{a}, \bar{b}, \bar{c} \rangle \in H^{11}(\mathfrak{M}, D)/\bar{a}H^7(\mathfrak{M}, D) + \bar{c}H^5(\mathfrak{M}, D)$. Obviously, (1) can not be a coboundary, because any non-zero coboundary in $(\mathfrak{M}, D)$ is of u_1- or u_2-degree greater than 1. Moreover, cohomology class $u_1 y_4 - u_2 y_3 \notin \bar{a}H^7(\mathfrak{M}, D) + \bar{c}H^5(\mathfrak{M}, D)$ (it is also evident, as far as otherwise there would hold the equality $u_1 y_4 - u_2 y_3 = u_1 h_1 + u_2 h_2 + \delta w$, and taking into consideration the degrees $\deg(u_i)$ and $\deg(y_j)$ one would come to the previous argument). Thus

$$\langle \bar{a}, \bar{b}, \bar{c} \rangle \neq 0$$

and the theorem is proved in the case $SU(6)/SU(3) \times SU(3)$. To finish the proof it is enough to notice, that for $n \geq 5$ the proof in fact does not differ from the particular case above, because the Cartan algebra in the general case differs from (C, δ) only by the exterior variables $y_6, \ldots$ for which $\delta y_6, \ldots, \ldots \in (f_3, f_4, f_5)$, but the latter do not change $\langle \bar{a}, \bar{b}, \bar{c} \rangle$ (here we denote by (, ,) the ideal generated by f_3, f_4, f_5. $\square$

Acknowledgements

The author is grateful to Professor S. Salamon for encouragement and valuable discussions during the Conference in Opava.

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Received 28 September 1992