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ON SYMPLECTIC AND CONTACT GROUPOIDS

Paulette Libermann

Abstract. This paper deals with Lie groupoids, in particular symplectic and contact groupoids. We formulate a definition of contact groupoids; we give examples and a counter-example (groupoid with a contact form which is not a contact groupoid).

Keywords. Lie groupoid, symplectic groupoid, contact groupoid, slidings, invertible sections, Lie algebroid.

0. Introduction

The theory of differentiable groupoids (called now Lie groupoids) has been introduced by C. Ehresmann [E] in 1950 in his paper on connections (where he defined the groupoid of paths, the groupoid associated with a principal bundle, the parallel displacement); then he utilized groupoids to define Lie pseudogroups by means of jets; he defined prolongations of groupoids. His pupils, the author [L], A. Haefliger [H], J. Pradines [P] etc. worked on this subject. In particular, introducing the notion of Lie algebroid (which corresponds to Lie algebras in the case of Lie groups), J. Pradines proved the Lie third theorem (at least for local Lie groupoids). The works of I. Kolář are connected with this subject [Ko].

Utilizing different kinds of prolongations, Guillemin-Sternberg [G-S], Kumpera-Spencer [K-S] investigated the theory of Lie equations.

Then appeared the work of K. Mackenzie [M] on Lie groupoids and Lie algebroids in Differential Geometry and the paper by J. Renault [R] dealing with a groupoid approach to C^* algebras.

Recently A. Weinstein [W1] and independently M.V. Karasev [Ka] had defined symplectic groupoids and they have shown their importance in the theory of Poisson manifolds. For instance, one of principal motivations to introduce this notion, for A. Weinstein, was the quantization of Poisson manifolds.

During the last five years, the theory of symplectic groupoids has known a very large expansion with, among others, the works of A. Weinstein and his pupils, P. Dazord and his pupils (see [W2], [W3], [W-X], [C-D-W], [D-S]). C. Albert and P. Dazord [A-D] have given a geometric approach of symplectic groupoids.

Our paper, which tries to be self-contained, could serve as an introduction to the above-mentioned publications. In particular it borrows ideas from the work of Albert-Dazord (slidings, definition pseudogroup, definition sheaf).

This paper is in final form and no version of it will be submitted for publication elsewhere.

This paper is divided into 3 parts. Chapter I deals with properties of Lie groupoids (invertible sections, slidings, prolongations). Chapter II deals with symplectic groupoids; we explain the point of view of Albert-Dazord, using our theory of symplectically complete foliations [L5], [L6] which shows that the base of the groupoid is a Poisson manifold. In chapter III we develop a theory of contact groupoids; our definition is suggested by the theory of symplectic groupoids. But our theory of complete foliations with respect of a contact form shows that the base must be also a Poisson manifold. We give examples of contact groupoids; in particular we recover the extension of a symplectic groupoid equipped with a prequantization [W2], [W-X]. We also give an example of a Lie groupoid such that Γ and Γ_0 are endowed with contact forms but the projections from Γ onto Γ_0 are not Jacobi maps.

Contact groupoids are being studied by C. Albert [A].

In the following paper, differentiable means of class C^∞. The set of C^∞ functions on M will be denoted by $C^\infty(M)$.

I. Properties of Lie groupoids

I.1. **Definition.** A groupoid Γ is a set equipped with a partially defined multiplication satisfying the axioms

(1) for any $x \in \Gamma$, there exist in Γ an unique right unit $\alpha(x)$ and an unique left unit $\beta(x)$ such that
$$x\alpha(x) = x, \quad \beta(x)x = x,$$
(2) xy is defined if and only if $\alpha(x) = \beta(y)$,
(3) if xy and yz are defined, so are $(xy)z$ and $x(yz)$; then $(xy)z = x(yz)$,
(4) any $x \in \Gamma$ admits an unique inverse x^{-1}, such that
$$x^{-1}x = \alpha(x), \quad xx^{-1} = \beta(x).$$

I.2. The subset Γ_0 of units is called the *base* of the groupoid. We have the *source* map $\alpha : \Gamma \to \Gamma_0$, and the *target* $\beta : \Gamma \to \Gamma_0$ which are surjective. For any $u \in \Gamma_0$, the sets $\alpha^{-1}(u)$ and $\beta^{-1}(u)$ will be called a *α-fiber* and a *β-fiber*.

The *involution* $\Phi : \Gamma \to \Gamma$ which maps x onto x^{-1} exchanges the α-fibers and the β-fibers.

The set $\Gamma_2 \subset \Gamma \times \Gamma$ of composable elements is $(\alpha \times \beta)^{-1}(\Delta_{\Gamma_0})$, where Δ_{Γ_0} is the diagonal of $\Gamma_0 \times \Gamma_0$.

The groupoid Γ is said to be *transitive* if the map $\alpha \times \beta : \Gamma \to \Gamma_0 \times \Gamma_0$ is transitive.

I.3. We deduce from the axioms that, for any groupoid Γ,
$$\alpha(xy) = \alpha(y), \quad \beta(xy) = \beta(x).$$
So for $x \in \Gamma$, the *left translation* L_x defined by $L_x(y) = xy$ is a bijection from $\beta^{-1}(\alpha(x))$ onto $\beta^{-1}(\beta(x))$ whose inverse $(L_x)^{-1}$ is the left translation $L_{x^{-1}}$. Likewise the *right translation* R_x such that
$$R_x(y) = yx$$
is a bijection from $\alpha^{-1}(\beta(x))$ onto $\alpha^{-1}(\alpha(x))$.

Moreover if $\alpha(x') = \alpha(x)$, then $x' = L_{x'x^{-1}}(x)$; if $\beta(x') = \beta(x)$, then $x' = R_{x^{-1}x'}(x)$. So the set of all left translations (resp. right translation) operates transitively on the α-fibers (resp. the β-fibers).

I.4. **Examples.** 1) A groupoid with one unit is a group; for the left and right translations, we recover the usual notions.

2) For any set A, we define on $A \times A$ a transitive groupoid structure as follows:

$$\alpha(x,y) = (y,y), \quad \beta(x,y) = (x,x), \quad (x,y)(y,z) = (x,z), \quad (x,y)^{-1} = (y,x).$$

I.5. **Definition.** By a Lie groupoid we shall mean a (not necessarily Hausdorff) differentiable manifold Γ endowed with a groupoid structure such that

(1) the base Γ_0 is a connected, paracompact, Hausdorff submanifold,
(2) the mappings α and β are submersions,
(3) the multiplication $\mu : \Gamma_2 \to \Gamma$ is differentiable,
(4) the involution $\Phi : \Gamma \to \Gamma$ is a diffeomorphism.

The condition (3) makes sense because from (2) we deduce that $\Gamma_2 \subset \Gamma \times \Gamma$ is a submanifold (being the fibered product of α and β).

Remarks. 1) In the terminology of C. Ehresmann and other authors, these groupoids were called differentiable, the word “Lie groupoid” being reserved for transitive groupoids. 2) A Lie groupoid with one unit is a Lie group.

I.6. **Examples.** 1) Let M be a differentiable Hausdorff manifold. As in example I.4. 2), we define a groupoid structure on $M \times M$ which is a Lie groupoid structure.

2) Let $\pi : E \to M$ be a vector bundle on the manifold M. The addition in the fibers defines a Lie groupoid structure on E for which $\alpha = \beta$; so this groupoid is not transitive

3) *Gauge groupoids.* This notion was introduced by C. Ehresmann [E] under a different name. See also [L4]. Let $P(M,G)$ be a G-principal bundle, with base M. Let us consider the quotient Γ of $P \times P$ by the diagonal action of G (in other words $\Gamma = P \times P/\rho$ where ρ is the equivalence relation such that $(z,z') \sim (zs, z's)$ for any $s \in G$). Then Γ is a transitive Lie groupoid whose α-fibers are principal bundles.

Conversely for any transitive Lie groupoid Γ, the α-fibers are principal bundles; so Γ may be considered as a gauge groupoid (see [L4]).

When M is a homogeneous space i.e. P is a Lie group, G a subgroup and $M = P/G$, then the corresponding gauge bundle is diffeomorphic to $P \times P/G$. This is the situation in the theory of the “moving frame”.

4) More generally the left action of a Lie group $\mathcal{G}$ on a manifold M defines a Lie groupoid structure Γ on $\mathcal{G} \times M$ such that: $\Gamma_0 = \{e\} \times M$; $\alpha(g,x) = (e,x)$, $\beta(g,x) = (e,gx)$, the product $(g_2, x_2 = g_1x_1)(g_1,x_1)$ being equal to (g_2g_1, x_1)

5) *Jets groupoids.* Given a pseudogroup of local diffeomorphisms Θ acting on a manifold M, the set $J^q(\Theta)$ of q-jets of these diffeomorphisms is a Lie groupoid with base M.

We could consider also the set $J^\lambda(\Theta)$ of the germs of the local diffeomorphisms belonging to Θ; in particular we have the groupoid $J^\lambda(M)$ of germs of all local diffeomorphisms. These groupoids are not Hausdoforff.

6) *Fundamental groupoid.* Let M be a connected manifold; the set $\pi(M)$ of homotopy classes of continuous paths with fixed extremities is endowed with a groupoid structure, the projections α and β on M being the origin and the extremity map. If $\pi(M)$ is equipped with the quotient topology of the compact open topology on the space of paths of M, then $\alpha \times \beta : \pi(M) \to M \times M$ is a covering map.

I.7. **Prolongations of groupoids** [E], [L4]. Let Γ be a Lie groupoid and Γ_0 its base. An *invertible local section* $s : U \subset \Gamma_0 \to \Gamma$ is a section of the α-projection such that $\beta \circ s$

is a local diffeomorphism. The image $s(U)$ is called by some authors a bissection; we prefer the name of *bi-transverse submanifold* because it is also the image of a β-section.

The set Θ of invertible sections of Γ is a pseudogroup for the multiplication defined as follows:

$$(s, s') \to s'', \quad \text{where } s'' \text{ is the section} \quad x \to s'(\beta s'(x))s(x).$$

The set Γ^k of all the k-jets of invertible sections is a groupoid (called the kth-prolongation of Γ); Γ^k has the same base Γ_0 as Γ. In particular $J^k(M)$, set of k-jets of all local diffeomorphisms is the prolongation of $J^0(M) = M \times M$. A Lie subgroupoid Γ of $J^k(M)$ is a *differential system*; its solutions are the local diffeomorphisms whose jets belong to Γ; Γ is said to be completely integrable if for each $X^k \in \Gamma$, there exists a solution f such that $j^k_x f = X^k$, where x is the source of X^k.

C. Ehresmann [E] has introduced the notions of jet and of groupoid to define the *Lie pseudogroups.* A Lie pseudogroup Θ of order k is a pseudogroup such that $J^k(\Theta)$ is a Lie subgroupoid of $J^k(M)$ and Θ is the set of solutions of $J^k(\Theta)$.

I.8. **Slidings** [A-D]. C. Albert and P. Dazord have introduced the notion of sliding which extends that of translation; in the case of a Lie group, it is the same notion.

A *left sliding* φ is a local diffeomorphism of a Lie groupoid Γ such that, if $\varphi \circ R_y$ or $R_y \circ \varphi$ is defined at $x \in \Gamma$, then they are equal. In other terms $\varphi(xy) = \varphi(x)y$. In particular $\varphi(\beta(y)y) = \varphi(y) = \varphi(\beta(y))y$. It follows the properties a) the source and target of φ are unions of β-fibers b) $\beta \circ \varphi = \beta \circ \varphi \circ \beta$ (so φ sends β-fibers on β-fibers) c) $\alpha \circ \varphi = \alpha$ (so each α-fiber is invariant by φ).

As $\varphi(y) = \varphi(\beta(y))y$, φ is determined by its restriction to $\Gamma_0 \cap U = V$, where U is the source of φ. From c) we deduce that $\varphi\,|_V$ is an α-section, and that this section is invertible. Conversely given an invertible α-section $s : V \to \Gamma$, it gives rise to the left sliding φ such that $\varphi(y) = s(\beta(y))y$ or $\varphi(y) = R_y s(\beta(y))$.

In the same way, *right slidings* ψ are defined by $\psi(xy) = x\psi(y)$. Reversing α and β by the involution Φ, we obtain similar properties for the right slidings. In particular the restriction of ψ to an open set of Γ_0 is an invertible β-section. As an invertible α-section $s : V \subset \Gamma_0 \to \Gamma$ determines an invertible β-section $\sigma = s \circ (\beta \circ s)^{-1}$, we deduce: *there exists a* (1,1)*-correspondence between right slidings, left slidings and invertible sections.*

The set P of all left slidings is a *pseudogroup along the projection* β i.e. the sources and targets of the elements of P are β-saturated sets. The pseudogroup Θ of invertible sections is in (1,1) correspondence with P. When Γ is the gauge groupoid of a principal bundle, these pseudogroups are in (1,1)-correspondence with the pseudogroup of local morphisms of the principal bundle.

Considering jets and germs, we see that $J^\lambda P$ and $J^q P$ are identified to the fibered products $\Gamma \times_\alpha \Gamma^\lambda$ and $\Gamma \times_\alpha \Gamma^q$.

In [A-D], it is shown that 1) P acts transitively on each α-fiber. 2) Moreover P *separates* the β-fibers in the following sense: if $\varphi \in P$ fixes $x \in \Gamma$, it fixes every x' in the β-fiber $\beta^{-1}(\beta(x))$; if y does not belong to this fiber, then $\varphi(y) \neq y$. In terms of invertible sections, the correspond section leaves $\beta(x)$ fixed; there exists a bi-transverse manifold passing through $\beta(x)$ and not through $\beta(y)$.

Any pseudogroup along β which satisfies these 2 properties is called a *definition pseudogroup.*

The authors have also shown that any left sliding commutes with any right sliding.

I.9. **Invariant vector fields and Lie algebroids.** Let Γ be a Lie groupoid with base Γ_0. Following C. Ehresmann [E], an *infinitesimal displacement* is a vector v tangent to a α-fiber, with origin at $u \in \Gamma_0$. The set depl Γ of infinitesimal displacements is a vector bundle. According to J. Pradines [P1], any local section $\sigma : U \subset \Gamma_0 \to \operatorname{depl}\Gamma$ extends to a vector field tangent to the α-fibers and invariant under the right translations; it is the vector field $X : x \to TR_{x^{-1}}(s(u'))$, where $R_{x^{-1}}$ is the right translation by x^{-1} mapping $\alpha^{-1}(u)$ onto $\alpha^{-1}(u')$, with $u = \alpha(x), u' = \beta(x)$. It results that the sheaf of germs of sections of depl Γ is a Lie algebra sheaf. The vector bundle depl Γ is a *Lie algebroid* in the sense of J. Pradines, as moreover the mapping $T\beta : \operatorname{depl}\Gamma \to T\Gamma_0$ induces on the sheaves of section a Lie algebra morphism. The involution Φ maps depl Γ on $\operatorname{depl}_\beta \Gamma$, composed of vectors tangent to the β-fibers.

When Γ is a Lie group, depl Γ is its Lie algebra and we recover the usual notions.

C. Albert and P. Dazord have proved that the vector fields tangent to the α-fibers and invariant under the right translations along the α-fibers are *right invariant* under the right slidings and conversely. As the fibers are Hausdorff, they have proved also that their flows is formed of left slidings.

Reversing α and β, we define the left invariant vector fields tangent to the β-fibers generating right slidings. It can be deduced that the bracket $[X, Y]$ (with X a right invariant vector field, Y a left invariant vector field) vanishes.

The problem of integrating Lie algebroids i.e. of finding a Lie groupoid whose Lie algebroid is given has been studied first by Pradines. The problem is not always possible; it leads to the notion of *local Lie groupoid* introduced by Van Est [VE]: roughly it is a Hausdorff manifold γ with projections α and β on a submanifold γ_0; the manifold γ is equipped with a partially defined multiplication whose set of units is γ_0; contrary to the Lie groupoids, the condition $\alpha(y) = \beta(x)$ is necessary but not sufficient for $y \cdot x$ to be defined. This integration problem is possible when dealing with local Lie groupoids (see [P1]).

I.10. **Definition sheaves** [A-D]. According to [A-D] a Lie algebra sheaf $\mathcal{L}$ separates the α-fibers if

(i) the commutator of $\mathcal{L}$ is transitive on the α-fibers,

(ii) two points x and y of Γ belong to the same α-fiber if and only if, for any local section X of $\mathcal{L}$ over an open set U such that the traces on U of the α-fibers are connected and contain x and y, the condition $X(x) = 0$ is equivalent to $X(y) = 0$.

A *definition sheaf* for a groupoid (with connected fibers) is a Lie algebra sheaf $\mathcal{L}$ satisfying the conditions

(a) $\mathcal{L}$ separates the α-fibers,

(b) the β-fibers are the transitive orbits of $\mathcal{L}$,

(c) any element of $\mathcal{L}$ is the germ of an α-saturated vector field on Γ.

This definition is used because given a manifold Γ with two projections α, β on a Hausdorff submanifold Γ_0, (with connected Hausdorff fibers), endowed with a Lie algebra sheaf satisfying (a),(b),(c), there exists on Γ a groupoid structure (with projections α and β), such that the elements of $\mathcal{L}$ are germs of left invariant vector fields.

II. Symplectic groupoids

II.1. **Definition.** According to A. Weinstein [W1], [C-D-W], a Lie groupoid Γ equipped with a symplectic form Ω is called a symplectic groupoid if the graph of the multiplication is a lagrangian submanifold of the symplectic manifold $\overline{\Gamma} \times \Gamma \times \Gamma$, where $\overline{\Gamma}$ is endowed with the form $-\Omega$.

As the graph is diffeomorphic to Γ_2, if $\dim \Gamma = 2n$, then $\dim \Gamma_2 = 3n$.

We shall formulate propositions whose proof is given in [A-D].

II.2. **Proposition.** *Let (Γ, Ω) be a symplectic groupoid. Then*

(1) *the base Γ_0 is a lagrangian submanifold of Γ,*
(2) *the involution Φ is an antisymplectomorphism ($\Phi^*\Omega = -\Omega$).*

Consequently the base and the α-fibers or β-fibers are n-dimensional.
A sliding is said to be *symplectic* if it is a local symplectomorphism.

II.3. **Proposition.** *Let (Γ, Ω) be a symplectic groupoid. Then*

(1) *A sliding φ is symplectic if and only if φ maps $(U \cap \Gamma_0)$ (where U is the source of φ) onto a lagrangian submanifold.*
(2) *The pseudogroup along β (resp. along α) of the left (resp. right) symplectic slidings is transitive on the α-fibers (resp. the β-fibers) and separates the β-fibers (resp. α-fibers).*
(3) *The sheaf $\mathcal{L}$ (resp. $\mathcal{R}$) of germs of left (resp. right) invariant hamiltonian vector fields is transitive on the β-fibers (resp. α-fibers).*
(4) *The foliations of Γ defined by the α-fibers and the β-fibers are symplectically orthogonal.*

II.4. The last proposition means that the tangent bundles to these foliations are symplectically orthogonal. We have proved [L5], [L6] that a foliation is *symplectically complete* (i.e. the Poisson bracket of two first integrals of the foliation is also a first integral) if and only the orthogonal vector bundle to the tangent bundle to the foliation is also completely integrable, defining a symplectically complete foliation. So for the α-foliation and the β-foliation, each one is locally generated by the hamiltonian vector fields corresponding to the first integrals of the other one.

In this situation, the pair (α, β) was called *dual pair* by A. Weinstein.

Any first integral of the α-foliation may be written $f = \alpha^*\overline{f}$ where $\overline{f}$ is a function on Γ_0, the correspondent hamiltonian vector field X_f (which is tangent to the β-leaves) is invariant under left translations. We have similar properties for the vector field $X_{\beta^*\overline{g}}$. If f is a first integral of the α-foliation, g a first integral of the β-foliation then $[f, g] = 0$ and hence $[X_f, X_g] = 0$.

In [A-D], it is also proved:

II.5. **Theorem.** *Let (Γ, Ω) be a Lie groupoid equipped with a symplectic form Ω, then Γ is a symplectic groupoid if and only if the two conditions are satisfied*

(1) *the submanifold Γ_0 is lagrangian,*
(2) *Γ admits a definition pseudogroup constituted of local symplectomorphisms.*

In the case of connected fibers, condition 2) may be replaced by 2′: Γ admits a definition sheaf constituted of germs of hamiltonian vector fields.

From [A-D] we may deduce also

II.6. **Proposition.** *Let (Γ, Ω) be a Lie groupoid equipped with a symplectic form Ω, with connected fibers, such that the α-foliations and β-foliations are symplectically orthogonal, then Γ is a symplectic groupoid if moreover Γ_0 is lagrangian.*

Proof. We shall show that the sheaf $\mathcal{L}$ of germs of vector fields X_{α^*f} is a definition sheaf. This sheaf is transitive on the β-fibers. Its commutator which contains the germs of vector fields $X_{\beta^*\overline{g}}$ is transitive on the α-fibers. Lastly $\mathcal{L}$ separates the α-fibers as $X_{\alpha^*\overline{f}}(x) = 0$ is equivalent to $d\overline{f}(\alpha(x)) = 0$.

From the theory of symplectically complete foliations developed in [L5], [L6], we deduce also

II.7. **Proposition.** *Let (Γ, Ω) be a symplectic groupoid. Then there exists on Γ_0 an unique Poisson structure Λ, such that the projection α is a Poisson morphism (and hence as $\Phi^*\Omega = -\Omega$, the projection β is an anti-Poisson morphism), the bracket $[\overline{f}_i, \overline{f}_j]_{\Gamma_0}$ of two functions on Γ_0 being defined by $\alpha^*[\overline{f}_i, \overline{f}_j]_{\Gamma_0} = [\alpha^*\overline{f}_i, \alpha^*\overline{f}_j]_{\Gamma}$, the last Poisson bracket being associated with the symplectic structure. In the same way, the hamiltonian vector fields $X_{\alpha^*\overline{f}_i}$ are projectable on $X_{\overline{f}_i} = \Lambda^{\#}(d\overline{f}_i)$.*

II.8. The problem of *symplectic integration* (set by Weinstein) is the following: given a Poisson manifold (P, Λ), find a symplectic groupoid (Γ, Ω) such that $\Gamma_0 = P$ and for which α is a Poisson morphism and β an anti-Poisson morphism. The problem of symplectic integration is not always possible; it has been studied by many mathematicians; for instance see [D-H]. If local symplectic groupoids are accepted, then the problem is always possible, as was proved by A. Weinstein and Albert-Dazord. We shall see examples of Poisson manifolds admitting a symplectic integration.

II.9. *Examples of symplectic groupoids* (see [A-D]).

Example 1. Let (M, Ω) be a symplectic manifold. Consider the manifold $M \times M$, endowed with the symplectic form η defined by $\eta = p_1^*\Omega - p_2^*\Omega$, where p_1 and p_2 are the projections of $M \times M$ on the first and second factor. Then the projections $\alpha : M \times M \to \Delta_M$, $\beta : M \times M \to \Delta_M$ (where Δ_M is the diagonal of $M \times M$) define a groupoid structure on M.

As Δ_M may be identified with M, the symplectic manifold (M, Ω) admits a symplectic integration $(M \times M, \eta)$.

Example 2. Let N be a manifold; then the cotangent bundle $\pi : T^*N \to N$ admits a symplectic groupoid structure Γ for which $\alpha = \beta = \pi$: the base Γ_0 (identified with the zero section), the leaves of the foliation defined by π are lagrangian i.e. self orthogonal.

N may be considered as a Poisson manifold with $\Lambda = 0$; then T^*N is a symplectic integration of N.

Example 3. (see [C-D-W] and [A-D]). Let Γ be a Lie groupoid. Then $T^*\Gamma$ is equipped with a second symplectic structure which makes the projection $\pi : T^*\Gamma \to \Gamma$ a Lie groupoid morphism.

The splittings

$$T_{\Gamma_0}\Gamma = T\Gamma_0 \oplus T\,\mathrm{depl}\,\Gamma, \quad T_{\Gamma_0}\Gamma = T\Gamma_0 \oplus T\,\mathrm{depl}_\beta\,\Gamma$$

where $T_{\Gamma_0}\Gamma$ is the tangent bundle to Γ along Γ_0, $\mathrm{depl}\,\Gamma$ and $\mathrm{depl}_\beta\,\Gamma$ have been defined in I.9) induce by duality projections p_α and p_β from $T^*_{\Gamma_0}\Gamma$ onto $(\mathrm{depl}\,\Gamma)^*$ and $(\mathrm{depl}_\beta\,\Gamma)^*$; combining these projections with suitable left and right slidings, we obtain projections $\hat{\alpha}$ and $\hat{\beta}$ on $(\mathrm{depl}\,\Gamma)^*$. The isomorphic vector bundles $\mathrm{depl}\,\Gamma$ and $\mathrm{depl}_\beta\,\Gamma$ may be identified with the normal bundle $T_{\Gamma_0}\Gamma/T\Gamma_0$; so $\hat{\alpha}$ and $\hat{\beta}$ may be considered as projections onto the conormal bundle $\mathcal{N}^*\Gamma_0$. It is proved that $\hat{\alpha}$ and $\hat{\beta}$ define a symplectic groupoid structure on $T^*\Gamma$.

It follows that the manifold $\mathcal{N}^*\Gamma_0$ (isomorphic to $(\operatorname{depl}\Gamma)^*$) is endowed with a Poisson structure. This last result has been obtained differently in [D-S] where it is proved that the dual of a Lie algebroid is endowed with a homogeneous Poisson structure and conversely.

This shows the links between integration of Poisson manifolds and integration of Lie algebroids.

When Γ is a Lie group G, then $\mathcal{N}^*\Gamma_0$ is the dual $\underline{g}^*$ of the Lie algebra $\underline{g}$ of G. We recover the Lie-Poisson structure on $\underline{g}^*$ (cf. A. Kirillov, B. Kostant, J.M. Souriau). In virtue of the third Lie theorem, any Lie-Poisson vector space $\underline{g}^*$ is integrable by the groupoid T^*G (where G is a Lie group with Lie algebra $\underline{g}$).

II.10. **Remark.** Let (Γ, Ω) be a symplectic groupoid with Poisson base (Γ_0, Λ). The form induced by Ω on each α-leave (resp. β-leave) being invariant under left (resp. right) translations is of constant rank. The theory of symplectically complete foliations [L6] shows that the rank of the induced form on the α-fiber at $x \in \Gamma$ is equal to the rank of Λ at $\alpha(x)$. So if the Poisson structure is regular (i.e. Λ of constant rank), then the form induced on all the fibers is constant. In this case, using the results of [L5], [L6] we show that locally, if rank $\Lambda = 2s$, the form Ω may be written

$$\begin{aligned}\Omega =& df_1 \wedge df_2 + \cdots + df_{2s-1} \wedge df_{2s} \\ &- (dg_1 \wedge dg_2 + \cdots + dg_{2s-1} \wedge dg_{2s}) \\ &+ df_{2s+1} \wedge dh_1 + \cdots + df_n \wedge dh_{n-2s},\end{aligned}$$

with $f_1, \ldots, f_{2s}$ first integral of the α-foliation, $g_1, \ldots, g_{2s}$ first integral of the β-foliation, $f_{2s+1}, \ldots, f_n$ first integrals of both foliations. This formula which is valid in the neighbourhood of a point $u \in \Gamma_0$ may be interpreted in the following way: as was proved by A. Lichnerowicz [Lic], the Poisson manifold $\Gamma_0 \cap U$ is diffeomorphic to the product $V_1 \times V_2$ where V_1 is a symplectic manifold of dimension $2s$ and V_2 (of dimension $n-2s$) has a null Poisson tensor; the manifold U is symplectomorphic to $V_1 \times V_1 \times T^*V_2$; the submanifold $\Gamma_0 \cup U$ is defined by the equations $f_i = g_i$ for $i = 1, \ldots, 2s$, $h_j =$ constant for $j = 1, \ldots n-2s$.

II.11. **Remark.** Let (Γ, Ω) be a symplectic groupoid with base Γ_0. Properties of this groupoid could be expressed in terms of invertible sections (cf. I.7). Let us call *adapted section* an invertible sections whose image is a lagrangian submanifold.

From II.3, we deduce that the set of adapted sections constitute a pseudogroup Θ; it is the set of solutions of a differential system Σ of order 1; this system is a subset of the prolongation Γ^1 of Γ. As Γ^1 may be identified with the tangent spaces to the bi-transverse submanifolds in the sense of I.7, Σ is the subset of these tangent spaces which are lagrangian.

Moreover 1) if $J^\lambda(\Theta)$ is the set of germs of elements of (Θ), the mapping $J^\lambda(\Theta) \to \Gamma$ is surjective, 2) for any $u \in \Gamma_0$ there are germs belonging to $J^\lambda(\Theta)$ satisfying $s(u) = u$ and distinct from the germ of the identity mapping of Γ_0.

III. Lie groupoids equipped with a contact form

III.1. *On contact manifolds* (see [L2], [L-M]). By a contact manifold (M, ω) we shall mean a connected $(2n+1)$-dimensional manifold equipped with a contact form ω i.e. a Pfaffian form such that $\omega \wedge (d\omega)^n \neq 0$.

The Reeb vector field E is uniquely defined by

$$i(E)\omega = 1, \quad i(E)d\omega = 0.$$

The tangent bundle TM and the cotangent bundle T^*M may be decomposed into

$$TM = \mathcal{E} \oplus \mathcal{H}, \quad T^*M = \mathcal{E}^0 \oplus \mathcal{H}^0,$$

where $\mathcal{E}$ (generated by E) is the kernel of $d\omega$, $\mathcal{H}$ (called the horizontal vector subbundle) is the kernel of ω, $\mathcal{E}^0$ is the annihilator of $\mathcal{E}$, $\mathcal{H}^0$ (generated by ω) is the annihilator of $\mathcal{H}$. The sections of $\mathcal{E}^0$ are called *semi-basic forms* (they satisfy the condition $i(E)\varphi = 0$). The horizontal subbundle is symplectic.

Any vector field X may be decomposed into

$$X = (i(X)\omega)E + \hat{X},$$

where $\hat{X}$ is horizontal.

Every Pfaffian form η may be decomposed into

$$\eta = (i(E)\eta)\omega + \hat{\eta},$$

where $\hat{\eta}$ is semi-basic.

The mapping $\omega^b : X \to -i(X)d\omega$ carries the vector field X onto a semi-basic form. The restriction of ω^b to horizontal vector fields is an isomorphism whose inverse $(\omega^b)^{-1}$ will be denoted by $\omega^\#$.

It was proved in [L2] that there exists an isomorphism Φ from the vector space $\mathcal{N}$ of infinitesimal conformal automorphism of ω (i.e. vector fields X such that $\mathcal{L}(X)\omega = \rho\omega$) onto the set $C^\infty(M)$ of all differentiable functions on M; this isomorphism is defined by

$$\Phi(X) = i(X)\omega;$$

then $\Phi^{-1}(f) = fE + \omega^\#(\widehat{df})$. The vector field $X_f = \Phi^{-1}(f)$ is called the *contact hamiltonian* vector field corresponding to f. In particular, as $\rho = i(E)df$, the contact hamiltonian vector fields which are infinitesimal automorphism of ω correspond to functions such that df is semi-basic (or 0). We notice that $\Phi(E) = 1$.

The isomorphism Φ carries the Lie algebra structure of $\mathcal{N}$ (defined by the usual bracket) onto a Lie algebra structure on $C^\infty(M)$; we define the bracket $[f, g]$ *(Jacobi bracket)* by

$$[f, g] = i[X_f, X_g]\omega;$$

then

$$X_{[f,g]} = [X_f, X_g], \quad X_{[1,f]} = [E, X_f],$$
$$[f, g] = d\omega(X_f, X_g) + f(i(E)dg) - g(i(E)df).$$

In particular when both df and dg are semi-basic,

$$[f, g] = d\omega(X_f, X_g).$$

The bivector field Λ such that

$$\Lambda(\alpha, \beta) = d\omega(\omega^\#\hat{\alpha},\ \omega^\#\hat{\beta})$$

for any pair of Pfaffian forms, satisfies the relations (as was proved by A. Lichnerowicz [Lic])

$$[\Lambda,\Lambda]_s = 2E\wedge\Lambda, \quad [E,\Lambda]_s = \mathcal{L}(E)\Lambda = 0,$$

where $[\ ]_s$ is the Schouten bracket. So M is endowed with a *Jacobi structure*.

Let $\Lambda^{\#} : T^*M \to TM$ be the morphism such that $\langle\Lambda^{\#}(\alpha),\beta\rangle = \Lambda(\alpha,\beta)$. We have: $\Lambda^{\#}(\alpha) = \omega^{\#}(\hat{\alpha})$. So the kernel of $\Lambda^{\#}$ is generated by ω and its image is horizontal.

We may write

$$X_f = fE + \Lambda^{\#}(df).$$

III.2. *On pseudo-orthogonal foliations* (see [L6]). Let $\mathcal{F}$ be a foliation on (M,ω), F its tangent bundle, F^0 the annihilator of F.

From III.1, it can be deduced that the distribution

$$F^{\perp} = \mathcal{E}\oplus\Lambda^{\#}(F^0)$$

is locally generated by the contact hamiltonian vector fields which correspond to the first integrals of $\mathcal{F}$; this distribution is said to be *pseudo-orthogonal* to F.

When the Reeb vector field (which is the contact hamiltonian vector field corresponding to $f = 1$) is tangent to the foliation, then we have the splitting

$$F = \mathcal{E}\oplus F\cap\mathcal{H}.$$

In this case, as F^0 is locally generated by semi-basic forms, the restrictions of $\Lambda^{\#}$ to F^0 is an isomorphism onto the orthogonal of $F\cap\mathcal{H}$ with respect of the symplectic subbundle $\mathcal{H}$; then we have

$$F^{\perp} = \mathcal{E}\oplus \mathrm{orth}_{\mathcal{H}}(F\cap\mathcal{H}),$$

rank $F^{\perp}$ = corank $F+1$, and $F^{\perp}$ is then a vector subbundle of TM with the relation

$$F^{\perp\perp} = F.$$

We shall say that F and $F^{\perp}$ are *pseudo-orthogonal* or that the pair $(F,F^{\perp})$ is *conjugate*.

Remark that if E is not tangent to the foliation, the distribution may have a rank which is not constant; even if $F^{\perp}$ has constant rank, $F^{\perp\perp}$ which contains the Reeb vector field is different form F.

A foliation $\mathcal{F}$ on (M,ω) is said to be ω-*complete* if for any pair (f_i,f_j) of first integrals of $\mathcal{F}$ (where f_i may be a constant), the bracket $[f_i,f_j]$ is also a first integral of $\mathcal{F}$ (eventually a constant).

It can be proved (see [L6]) the following:

Theorem. *A foliation $\mathcal{F}$ on (M,ω) containing the Reeb vector field E is ω-complete if and only if the pseudo-orthogonal subbundle $F^{\perp}$ is completely integrable, defining a foliation $\mathcal{F}^{\perp}$ which is also ω-complete. Then for any pair (f_i,g_α) of functions such that f_i is a first integral of $\mathcal{F}, g_\alpha$ a first integral of $\mathcal{F}^{\perp}$, we get the relation $[f_i,g_\alpha]=0$ or equivalently $[X_{f_i},X_{g_\alpha}]=0$.*

The proof uses the fact that the df_i's are semi-basic. Then the relations

$$d[f_i,f_j] = \sum_k \lambda_{ij}^k df_k, \quad d[1,f] = \sum_k \lambda^k df_k$$

are equivalent to

$$\hat{X}_{[f_i,f_j]} = \sum_k \lambda^k_{ij}\hat{X}_k, \quad \hat{X}_{[1,f]} = [\widehat{E, X_f}] = \sum \lambda^k \hat{X}_k,$$

where $\hat{X}$ is the horizontal part of X.

As $F^{\perp\perp} = F$, $\mathcal{F}^\perp$ is also ω-complete and the g_α are also semi-basic. The relations $[f_i, g_\alpha] = d\omega(X_f, X_g)$ show that $[f_i, g_\alpha] = 0$.

It also can be proved

Proposition. *When $\mathcal{F}$ is an ω-complete simple foliation ($M/\mathcal{F} = \overline{M}$is a manifold), the contact hamiltonian vector fields corresponding to the first integrals of $\mathcal{F}$ (in particular the Reeb vector field) are projectable on $\overline{M}$ as well as the bivector field Λ; so the projection $\pi : M \to \overline{M}$ is a Jacobi morphism, the Jacobi bracket being defined by $[\overline{f}, \overline{g}]_{\overline{M}}$such that*

$$\pi^*[\overline{f}, \overline{g}]_{\overline{M}} = [\pi^*\overline{f}, \pi^*\overline{g}].$$

This results from the fact that any first integral of $\mathcal{F}$ may be written $f = \pi^*\overline{f}$, where $\overline{f}$ is a function on $\overline{M}$; X_f is then projectable on $\overline{X}_{\overline{f}}$, which is the contact hamiltonian vector field on $\overline{M}$ corresponding to $\overline{f}$.

Remark. When the Reeb vector field belongs to the ω-complete foliation $\mathcal{F}$, then E projects onto 0 and the Jacobi structure on $\overline{M}$ is a *Poisson structure.*

For instance the foliation generated by the Reeb vector fields is ω-complete. If the foliation is simple, the quotient manifold $\overline{M}$ is symplectic and $d\omega$ is the transpose of the symplectic form on $\overline{M}$.

III.3. **Contact groupoids.** Let (Γ, ω) be a Lie groupoid equipped with a contact form ω. The conditions of compatibility can not be defined by means of the graph of the multiplication in $\Gamma \times \Gamma \times \Gamma$ because the form ω does not induce a contact form on $\Gamma \times \Gamma \times \Gamma$.

To simplify we shall assume that the α-fibers and the β-fibers of Γ are *connected.*

Definition. A groupoid Γ (with connected fibers) endowed with a contact form will be said to be a *contact groupoid* if the following conditions are satisfied

(1) The relation $\Phi^*\omega = -\omega$ holds for the involution $\Phi : f \to f^{-1}$.

(2) The α-fibers and the β-fibers are pseudo-orthogonal (i.e. each one of the two foliations is locally generated by the contact hamiltonian vector fields corresponding to the first integrals of the other one).

This definition, which is suggested by the theory of symplectic groupoids is justified by the following theorem (see II.6).

Theorem. *Let (Γ, ω) be a contact groupoid, then*

(1) *The manifold of units Γ_0 is a Legendre submanifold.*

(2) *Γ admits a definition sheaf composed of the germs of contact hamiltonian vector fields corresponding to all the germs of first integrals of the α-foliation (eventually the constants).*

Proof. 1) As $\Phi^*\omega = -\omega$, the form induced by ω on Γ_0 is null. The α-fibers and β-fibers must have the same dimension q; from III.2, it follows that codimension of the α-fibers

= dimension of the β-fibers $+1$; if $\dim\Gamma = 2n+1$, then we must have $q = n+1$; we deduce that $\dim\Gamma_0 = n$ and Γ_0 is a Legendre submanifold.

2) The proof is similar to that of proposition II.6. From III.2, we deduce that $[X_f, X_g] = 0$ for f (resp. g) a first integral of the α (resp. β) foliation. Using III.1, we see that $X_f(x) = 0$ if and only if $\overline{f}(\alpha(x)) = 0$, $d\overline{f}(\alpha(x)) = 0$ with $\overline{f}$ defined by $f = \alpha^*\overline{f}$.

III.4. **Counter-example.** We shall exhibit a Lie groupoid Γ with contact form such that the base Γ_0 is a Legendre submanifold, this base being also endowed with a contact structure. But the Reeb vector field is not tangent to the fibers; the α-foliation and the β-foliation are not ω-complete, so the projection along the fibers is not a Jacobi map.

Let (M, ω) be a contact manifold, of dimension $2n+1$. We consider the manifold $\Gamma = M \times M \times \mathbb{R}$ that we shall denote by $M_1 \times M_2 \times \mathbb{R}$ with the projections $p_1 : \Gamma \to M_1$, $p_2 : \Gamma \to M_2$; we shall write $\overline{\omega}^1 = p_1^*\omega_1$, $\overline{\omega}^2 = p_2^*\omega_2$, $\tilde{E}_1 = E_1 + 0$, $\tilde{E}_2 = 0 + E_2$, E_1 and E_2 being the Reeb vector fields of M_1 and M_2. The manifold Γ is of dimension $4n+3$.

Proposition. *The form $\eta = e^t\overline{\omega}_1 - e^{-t}\overline{\omega}_2$ is a contact form on Γ, with Reeb vector field $E = \frac{1}{2}(e^{-t}\tilde{E}_1 - e^t\tilde{E}_2)$.*

To prove that η is of class $4n+3$, we could calculate $\eta \wedge (d\eta)^{2n+1}$; we can also use local coordinates (x_a) on M_1, (y_b) on M_2. We get

$$\begin{aligned}\eta &= e^t\Big(dx_0 - \sum_{i=1}^{n} x_{i+n}dx_i\Big) - e^{-t}\Big(dy_0 - \sum_{j=1}^{n} y_{j+n}dy_j\Big)\\ &= dz_0 - (e^t x_0 + e^{-t}y_0)dt - e^t\Big(\sum_{i=1}^{n} x_{i+n}dx_i\Big) + e^{-t}\Big(\sum_{j=1}^{n} y_{j+n}dy_j\Big)\end{aligned}$$

with $z_0 = e^t x_0 - e^{-t}y_0$.

We check that $E = \frac{1}{2}(e^{-t}\tilde{E}_1 - e^t\tilde{E}_2)$ satisfies the relations $i(E)\eta = 1$, $i(E)d\eta = 0$.

We define on Γ a groupoid structure, the units being of the form $(x, y = x, 0)$ with $\alpha(x,y,t) = (y,y,0)$, $\beta(x,y,t) = (x,x,0)$; the product $(x_1,y_1,t_1)(x_2,y_2,t_2)$ is defined when $y_1 = x_2$; then it is equal to (x_1, y_2, t_1+t_2). We check that the manifold of units Γ_0 (which can be identified with M) is a Legendre submanifold of Γ. The first integrals of the α-fibers are $y_0, \ldots, y_{2n}$. The Jacobi bracket $[y_j, y_k]$ depends upon t; so we do not have an ω-complete foliation. Notice that the α-fibers are exact symplectic manifolds endowed with the form $e^t\overline{\omega}_1$.

Remark that if $\check{M} = M \times \mathbb{R}$ (endowed with the form $e^t\omega$) is the symplectification of M, then the symplectic groupoid $\check{M} \times \check{M}$ (endowed with the form $e^t\eta$) is the symplectification of Γ.

III.5. *Examples of contact groupoids.*

Example 1. Let Γ be a symplectic groupoid with an exact symplectic form ($\Omega = d\theta$).

Then $\tilde{\Gamma} = \Gamma \times \mathbb{R}$ is endowed with the contact form $\omega = dt - \theta$.

The groupoid structure on $\tilde{\Gamma}$ is defined by the projections $\tilde{\alpha}(x,t) = (\alpha(x), 0)$, $\tilde{\beta}(x,t) = (\beta(x), 0)$, the product $(x_1,t_1)(x_2,t_2) = (x_1 \cdot x_2,\ t_1+t_2)$.

So the base is $\tilde{\Gamma}_0 = \Gamma_0 \times \{0\}$.

Particular cases

a) M is any differentiable manifold, $\Gamma = T^*M \times \mathbb{R} = J^1(M, \mathbb{R})$,

b) M is an exact symplectic manifold and $\tilde{\Gamma} = M \times M \times \mathbb{R}$.

Example 2. (see [W2] and [W-X]) *Gauge groupoids of prequantizations.*

A *prequantization* of a symplectic manifold $(\mathcal{S}, \Omega)$ is a principal $\mathbb{T}$-bundle $\pi : P \to S$ together with a connection form ω whose curvature is $-\Omega$. Here $\mathbb{T} = S_1 = \mathbb{R}/Z$.

Lemma. *(Weinstein* [W2]*) Let P be a prequantization of the symplectic manifold $\mathcal{S}$. The gauge groupoid Γ_P, defined as the quotient of $P \times P$ by the diagonal action of $\mathbb{T}$, carries a 1-form which makes it a prequantization of $S \times S$. The manifold of units is a Legendre submanifold.*

Proof. With the same conventions which were used in the counter-example, we set $P \times P = P_1 \times P_2$ with the projections $p_1 : P_1 \times P_2 \to P_1$, $p_2 : P_1 \times P_2 \to P_2$, $\overline{\omega}^1 = p_1^*\omega_1$, $\overline{\omega}^2 = p_2^*\omega_2$. The form $\eta = \overline{\omega}_1 - \overline{\omega}_2$ defined on $P_1 \times P_2$ vanishes on the diagonal. This form is invariant under the 2-dimensional group generated by $\tilde{E}_1$ and $\tilde{E}_2$. The diagonal action of $\mathbb{T}$ is generated by the vector field $\tilde{E}_1 + \tilde{E}_2$ which satisfies the condition $i(\tilde{E}_1 + \tilde{E}_2)(\overline{\omega}_1 - \overline{\omega}_2) = 0$. So the form $\overline{\omega}_1 - \overline{\omega}_2$ passes to the quotient Γ_P; the vector fields $\tilde{E}_1$ and $\tilde{E}_2$ project on the same vector field X on Γ_P; X generates an action of $\mathbb{T}$ which makes Γ_P into a principal bundle over $\mathcal{S}_1 \times \mathcal{S}_2 = \mathcal{S} \times \mathcal{S}$. The set of units is annihilated by the connection form $\overline{\eta}$ on Γ_P and this submanifold has the "good" dimension to be legendrian.

Corollary. *The gauge groupoid Γ_P is a contact groupoid.*

Proof. It is easy to see that $\Phi^*\overline{\eta} = -\overline{\eta}$ (where Φ is the involution $f \to f^{-1}$ in Γ_P).

We have to prove that the projections α and β of Γ_P on $\mathcal{S}$ define pseudo-orthogonal foliations. Using the convention that $\mathcal{S} \times \mathcal{S} = \mathcal{S}_1 \times \mathcal{S}_2$, we factorize $\alpha : \Gamma_P \to \mathcal{S}_1$ and $\beta : \Gamma_P \to \mathcal{S}_2$ into $\alpha = q_1 \circ \pi$ and $\beta = q_2 \circ \pi$, where π is the projection $\Gamma_P \to \mathcal{S}_1 \times \mathcal{S}_2$ and q_1 (resp. q_2) is the projection $\mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_1$ (resp. $\mathcal{S}_1 \times \mathcal{S}_2 \to \mathcal{S}_2$). The horizontal bundle $\mathcal{H} = \ker \overline{\eta}$ is the pull-back $\pi^*T(\mathcal{S}_1 \times \mathcal{S}_2)$. If F_α and F_β are the tangent bundles to the α-foliation and the β-foliation, we deduce

$$F_\alpha \cap \mathcal{H} = \pi^*(T\mathcal{S}_1), \qquad F_\beta \cap \mathcal{H} = \pi^*(T\mathcal{S}_2).$$

As $T\mathcal{S}_1$ and $T\mathcal{S}_2$ are orthogonal subbundles of the symplectic bundle $T(\mathcal{S}_1 \times \mathcal{S}_2)$, the bundles $F_\alpha \cap \mathcal{H}$ and $F_\beta \cap \mathcal{H}$ are orthogonal with respect of the symplectic bundle $\mathcal{H}$. So using the notations and results of III.2 we deduce $F_\beta = \mathcal{E} \oplus \mathrm{orth}_{\mathcal{H}}(F_\alpha \cap \mathcal{H}) = F_\alpha^\perp$ (where $\mathcal{E}$ is generated by $\overline{E}$).

Remark. In [L4], we have considered the particular case of the principal bundle: $S_{2n+1} \to \mathbb{P}_n(\mathbb{C})$, where S_{2n+1} is the $(2n+1)$-sphere and $\mathbb{P}_n(\mathbb{C})$ the complex projective space. We have shown the existence of a contact form on the gauge groupoid, using special properties of the product of spheres.

In particular for $n = 1$, S_3 is a homogeneous space ($S_2 = S_3/S_1$) and the gauge groupoid is diffeomorphic to $S_3 \times S_2$.

More generally A. Weinstein [W2], [W3], A. Weinstein and P. Xu [W-X] have studied the prequantization of symplectic groupoids, which is a step (through the integration of Poisson manifolds) in the quantization of Poisson manifolds.

Example 3. This example is connected with the use of the Legendre transformation in Mechanics.

Let $J^1(\mathbb{R}^n, \mathbb{R})$, identified with $T^*\mathbb{R}^n \times \mathbb{R}$ or $\mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}$, equipped with the usual contact form, which by means of coordinates $y_1, \ldots, y_n$ in $\mathbb{R}^n, p_1, \ldots, p_n$ in $\mathbb{R}^{n*}$, may be written

$$\omega = dt - \sum_1^n p_i dy_i.$$

There exists a first groupoid structure on $J^1(\mathbb{R}^n, \mathbb{R})$ identified with $T^*\mathbb{R}^n \times \mathbb{R}$ (see example 1.a).

We shall introduce a second one using the Legendre involution and a Legendre transformation, with the terminology of [L-M] (chapters II and V).

The Legendre involution is the mapping $\Psi : \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R} \to \mathbb{R}^{n*} \times \mathbb{R}^n \times \mathbb{R}$ defined by

$$\Psi(v, \xi, t) = (\xi, v, -t + \langle \xi, v \rangle).$$

Setting $\tau(v, \xi) = t - \frac{1}{2}\langle \xi, v \rangle$, we may write

$$\Psi(v, \xi, \tau) = (\xi, v, -\tau).$$

As $\tau = t - \frac{1}{2}\sum_1^n p_i y_i$, we obtain

$$\omega = d\tau - \frac{1}{2}\sum_1^n (p_i dy_i - y_i dp_i)$$

which shows that $\Psi^*\omega = -\omega$.

Now we consider a function $L : \mathbb{R}^n \to \mathbb{R}$ which defines a diffeomorphism $\mathcal{L} = dL$ from $\mathbb{R}^n$ to $\mathbb{R}^{n*}$; $\mathcal{L}$ is called a Legendre transformation and L is said to be hyperregular.

By means of the coordinates (y_i, p_i) we write

$$\mathcal{L}(y_1, \ldots, y_n) = (p_1, \ldots, p_n) \quad \text{with} \quad p_i = \frac{\partial L}{\partial y_i}.$$

It is known (see [L-M] ch. II) that $\mathcal{L}^{-1}$ is associated with the function $H = \sum_1^n p_i y_i - L$;

$$\mathcal{L}^{-1}(p_1, \ldots, p_n) = (y_1, \ldots, y_n) \quad \text{with} \quad y_i = \frac{\partial H}{\partial p_i}.$$

Combining Ψ and $\mathcal{L}$, we obtain the diffeomorphism

$$\Phi : \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}$$

such that

$$\Phi(v, \xi, t) = (v', \xi', t')$$

with

$$v' = \mathcal{L}^{-1}(\xi)$$

$$\xi' = \mathcal{L}(v)$$

$$\tau(v', \xi', t') = -\tau(v, \xi, t)$$

or

$$t' = -t + \tfrac{1}{2}\langle \xi, v \rangle + \tfrac{1}{2}\langle \mathcal{L}(v), \mathcal{L}^{-1}(\xi) \rangle.$$

The set of elements of $\mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}$ which are invariant under Φ (and on which the function τ vanishes) is the submanifold

$$W = \{(v, \xi, t) \in \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}; \quad \xi = \mathcal{L}(v), \quad t = \tfrac{1}{2}\langle \mathcal{L}(v), v\rangle\}$$

or equivalently

$$W = \{(v, \xi, t) \in \mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}; \quad v = \mathcal{L}^{-1}(\xi), \quad t = \tfrac{1}{2}\langle \xi, \mathcal{L}^{-1}(\xi)\rangle\}.$$

We define a groupoid structure on $J^1(\mathbb{R}^n, \mathbb{R})$, setting

$$\alpha(v, \xi, t) = (\mathcal{L}^{-1}(\xi),\ \xi,\ \tfrac{1}{2}\langle \xi, \mathcal{L}^{-1}(\xi)\rangle)$$

$$\beta(v, \xi, t) = (v,\ \mathcal{L}(v),\ \tfrac{1}{2}\langle \mathcal{L}(v), v\rangle).$$

The product $(v_2, \xi_2, t_2)(v_1, \xi_1, t_1)$ is defined if and only if $\xi_2 = \mathcal{L}(v_1)$; this product must satisfy the condition

$$\tau\big((v_2, \xi_2, t_2)(v_1, \xi_1, t_1)\big) = \tau(v_2, \xi_2, t_2) + \tau(v_1, \xi_1, t_1);$$

so it is equal to

$$(v_2, \xi_1, t_2 + t_1 - \tfrac{1}{2}\langle \xi_2, v_2\rangle - \tfrac{1}{2}\langle \xi_1, v_1\rangle + \tfrac{1}{2}\langle \xi_1, v_2\rangle).$$

We check that the inverse of (v, ξ, t) is $\Phi(v, \xi, t)$; so Φ exchanges the α-fibers and the β-fibers. The α-fibers (resp. β-fibers) are diffeomorphic to $\mathbb{R}^n \times \mathbb{R}$ (resp. $\mathbb{R}^{n*} \times \mathbb{R}$);they contain the Reeb vector field; as $[y_i, y_j] = 0$ and $[p_i, p_j] = 0$, the two foliations are pseudo-orthogonal. But W is not necessarily a Legendre submanifold.

As $\langle \mathcal{L}(v), v\rangle = i(Z)dL$, where Z is the field of delations of $\mathbb{R}^n$, when the function L is homogeneous of degree 2, then $\frac{1}{2}\langle \mathcal{L}(v), v\rangle = L$ and

$$W = \{v, dL(v), L(v)\}, \quad v \in \mathbb{R}^n;$$

then W is the image of $\mathbb{R}^n$ by the section j^1L of $J^1(\mathbb{R}^n, \mathbb{R})$; it is known (see L-M]) that is a legendrian. We remark that in this case, H is also homogeneous of degree 2 and W is also the image of $\mathbb{R}^{n*}$ by the section j^1H of $J^1(\mathbb{R}^{n*} \times \mathbb{R})$.

We check that $\Phi^*\omega = -\omega$. So

Proposition. *If L and consequently H are homogeneous of degree 2, then we have defined a contact groupoid structure on $J^1(\mathbb{R}^n, \mathbb{R})$.*

Remark. Dealing with a manifold M, we may define the Legendre involution (see [L-M] chapter V)

$$\Psi : TM \times_M T^*M \times \mathbb{R} \to T^*M \times_M TM \times \mathbb{R}.$$

Following the same line as we did for $\mathbb{R}^n \times \mathbb{R}^{n*} \times \mathbb{R}$ but replacing the differential and the jets by the vertical differential and the vertical jets, by means of a Legendre transformation we define a groupoid structure on $TM \times_M T^*M \times \mathbb{R}$; for instance the diffeomorphism $\mathcal{L} : TM \to T^*M$ (Legendre transformation) is defined by means of the vertical differential of the function $L : TM \to \mathbb{R}$. We obtain the same formulae but v, ξ belong to TM and T^*M.

The manifold $TM \times_M T^*M \times \mathbb{R}$ is no more a contact manifold; the projection $TM \times_M T^*M \times \mathbb{R}$ on M is a vector bundle each of whose fibers is equipped with a contact form; this vector bundle could be called a *contact vector bundle.* A fibered Legendre submanifold could be a submanifold whose intersection with each fiber is a Legendre submanifold.

If $L = T - U$, where the restriction of T to each fiber is a non degenerate quadratic form, and U is a function on M, then we could define the notion of *fibered contact groupoid.*

This is the case of a so-called holonomic dynamical system with n degrees of freedom; T is then positive (kinetic energy), U represents the potential energy (see [L-M] ch. II).

It would be interesting to investigate the Lie groupoids endowed with a general contact structure (i.e. with a symplectic vector subbundle of corank 1 but without a global contact form) such that the symplectification is a symplectic groupoid.

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Paulette Libermann

Université Paris VII
U.F.R. de Mathématiques
2 Place Jussieu
75251 Paris Cedex 05
France

and

116 Avenue du Général Leclerc
75014 Paris
France

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