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MEASURE OF LINEAR SPACES AND TOTAL CURVATURES OF COMPACT ANTI–INVARIANT SUBMANIFOLDS IN $\mathbb{C}^n$.

Ximo Gual Arnau

Abstract. Let M be a totally real compact manifold, immersed in the Kaehler manifold $\mathbb{R}^{2n}$. We define the total curvatures of M and we get a local interpretation of them. Previously, we obtain an invariant measure for the different linear spaces in $\mathbb{C}^n$.

Keywords. Total curvatures, densities, subspaces in $\mathbb{C}^n$, moving frames.
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0.- Introduction

The total absolute curvatures of compact differentiable manifolds immersed in the euclidean space $\mathbb{R}^n$, have been defined by L. A. Santaló as curvatures of global character in the manifold, [SA-1]; moreover, he expressed them as the integral of certain local curvatures defined in every point of the manifold, [SA-2].

In this work, we will consider totally real (anti-invariant) submanifolds in the complex space $\mathbb{C}^n$, or more specifically in $\mathbb{R}^{2n}$ as Kaehler manifold [K-M], and following a similar process as in [SA-1, 2], we will define the total absolute curvatures of these submanifolds and we will present them as the integral over the submanifold of certain local curvatures. These definitions allow us to compare the new curvatures with the total curvatures for submanifolds in $\mathbb{R}^{2n}$ as real euclidean space.

In order to obtain these goals we need some results of Integral Geometry for linear spaces in $\mathbb{C}^n$, which permit us to define a density (measure) of them, invariant under the group of complex motions. These measures will be obtained following the general method for densities in homogeneous spaces, [SA-3].

The linear spaces that we will consider are basically the isotropic, coisotropic and symplectic subspaces of $(\mathbb{R}^{2n}, \omega)$, where ω is the symplectic (Kaehler) form defined by the complex operator J, [L-R].

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1.- Review of hermitian geometry

From now on, $\mathbb{C}^n$ will be identified with $\mathbb{R}^{2n}$ and its canonical complex structure J, and we will use the canonical inner product in $\mathbb{R}^{2n}$ as Hermitian inner product with respect to J.

In order to obtain the density for the different linear spaces in $\mathbb{C}^n$, we will use the method of *moving frames,* [SH].

1.1. Definition. A moving frame on an open set $U \subset \mathbb{C}^n$ is a set $(e_1(p), \ldots, e_n(p), e_{1*}(p), \ldots, e_{n*}(p))$ of smooth $\mathbb{R}^{2n}$-valued functions such that for each $p \in U$, $(e_1, \ldots, e_n, e_{1*}, \ldots, e_{n*})$ is a unitary frame for $\mathbb{R}^{2n} \cong T_p\mathbb{R}^{2n}$; where $e_{i*} = Je_i,\ 1 \leq i \leq n$.

Let $p : U \hookrightarrow \mathbb{R}^{2n}$ be the inclusion map, so that the vector valued 1-form dp is the identity at each point of U, i.e., $dp\,(e_i) = e_i$.

We define the following 1-forms:

$$\omega_i = dp\, e_i, \quad \omega_{i*} = dp\, e_{i*}, \quad 1 \leq i \leq n, \tag{1.1}$$

$$\begin{matrix} \omega_{ij} = de_i\, e_j, & \omega_{i*j*} = de_{i*}\, e_{j*}, \\ \omega_{i*j} = de_{i*}\, e_j, & \omega_{ij*} = de_i\, e_{j*}, \end{matrix} \quad 1 \leq i,j \leq n. \tag{1.2}$$

1.2. Note. The 1-forms (ω_i, ω_{i*}) are nothing but the dual basis of (e_i, e_{i*}), since

$$\omega_i(e_j) = dp\, e_i(e_j) = dp(e_j)e_i = e_j\, e_i = \delta_{ij}.$$

Taking this into account it is clear that $\omega_{ij} = \omega_{i*j*}$ and $\omega_{i*j} = -\omega_{ij*}$, so we have the structure equations

(1.3)

$$\begin{aligned} d\omega_i &= \sum_{k=1}^n \omega_{ik} \wedge \omega_k + \sum_{k=1}^n \omega_{ik*} \wedge \omega_{k*}, \qquad & d\omega_{i*} &= \sum_{k=1}^n \omega_{i*k} \wedge \omega_k + \sum_{k=1}^n \omega_{ik} \wedge \omega_{k*}, \\ d\omega_{ij} &= \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj} + \sum_{k=1}^n \omega_{ik*} \wedge \omega_{k*j}, \qquad & d\omega_{ij*} &= \sum_{k=1}^n \omega_{ik} \wedge \omega_{kj*} + \sum_{k=1}^n \omega_{ik*} \wedge \omega_{k*j*}. \end{aligned}$$

1.3. Definition. A vector subspace π of $\mathbb{R}^{2n}$ is said to be *holomorphic* relative to the complex structure J if π is invariant under J, i.e., $J\pi = \pi$.

A vector subspace π of $\mathbb{R}^{2n}$ is said to be *antiholomorphic* or *totally real* if $J\pi \subset \pi^\perp$.

By L_{2r+t} we will denote a subspace of $\mathbb{R}^{2n}$ that has a $2r$-dimensional holomorphic section and a t-dimensional antiholomorphic section.

1.4. Note. If we consider the symplectic structure of $\mathbb{R}^{2n}$ inherits from its complex structure we find that holomorphic subspaces correspond to *symplectic* subspaces, antiholomorphic subspaces to *isotropic* subspaces and L_{2r+t} with $r + t = n$ to *coisotropic* sub-spaces; [L-R].

1.5. Definition. A differentiable manifold M immersed in the euclidean space $\mathbb{R}^{2n}$, is called *anti-invariant (totally real)* submanifold of $\mathbb{C}^n$ if

$$\forall p \in M, \qquad J(T_pM) \subset (T_pM)^\perp.$$

An anti-invariant submanifold of $\mathbb{R}^{2n}$ is *isotropic* with respect to the symplectic structure.

1.6. Note. The complex motions form a group $\mathcal{M}^c$ defined by the equations,

$$x' = ax + b, \qquad a = A + iB \in U(n), \quad b = b_1 + ib_2 \in \mathbb{C}^n.$$

2.- Densities for linear spaces in $\mathbb{C}^n$

In order to define the total curvature for anti-invariant submanifolds immersed in $\mathbb{C}^n$, we need, as in the real case, to define a measure of the different linear spaces introduced before, invariant by $\mathcal{M}^c$.

These invariant densities will be obtained following the general method for homogeneous spaces, [SA-3].

2.1.- Density for holomorphic linear spaces.

We seek a density for holomorphic planes L_{2r} of real dimension $2r$, invariant under the group $\mathcal{M}^c$.

Let L^0_{2r} be a fixed holomorphic plane, spanned by the point p and the unit vectors $(e_1, \ldots, e_r, e_{1*}, \ldots, e_{r*})$; if we consider the subgroup of $\mathcal{M}^c$ formed by the complex motions that leave L^0_{2r} invariant, and that will be denoted by $\mathcal{H}_{2r}$; evidently, we have a bijective correspondence between the set of holomorphic subspaces and the elements of the homogeneous space $\mathcal{M}^c/\mathcal{H}_{2r}$, therefore we find that the density for holomorphic planes L_{2r} will be the density on the homogeneous space $\mathcal{M}^c/\mathcal{H}_{2r}$.

Since L^0_{2r} must keep invariant under $\mathcal{M}^c$, then

$$\omega_\alpha = \omega_{\alpha*} = \omega_{i\alpha} = \omega_{i\alpha*} = 0, \qquad \text{for } 1 \le i \le r, \quad r+1 \le \alpha \le n. \tag{2.1.1}$$

These forms are a completely integrable system and by general theory [SA-3], it is known that the invariant density for $\mathcal{M}^c/\mathcal{H}_{2r}$ becomes

$$dL_{2r} = \bigwedge_\alpha (\omega_\alpha \wedge \omega_{\alpha*}) \bigwedge_{i,\alpha} (\omega_{i\alpha} \wedge \omega_{i\alpha*}), \tag{2.1.2}$$

where $1 \le i \le r$, and $r+1 \le \alpha \le n$.

It is the unique form invariant under $\mathcal{M}^c$, up to a constant factor.

Density for holomorphic planes about a fixed plane.

The density for linear holomorphic spaces which contain a fixed holomorphic space; that is, we have that L_{2q} is invariant and L_{2r} variable with $L_{2q} \subset L_{2r}$, will be equal to the invariant volume element of $\mathcal{H}_{2q}/\mathcal{H}_{2r[2q]}$; where $\mathcal{H}_{2q}$ is the group of complex motions that maintain a fixed L^0_{2q} invariant and $\mathcal{H}_{2r[2q]}$ is the subgroup of $\mathcal{H}_{2q}$ formed by the elements that keep a fixed L^0_{2r} invariant, with $L^0_{2q} \subset L^0_{2r}$. Following the procedure we have used before, this density is

$$dL_{2r[2q]} = \bigwedge_{i,\alpha}(\omega_{i\alpha} \wedge \omega_{i\alpha *}), \quad \text{with} \quad q+1 \leq i \leq r; \quad r+1 \leq \alpha \leq n. \tag{2.1.3}$$

2.1. Note. For $q = 0$ we get the density for holomorphic $2r$-planes about a fixed point, which is the invariant volume element of the Grassmann manifold $G^c_{r,n}$,

$$\frac{U(n)}{U(r) \times U(n-r)}. \tag{2.1.4}$$

2.2. Note. The holomorphic planes L_{2r} around a totally real plane L_q with $q < r$, have the invariant density given by $dL_{2r[2q]}$; because J is an automorphism, and then, if we keep the vectors e_i fixed for $1 \leq i \leq q$, the vectors e_{i*} will also be fixed, for $1 \leq i \leq q$.

2.2.- Density for totally real subspaces.

Using a similar procedure as in 2.1, the invariant volume element for totally real subspaces in $\mathbb{C}^n$ will be the density on the homogeneous space $\mathcal{M}^c/\mathcal{H}_r$, where $\mathcal{H}_r$ is the subgroup of $\mathcal{M}^c$ whose elements are the complex motions that leave invariant a totally real plane L_r fixed, spanned by the vectors $(e_1, \ldots, e_r)$.

In order to keep L_r fixed, it is necessary that

$$\omega_\alpha = \omega_{\alpha *} = \omega_{j*} = \omega_{i\alpha} = \omega_{i\alpha *} = \omega_{ij*} = 0, \qquad \text{for} \quad 1 \leq i \leq j \leq r, \quad r+1 \leq \alpha \leq n. \tag{2.2.1}$$

And then, the density for anti-invariant linear spaces will be:

$$dL_r = \bigwedge_{\alpha}(\omega_\alpha \wedge \omega_{\alpha *}) \bigwedge_{j} \omega_{j*} \bigwedge_{i \leq j} \omega_{ij*} \bigwedge_{i,\alpha}(\omega_{i\alpha} \wedge \omega_{i\alpha *}), \tag{2.2.2}$$

where $1 \leq i, j \leq r$, and $r+1 \leq \alpha \leq n$.

Density for totally real r-planes about a fixed real q-plane.

The density for totally real planes L_r around a totally real plane L_q is given by

$$dL_{r[q]} = \bigwedge_{i \leq j} \omega_{ij*} \bigwedge_{i,\alpha}(\omega_{i\alpha} \wedge \omega_{i\alpha *}), \tag{2.2.3}$$

where $q+1 \leq i, j \leq r$, and $r+1 \leq \alpha \leq n$.

For $q = 0$ we get the density for totally real r-planes through the origin, which is the invariant volume element of the mixed Grassmann manifold

$$\frac{U(n)}{U(n-r) \times O(r)}. \tag{2.2.4}$$

2.3.- Density for subspaces L_{2r+t}.

Under the reasonings of the preceding sections, this density corresponds to the homogeneous space $\mathcal{M}^c/\mathcal{H}_{2r+t}$, where $\mathcal{H}_{2r+t}$ is the subgroup of $\mathcal{M}^c$ of all motions that keep a subspace L_{2r+t}, determined by the unit vectors $(e_1,\ldots,e_{r+t},e_{1*},\ldots,e_{r*})$, invariant; which is,

(2.3.1)
$$dL_{2r+t} = \bigwedge_{\alpha}(\omega_\alpha\wedge\omega_{\alpha*})\bigwedge_{j}\omega_{j*}\bigwedge_{i\le j}\omega_{ij*}\bigwedge_{i,\alpha}(\omega_{i\alpha}\wedge\omega_{i\alpha*})\bigwedge_{u,\alpha}(\omega_{u\alpha}\wedge\omega_{u\alpha*})\bigwedge_{u,i}(\omega_{ui}\wedge\omega_{ui*}),$$

where $1\le u\le r,\quad r+1\le i,j\le r+t$, and $r+t+1\le\alpha\le n$.

Density for linear spaces L_{2r+t} that leave a holomorphic $2q$-plane fixed.

Let L^0_{2q} be a fixed holomorphic subspace with $q<r$. The density for linear spaces L_{2r+t} that leave L^0_{2q} invariant is

$$dL_{2r+t[2q]} = \bigwedge_{i\le j}\omega_{ij*}\bigwedge_{i,\alpha}(\omega_{i\alpha}\wedge\omega_{i\alpha*})\bigwedge_{u,\alpha}(\omega_{u\alpha}\wedge\omega_{u\alpha*})\bigwedge_{u,i}(\omega_{ui}\wedge\omega_{ui*}), \tag{2.3.2}$$

where $q+1\le u\le r,\quad r+1\le i,j\le r+t$, and $r+t+1\le\alpha\le n$.

Density for linear spaces L_{2r+t} about a fixed totally real q-plane L^0_q.

We will distinguish two cases:

A) L^0_q is contained in the holomorphic section of L_{2r+t}; then, as in Note 2.2

$$dL_{2r+t[q]} = dL_{2r+t[2q]}. \tag{2.3.3.}$$

B) L^0_q is in the totally real section of L_{2r+t}, then

$$dL_{2r+t[q]} = \bigwedge_{i\le j}\omega_{ij*}\bigwedge_{i,\alpha}(\omega_{i\alpha}\wedge\omega_{i\alpha*})\bigwedge_{u,\alpha}(\omega_{u\alpha}\wedge\omega_{u\alpha*})\bigwedge_{u,i}(\omega_{ui}\wedge\omega_{ui*}), \tag{2.3.4}$$

where $1\le u\le r,\qquad r+q+1\le i,\qquad j\le r+t$, and $r+t+1\le\alpha\le n$.

In both cases, for $q=0$, we have the density for the linear spaces L_{2r+t} through the origin, that is, the invariant volume element of the mixed Grassmannian

$$\frac{U(n)}{U(r)\times U(n-r-t)\times O(t)}. \tag{2.3.5}$$

Note 2.3. If $r+t=n$, we have the duality between the totally real r-planes and their orthogonal complement L_{2r+t},

$$dL_{2r+t[0]} = dL_{t[0]}, \tag{2.3.6}$$

as follows immediately from (2.2.4) and Note 1.2.

The verification that all the forms defined in this section are densities is immediate using the structure equations (1.3) to verify that their exterior differentials vanish, which is a necessary and sufficient condition, [SA-3].

3.- Total curvatures of compact anti-invariant submanifolds in $\mathbb{C}^n$

The purpose of this section is to define the total curvatures of a compact totally real (anti-invariant) manifold M, in the euclidean space $\mathbb{C}^n$, and to express them as the integral over the manifold of the absolute value of certain curvatures defined in each point of M.

In [SA-1,2], we can find these curvatures, for compact manifolds immersed in the real euclidean space $\mathbb{R}^n$.

3.1.- The total absolute curvatures.

Let M be a compact, totally real C^∞-differentiable manifold of dimension q, without boundary, immersed in euclidean complex space $\mathbb{C}^n$.

In order to define the r-th total absolute curvature of M, we must distinguish several cases.

Case I.- We are going to consider that the tangent space at each point of M, either contains a *totally real linear space,* or it is included in a *totally real linear space.*

I.a) Case $1 \leq r \leq q$.

Let $L_{2n-r[0]}$ be a $(2n-r)$-dimensional linear space through the origin, with a holomorphic section of dimension $2(n-r)$ and a totally real section of dimension r.

Let Γ_r denotes the set of all totally real linear subspaces L_r in $\mathbb{C}^n$ which are contained in some tangent space T_pM, passing through p and orthogonal to $L_{2n-r[0]}$.

Since M is compact the intersection $\Gamma_r \cap L_{2n-r[0]}$ will be a compact submanifold of $L_{2n-r[0]}$. Let $\mu(\Gamma_r \cap L_{2n-r[0]})$ be the measure of this manifold as submanifold of $L_{2n-r[0]}$. If the dimension of this submanifold is 0, then μ means the number of intersection points of $\Gamma_r \cap L_{2n-r[0]}$.

3.1. Definition. We define the r-th total absolute curvature of $M \subset \mathbb{C}^n$ as the mean value of the measures μ for all $L_{2n-r[0]}$; that is,

$$K_r(M) = \frac{\int \mu(\Gamma_r \cap L_{2n-r[0]}) dL_{2n-r[0]}}{\int dL_{2n-r[0]}}, \tag{3.1.1}$$

where the integration is over the mixed Grassmannian given by (2.2.4).

I.b) Case $q < r \leq n$.

Using the same notations as above, denoting now Γ_r the set of all totally real r-planes which contain some T_pM and are orthogonal to $L_{2n-r[0]}$, the total absolute curvatures are defined by the same formulae (3.1.1).

Case II.- We are going to consider that the tangent space at each point of M is contained in a *holomorphic linear space.*

II.a) $q \leq r < n$.

In this case Γ_r denotes all the holomorphic $2r$-dimensional planes which contain some tangent space T_pM and are orthogonal to a holomorphic subspace $L_{2(n-r)[0]}$, and so the definition of $K_{2r}(M)$ is the same as in (3.1.1), but integrating over the complex Grassmann manifold (2.1.4).

Case III.- We consider that the tangent space at each point of M is contained in a *linear space* L_{2s+t}.

III.a) $q < s + t < n$.

Now, Γ_r with $r = 2s + t$, denotes the set of all the planes L_{2s+t} that contain some T_pM an are orthogonal to a space of complementary dimension, and the total absolute curvatures are defined by (3.1.1) having to account that $r = 2s + t$.

3.2 Note. Since the measures are non negative, all the totally absolute curvatures are greater or equal than zero. We are interested only in the non-vanishing curvatures, which correspond to the cases that dimension of $\Gamma_r \cap L_{2n-r[0]}$ is non negative.

3.3 Proposition. *The totally absolute curvatures K_r, defined in case I.a), are greater than zero if $q = n$, and $r = 1$.*

In cases I.b), II.a) and III.a) all the curvatures K_r vanish.

Proof. We have seen in note 3.2 that $K_r \geq 0$ in every case. We now proceed to compute the dimension of $\Gamma_r \cap L_{2n-r[0]}$.

Since to each r-plane L_r orthogonal to $L_{2n-r[0]}$ corresponds one and only one intersection point with $L_{2n-r[0]}$, the dimensions of $\Gamma_r \cap L_{2n-r[0]}$ coincides with the dimension of Γ_r as submanifold of the set of totally real linear spaces in $\mathbb{C}^n$.

In order to obtain the dimension of Γ_r in case *I.a)*, we consider the following manifolds;

Let G_r be the set of all totally real r-planes in $\mathbb{C}^n$,then

$$dim(G_r) = \frac{1}{2}(r(r+1)) + 2r(n-r) + r + 2(n-r). \tag{3.1.2}$$

Let H_r be the set of all r-planes L_r in $\mathbb{C}^n$ which are contained in some T_pM passing through p. If we consider T_pM as a real linear space, all the r-planes through p in T_pM form the real Grassmann manifold $G_{r,q}$ so the dimension of H_r is $r(q-r) + q$.

We denote by P_r, the totally real subspaces L_r that are orthogonal to $L_{2n-r[0]}$; then, $dim(P_r) = 2n - r$.

H_r and P_r are submanifolds of G_r; and Γ_r is the intersection of H_r and P_r consequently, the dimension of Γ_r, is

$$dim(\Gamma_r) = 2n - r + r(q-r) + q - (\frac{1}{2}(r(r+1)) + 2r(n-r) + r + 2(n-r)). \tag{3.1.3}$$

Simplifying this expression we get, $dim(\Gamma_r) = \frac{1}{2}r^2 - \frac{1}{2}r + rq + q + 2rn$, which is zero if $q = n$, $r = 1$ and negative for the rest of values of r, q and n.

Hence, in order that $K_r \neq 0$, it is necessary and sufficient that $q = n$, $r = 1$.

Now, we will study the dimension of Γ_r in the other cases. In case *I.b)*, we will consider G_r and P_r as in the preceding case, and H_r will be the set of all r-planes L_r in $\mathbb{C}^n$ which contain a fixed T_pM as q-linear space, then, the dimension of H_r is the degree of the form (2.2.3), i.e.

$$2(r-q)(n-r) + \frac{1}{2}(r-q)(r-q-1) + q,$$

so the dimension of Γ_r is

$$(3.1.4)\ 2n-r+2(r-q)(n-r)+\frac{1}{2}(r-q)(r-q-1)+q-(\frac{1}{2}(r(r+1))+2r(n-r)+r+2(n-r)),$$

which is greater or equal than zero if and only if $2r-4n+q+1\geq 0$, but this is not possible when one takes into account that $q<r\leq n$.

For case $II.a)$, the submanifolds G_r, P_r and H_r are the same as in case $I.b)$, but using holomorphic planes L_{2r}, so we are left with $dim(G_r)=2(r+1)(n-r)$, $dim(P_r)=2(n-r)$, $dim(H_r)=2(r-q)(n-r)+q$ and we get

$$(3.1.5)\qquad dim(\Gamma_r)=2(r-q)(n-r)+q+2n-2r-(2r(n-r)+2(n-r)),$$

which is non-negative if $2r-2n+1\geq 0$, but this is impossible for $q\leq r<n$.

Finally, using similar procedures in case $III.a)$ as in the preceding cases we get that $dim(\Gamma_r)$ is negative if $q<s+t<n$. $\square$

Expression of the total absolute curvature $K_1(M)$.

From definition (3.1.1) and proposition (3.3), and using the equivalence (2.3.6), we deduce that only one total absolute curvature doesn't vanish; and this is

$$(3.1.6)\qquad K_1(M)=\frac{\int_G \mu dL_{1[0]}}{\int_G dL_{1[0]}};\qquad \text{where}\qquad G=\frac{U(n)}{U(n-1)\times O(1)},$$

and Γ_1 is the set of real lines in $\mathbb{C}^n$, which are contained in some tangent space $T_p(M)$, passing through p and orthogonal to $L_{2n-1[0]}$; and μ is the number of intersection points of Γ_1 with $L_{2n-1[0]}$.

We now compute the volume of G, in order to simplify this expression, i.e.

$$(3.1.7)\qquad Vol(G)=\int_G dL_{1[0]}=\frac{Vol(U(n))}{Vol(U(n-1))\times Vol(O(1))}.$$

Let $(e_1,\ldots,e_n,e_{1*},\ldots,e_{n*})$ be a unitary frame in $\mathbb{R}^{2n}$, the volume element of $U(n)$ is given by

$$(3.1.8)\qquad \Omega=\bigwedge_{1\leq i<j\leq n}\omega_{ij}\wedge\omega_{ij*}\bigwedge_{1\leq k\leq n}\omega_{kk*}.$$

We have the following relation between Ω and the density for $U(n)$, obtained in [SA-4], Ω^c:

$$(3.1.9)\qquad \Omega^c=\left(\sqrt{-1}\right)^{\frac{n(n+1)}{2}}\,2^{\frac{n(n-1)}{2}}\Omega.$$

Taking this into account, we find that the volume of $U(n)$ with respect to the form Ω is

$$(3.1.10)\qquad Vol(U(n))=\int_G\Omega=2^n\prod_{r=1}^{n}\frac{\pi^r}{(r-1)!}.$$

On the other hand, $Vol(O(1))=1$ if the orientation is considered.

Then, we can express the total curvature $K_1(M)$ as

$$(3.1.11)\qquad K_1(M)=\frac{(n-1)!}{2\pi^n}\int_G\mu dL_{1[0]}.$$

3.2.- Local representation of $K_1(M)$. Total (non absolute) curvatures.

Let $(e_1, \ldots, e_n, e_{1*}, \ldots, e_{n*})$ be a unitary frame in $\mathbb{R}^{2n}$, where $(e_1, \ldots, e_n)$ span the tangent space to M in the point x; then, the density for totally real lines through x is given by (2.2.4) which we will write now

$$dL_{1[x]} = \omega_{1i*} \wedge \omega_{1j}, \qquad i = 1, \ldots, n, \qquad j = 2, \ldots, n.$$

By rotation about the point x, the end point of the vector e_1 moves on the unit sphere S^{2n-1} centered at x.

The scalar products $de_1 e_j = \omega_{1j}$, $de_1 e_{i*} = \omega_{1i*}$, are equal to the arc elements of S^{2n-1} at the end point of e_1 in the direction of e_j and e_{i*}, respectively; so that, denoting by dS^{2n-1} the $(2n-1)$-dimensional volume element of S^{2n-1} at the end point of e_1, we have

$$dS^{2n-1} = \omega_{1i*} \wedge \omega_{1j}, \qquad i = 1, \ldots, n, \qquad j = 2, \ldots, n.$$

and $K_1(M)$ is the mean value of μ over the unit sphere S^{2n-1}; which is the same expression that L. A. Santaló has obtained for total curvatures of n-dimensional submanifolds in $\mathbb{R}^{2n}$, [SA-2].

On the other hand, the assumption that $(e_1, \ldots, e_n)$ are tangents to M gives $\omega_{i*} = 0$. Taking this and formula (1.3) into account, and according to Cartan's lemma, we have

$$\omega_{ij*} = h^{j*}_{ik}\,\omega_k, \qquad h^{j*}_{ik} = h^{j*}_{ki}, \tag{3.2.1}$$

where h^{j*}_{ik} are the coefficients of the second fundamental form with respect to e_{j*}, which satisfies,

$$h^i_{jk} = h^j_{ik} = h^k_{ij}; \tag{3.2.2}$$

where we write h^{j*}_{ik} as h^j_{ik} following [K-M].

Moreover,

$$R_{ijhk} = R_{i*j*hk} = \sum_{r=1}^{n} \left(h^r_{ik}\, h^r_{jh} - h^r_{ih}\, h^r_{jk} \right). \tag{3.2.3}$$

The local representation of $K_1(M)$ will be, as in the real case, the integral over the manifold M of

$$Q_1(x) = \frac{(n-1)!}{2\pi^n} \int_{G_{1,n}} |f(x, e_1)| dL_{1[x]}, \tag{3.2.4}$$

where $f(x, e_1)$ is, using (3.2.2), the determinant of the symmetric matrix

$$\begin{pmatrix} h^1_{11} & h^1_{12} & \ldots & h^1_{1n} \\ h^1_{12} & h^1_{22} & \ldots & h^1_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h^1_{1n} & h^1_{2n} & \ldots & h^1_{nn} \end{pmatrix}.$$

The total (no absolute) curvature $K_1^*(M)$ is defined as the integral over the manifold M of

$$Q_1^*(x) = \frac{(n-1)!}{2\pi^n} \int_{G_{1,n}} f(x, e_1)\, dL_{1[x]}. \tag{3.2.5}$$

By the same reasoning as in [SA-2], and using (3.2.3), we obtain that

$$Q_1^*(x) = \frac{1}{2^n\,(2\pi)^{n/2}\,(n/2)!} \delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n} R_{i_1 i_2 j_1 j_2} \ldots R_{i_{n-1} i_n j_{n-1} j_n} \qquad (n \text{ even}), \tag{3.2.6}$$

where $\delta^{i_1 i_2 \ldots i_n}_{j_1 j_2 \ldots j_n}$ is equal to $+1$ or -1 according as $(i_1 i_2 \ldots i_n)$ is an even or odd permutation of $(j_1 j_2 \ldots j_n)$ and is otherwise zero. The summation of all $i_1, i_2, \ldots i_n$ and $j_1, j_2, \ldots j_n$ is from 1 to n.

If n is odd, then $Q_1^*(x) = 0$.

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Departament de Matemàtiques i Informàtica, Universitat Jaume I, 12071 CASTELLON, SPAIN.

E-mail: GUAL@ GREGAL.UJI.ES

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