THE OVERALL PROPERTIES OF PIEZOCOMPOSITE MATERIALS WITH CYLINDRICAL FIBRES

J. B. Castillero, J. A. Otero, R. R. Ramos, R. G. Diaz
Institute of Cybernetics, Mathematics and Physics (ICIMAF)
Acoustic Group, Calle D, No. 353 e/t 15 y 17
Vedado, Havana, Cuba.

Abstract

The purpose of this paper is to apply the technique of asymptotic homogenization to determine the effective elastic, piezoelectric and dielectric moduli of reinforced piezoelectric composite materials with unidirectional cylindrical fibres with periodic structure. Each periodic cell of the medium is a binary piezoelectric composite wherein both constituents are homogeneous piezoelectric materials with hexagonal symmetry (6mm). The analytical solution made use of series expansions of the doubly periodic Weierstrass elliptic functions to obtain overall averaged coefficients of the piezocomposite material. This result is a generalization of the main formulae published by [11] (Chapter 6) for the purely elastic case.

1 Introduction

In recent years, new piezoelectric composite materials have been intensively developed to join the requirements of electro- and hydroacoustic for more effective transducer materials. By varying the proportion of several homogeneous constituents one should be able to obtain effective macroscopic materials of desired homogenized properties, e.g. a piezoelectric inactive polymer matrix - piezoelectric ceramic fibres, as in [6], [3], [12], [13] and more recently, piezoelectric ceramic - piezoelectric polymer composites in [14] .
Different techniques have been adopted in the literature to estimate the effective electroelastic properties of fibrous composites. For instance, [5] derived the effective moduli of a transversaly isotropic piezocomposite with cylindrical inclusions based on the hypothesis of equivalent homogeneity. Benveniste and Dvorak ([2]) derived the effective behaviour of a binary piezoelectric composite with unidirectional fibres based on the theory of the creation of uniform fields

in heterogeneous media by proper boundary conditions. Moreover, the reader can find an extensive list of references on others methods in [4] .
The asymptotic averaging method ([1]) have been used for the determination of the effective coefficients of fibrous composites with isotropic phases by [11] , [8], [9]. The aim of this contribution is to present our preliminary results concerning effective properties of piezocomposite. In the present paper by means of this method of homogenization the analytical expressions for the effective coefficients of binary homogeneous piezoelectric composite materials with unidirectional cylindrical fibres have been obtained using the theory of functions of complex variable.

2 Local Problems and Effective Coefficients

The main problem of the asymptotic homogenization, for piezoelectric composite materials, consists finding the local periodic functions N_{pmn}, M_{mn}, Φ_{pn} and P_n from the following local problems (see [15], [16], Galka et al. (1992), [10]):
P^1_{loc}

$$\begin{aligned}(C_{ijpq}(x)N_{pmn,q}(\xi)+e_{pij}(\xi)M_{mn,p}(\xi))_{,j} &= -C_{ijmn,j}(\xi)\\ (e_{ipq}(\xi)N_{pmn,q}(\xi)-\epsilon_{ip}(\xi)M_{mn,p}(\xi))_{,i} &= -e_{imn,i}(\xi)\end{aligned} \tag{1}$$

P^2_{loc}

$$\begin{aligned}(C_{ijpq}(\xi)\Phi_{pn,q}(\xi)+e_{pij}(\xi)P_{n,p}(\xi))_{,j} &= -e_{nij,j}(\xi)\\ (-e_{ipq}(\xi)\Phi_{pn,q}(\xi)+\epsilon_{ip}(\xi)P_{n,p}(\xi))_{,i} &= -\epsilon_{in,i}(\xi)\end{aligned} \tag{2}$$

The moduli: $C_{ijpq}(\xi)$ (elastic), $e_{pij}(\xi)$ (piezoelectric), $\epsilon_{ip}(\xi)$ (dielectric) satisfy the usual symmetry conditions

$$C_{ijkl}=C_{klij}=C_{jikl}=C_{ijlk},\quad e_{kij}=e_{kji},\quad \epsilon_{ij}=\epsilon_{ji}. \tag{3}$$

$\xi=(\xi_1,\xi_2,\xi_3)$ is the local variable on the periodic cell. The subscripts assume the values 1-3 and the comma denotes partial differentiation. The uniqueness of the solution of the above local problems is achieved by imposing the conditions:

$$\langle N_{pmn}(\xi)\rangle=0\ ,\ \langle M_{mn}(\xi)\rangle=0\ ,\ \langle \Phi_{pn}(\xi)\rangle=0\ ,\ \langle P_n(\xi)\rangle=0 \tag{4}$$

where $\langle f\rangle=\frac{1}{|Y|}\int_Y f dY$, Y is the periodic cell.
Once solved the local problems, it is possible to determine the homogenized (effective) moduli by using the following expressions:

$$\overline{C}_{ijmn} = \langle C_{ijmn}+C_{ijpq}N_{pmn,q}+e_{pij}M_{mn,p}\rangle, \tag{5}$$

$$\overline{e}_{imn} = \langle e_{imn}+e_{ipq}N_{pmn,q}-\epsilon_{ip}M_{mn,p}\rangle, \tag{6}$$

$$\overline{\epsilon}_{in} = \langle \epsilon_{in}+\epsilon_{ip}P_{n,p}-e_{ipq}\Phi_{pn,q}\rangle. \tag{7}$$

3 Formulation of the problem

The previously described homogenization technique is capable of treating the three-dimensional piezoelectric behaviour of reinforced piezocomposite materials with periodic structure. In order to demonstrate the applicability of this method, the unidirectional fibre-reinforced composite shown in Fig. 1 is considered.
In Fig. 1, the periodic cell of the fibre-reinforced piezocomposite material consists of a single circular fibre " F" embeded into a square matrix " M". In the regions " F" and " M" in virtue of the homogeneity of the components of the piezocomposite we can obtain, from the equations (1), (2), the following systems:
Equations I_{pq}:

$$\begin{aligned}(C_{ijkl}(\xi)N_{kpq,l}(\xi) + e_{kij}(\xi)M_{pq,k}(\xi))_{,j} = 0\\ (e_{ikl}(\xi)N_{kpq,l}(\xi) - \epsilon_{ik}(\xi)M_{pq,k}(\xi))_{,i} = 0\end{aligned} \tag{8}$$

Equations I_n:

$$\begin{aligned}(C_{ijkl}(\xi)\Phi_{kn,l}(\xi) + e_{kij}(\xi)P_{n,k}(\xi))_{,j} = 0\\ (e_{ikl}(\xi)\Phi_{kn,l}(\xi) - \epsilon_{ik}(\xi)P_{n,k}(\xi))_{,i} = 0\end{aligned} \tag{9}$$

The solutions of I_{pq} and I_n must be double periodic in ξ_1 and ξ_2 subjected to the following perfect bonding conditions at the interface Γ:
For equations I_{pq}:

$$N^1_{kpq}|_\Gamma = N^2_{kpq}|_\Gamma\ , \qquad M^1_{pq}|_\Gamma = M^2_{pq}|_\Gamma, \tag{10}$$

$$\begin{aligned}(\sigma^1_{ij\,(pq)} + C^1_{ijpq}n^1_j|_\Gamma = -(\sigma^2_{ij\,(pq)} + C^2_{ijpq})n^2_j|_\Gamma,\\ (D^1_{j(pq)} + e^1_{jpq})n^1_j|_\Gamma = -(D^2_{j(pq)} + e^2_{jpq})n^2_j|_\Gamma,\end{aligned} \tag{11}$$

For equations I_n:

$$\Phi^1_{pn}|_\Gamma = \Phi^2_{pn}|_\Gamma\ , \qquad P^1_n|_\Gamma = P^2_n|_\Gamma, \tag{12}$$

$$\begin{aligned}(\sigma^1_{ij\,(n)} + e^1_{ij\,n})n^1_j|_\Gamma &= -(\sigma^2_{ij\,(n)} + e^2_{ij\,n})n^2_j|_\Gamma,\\ (D^1_{j(n)} - \epsilon^1_{jn})n^1_j|_\Gamma &= -(D^2_{j(n)} - \epsilon^2_{jn})n^2_j|_\Gamma.\end{aligned} \tag{13}$$

where

$$\begin{aligned}
\sigma_{ij(pq)} &= C_{ijk\ell}N_{kpq,\ell} + e_{kij}M_{pq,k}\,,\\
\sigma_{ij(n)} &= C_{ijk\ell}\Phi_{kn,\ell} + e_{kij}P_{n,k}\,,\\
D_{j(pq)} &= e_{ik\ell}N_{kpq,\ell} - \epsilon_{ik}M_{pq,k}\,,\\
D_{j(n)} &= e_{ik\ell}\Phi_{kn,\ell} - \epsilon_{ik}P_{n,k}\,,
\end{aligned}$$

n_j are components of the outward unit normal vector to the interface, the superscripts "1" and "2" denotes the values of the functions over the matrix and the fibre respectively.
In general, the exact number of problems of the type I_{pq} and I_n that must be solved in each particular case will depend on the symmetry properties of the matrix and fibre materials. If both are homogeneous piezoelectric materials with hexagonal symmetry (6mm), then, it is possible to obtain the overall averaged coefficients by solving five similar strain problems involving $I_{\beta\beta}(\beta = 1,2,3)$, I_{12} and I_3; and two similar "antiplane" problems involving I_{13} and I_1. The materials with hexagonal symmetry (6mm) are characterized by the following independent constants: five elastic constants $(C_{1111} = C_{2222}$, C_{1122}, $C_{1133} = C_{2233}$, C_{3333}, $C_{2323} = C_{1313}$, $C_{1212} = \frac{(C_{1111}-C_{1122})}{2})$, three piezoelectric constants $(e_{311} = e_{322}$, e_{333}, $e_{113} = e_{223})$, and two dielectric constants $(\epsilon_{11} = \epsilon_{22}$, $\epsilon_{33})$.

4 The antiplane problems

Let us consider the **problem** I_{13}. For this problem we can assume that $N^\alpha_{313} \equiv N^\alpha_3$ and $M^\alpha_{13} \equiv M^\alpha$ are independent of ξ_3, and $N^\alpha_{113} = N^\alpha_{213} = 0$. The corresponding problem can be expressed as:

$$C^\alpha_{1313}N^\alpha_{3,KK} = 0, \qquad e^\alpha_{113}M^\alpha_{,KK} = 0, \tag{14}$$

$$N^1_3|_\Gamma = N^2_3|_\Gamma\,, \qquad M^1|_\Gamma = M^2|_\Gamma, \tag{15}$$

$$\begin{aligned}
(\sigma^1_{3J(13)}n^1_J + C^1_{1313}n^1_1)|_\Gamma &= -(\sigma^2_{3J(13)}n^2_J + C^2_{1313}N^2_1)|_\Gamma,\\
(D^1_{J(13)}n^1_J + e^1_{113}n^1_1)|_\Gamma &= -(D^2_{J(13)}n^2_J + e^2_{113}n^2_1)|_\Gamma,
\end{aligned} \tag{16}$$

with $\alpha, J, K = 1,2$ and

$$\sigma_{3J(13)} = C_{1313}N_{313,J} + e_{113}M_{13,J}\ ,\ \sigma_{33(13)} = 0\ ,\ \sigma_{IJ(13)} = 0,$$
$$D_{J(13)} = e_{113}N_{313,J} - \epsilon_{11}M_{13,J}\ ,\ D_{3(13)} = 0,$$

$I, J = 1, 2$. The solution of (14)-(16) can be achieved by representing the harmonic functions N_3^α and M^α by the following series expansion:

$$N_3^1 = Re\left\{A\frac{z}{R} + \frac{1}{C_{1313}^1} f_1(z)\right\}, \tag{17}$$

$$N_3^2 = \frac{1}{C_{1313}^2} Re\{f_2(z)\}, \tag{18}$$

$$M^1 = Re\left\{B\frac{z}{R} + \frac{1}{e_{113}} g_1(z)\right\}, \tag{19}$$

$$M^2 = \frac{1}{e_{113}^2} Re\{g_2(z)\}, \tag{20}$$

where

$$f_1(z) = \sum_{k=1}^{\infty}{}^{*} a_k R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!}\ ,\ f_2(z) = \sum_{k=1}^{\infty}{}^{*} c_k \left(\frac{z}{R}\right)^k,$$

$$g_1(z) = \sum_{k=1}^{\infty}{}^{*} b_k R^k \frac{\zeta^{(k-1)}(z)}{(k-1)!}\ ,\ g_2(z) = \sum_{k=1}^{\infty}{}^{*} d_k \left(\frac{z}{R}\right)^k. \tag{21}$$

Here, $\zeta(z)$ is Weierstrass's zeta function, satisfying the quasiperiodicity conditions: $\zeta(z+1) - \zeta(z) = \pi$, $\zeta(z+\mathbf{i}) - \zeta(z) = -\mathbf{i}\pi$. The constants A, B, a_k, c_k, b_k, $d_k (k = 1, 3, 5, ..)$ are real, $\sum^*$ means the sum for the odd subscripts,

$$\zeta(z) = \frac{1}{z} + \sum_{m,n}{}' \left\{\frac{1}{z - \beta_{mn}} + \frac{1}{\beta_{mn}} + \frac{z}{\beta_{mn}^2}\right\}$$

where $\sum'_{m,n}$ means that m, n are integers except m=n=0, $\beta_{mn} = m + \mathbf{i}n$, and $\mathbf{i}^2 = -1$.

Taking into account, the periodicity conditions for N_3 and M the constants A and B are related to a_1 and b_1 respectively by the formulae

$$A = -\frac{\pi R^2 a_1}{C_{1313}^1}\ , \qquad B = -\frac{\pi R^2 b_1}{e_{113}^1}$$

The perfect bonding conditions at the interface Γ, (15) and (16), can be written in the following form:

$$A\frac{t+\overline{t}}{2R} + \frac{1}{C_{1313}^1} Re\{f_1(t)\} = \frac{1}{C_{1313}^1} Re\{f_2(t)\}, \tag{22}$$

$$B\frac{t+\overline{t}}{2R} + \frac{1}{e_{113}^1} Re\{g_1(t)\} = \frac{1}{e_{113}^2} Re\{g_2(t)\}, \tag{23}$$

$$Im\{f_1(t)+g_1(t)\}+\left(\frac{C^1_{1313}}{R}A-\frac{e^1_{113}}{R}B+C^1_{1313}-C^2_{1313}\right)\frac{t-\overline{t}}{2\mathbf{i}}= Im\{f_2(t)+g_2(t)\}, \quad (24)$$

$$Im\left\{\frac{e^1_{113}}{C^1_{1313}}f_1(t)-\frac{\epsilon^1_{11}}{e^1_{113}}g_1(t)\right\} + \left(\frac{e^1_{113}}{R}A-\frac{\epsilon^1_{11}}{R}B+e^1_{113}-e^2_{113}\right)\frac{t-\overline{t}}{2\mathbf{i}}= Im\{\frac{e^2_{113}}{C^2_{1313}}f_2(t)-\frac{\epsilon^2_{11}}{e^2_{113}}g_2(t)\}, \quad (25)$$

where $t = Re^{\mathbf{i}\theta}$, "Re" means the real part and "Im" means the imaginary part.
The Laurent series expansions of the functions N^1_3 and M^1 about the point $z = 0$ are given by

$$N^1_3 = Re\left\{z\frac{A}{R}+\frac{1}{C^1_{1313}}\left[\sum_{k=1}^{\infty\ *}a_k\left(\frac{R}{z}\right)^k+\sum_{p=1}^{\infty\ *}\sum_{k=1}^{\infty\ *}a_k\eta_{kp}\left(\frac{z}{R}\right)^p\right]\right\} \quad (26)$$

$$M^1 = Re\left\{z\frac{B}{R}+\frac{1}{e^1_{113}}\left[\sum_{k=1}^{\infty\ *}b_k\left(\frac{R}{z}\right)^k+\sum_{p=1}^{\infty\ *}\sum_{k=1}^{\infty\ *}b_k\eta_{kp}\left(\frac{z}{R}\right)^p\right]\right\}, \quad (27)$$

where,

$$\eta_{kp} \equiv \frac{(-1)^k(p+k-1)!}{p!(k-1)!}R^{p+k}S_{p+k},$$

$$S_{p+k} \equiv \sum_{m,n}\frac{1}{(\beta_{mn})^{p+k}}.$$

Substituting the expansions (18), (20), (26) and (27) into the conditions (22)-(25) on the interface Γ, the following system of algebraic equations for the constants a_k and b_k (k=1,3,5,...) can be obtained:

$$\frac{C^2_{1313}+C^1_{1313}}{C^1_{1313}}a_k+\frac{e^2_{113}+e^1_{113}}{e^1_{113}}b_k+$$
$$\frac{C^2_{1313}-C^1_{1313}}{C^1_{1313}}(\sum_{n=1}^{\infty}{}^{*}a_n\eta_{nk}-$$
$$\pi R^2a_1\delta_{1k})+\frac{e^2_{113}-e^1_{113}}{}$$

$$e^1_{113}(\sum_{n=1}^{\infty}{}^{*}b_n\eta_{nk}-\pi R^2b_1\delta_{1k})=(C^1_{1313}-C^2_{1313})R\delta_{1k},$$

$$\frac{e^1_{113}+e^2_{113}}{C^1_{1313}}a_k-\frac{\epsilon^1_{11}+\epsilon^2_{11}}{e^1_{113}}b_k+\frac{e^2_{113}-e^1_{113}}{C^1_{1313}}(\sum_{n=1}^{\infty}{}^{*}a_n\eta_{nk}-$$

$$\pi R^2a_1\delta_{1k})-\frac{\epsilon^2_{11}-\epsilon^1_{11}}{e^1_{113}}(\sum_{n=1}^{\infty}{}^{*}b_n\eta_{nk}-\pi R^2b_1\delta_{1k})=(e^1_{113}-e^2_{113})R\delta_{1k}. \quad (28)$$

Having solved the above system, one can determine the constants c_k and d_k (k=1, 3, 5, ...) by

$$c_k = \frac{C^2_{1313}}{C^1_{1313}}(-\pi R^2a_1\delta_{1k}+a_k+\sum_{k=1}^{\infty}{}^{*}a_n\eta_{nk}),$$
$$d_k = \frac{e^2_{113}}{e^1_{113}}(-\pi R^2b_1\delta_{1k}+b_k+\sum_{k=1}^{\infty}{}^{*}b_n\eta_{nk}).$$

On the other hand, from formulae (5), (6) the following expressions for the effective coefficients $\overline{C}_{1313}$ and $\overline{e}_{113}$ can be obtained:

$$\overline{C}_{1313} = \langle C_{1313}+C_{1313}N_{3,1}+e_{113}M_{,1}\rangle, \quad (29)$$
$$\overline{e}_{113} = \langle e_{113}+e_{113}N_{3,1}-\epsilon_{11}M_{,1}\rangle. \quad (30)$$

Now, solving the above integrals (29), (30) and substituting the expansions (26) and (27) we obtain:

$$\overline{C}_{1313}=$$
$$\langle C_{1313}\rangle+\frac{C^2_{1313}-C^1_{1313}}{C^1_{1313}}\pi R(a_o+\overline{A})+\frac{e^2_{113}-e^1_{113}}{e^1_{113}}\pi R(b_o+\overline{B}),$$
$$\overline{e}_{113}=$$
$$\langle e_{113}\rangle+\frac{e^2_{113}-e^1_{113}}{C^1_{1313}}\pi R(a_o+\overline{A})-\frac{\epsilon^2_{11}-\epsilon^1_{11}}{e^1_{113}}\pi R(b_o+\overline{B}),$$

where $a_o \equiv -a_1 R^2\pi$, $b_o \equiv -b_1 R^2\pi$ and $\overline{A} \equiv a_1 + \sum_{n=1}^{\infty}{}^{*} a_n \eta_{n1}$, $\overline{B} \equiv b_1 + \sum_{n=1}^{\infty}{}^{*} b_n \eta_{n1}$.
Finally, by using the equations (28) for $k = 1$, we can obtain

$$\overline{C}_{1313} = C^1_{1313} - 2\pi R(a_1 + b_1), \tag{31}$$

$$\overline{e}_{113} = e^1_{113} - 2\pi R(\frac{e^1_{113}}{C^1_{1313}} a_1 - \frac{\epsilon^1_{11}}{e^1_{113}} b_1). \tag{32}$$

The problem I_1: For this problem we assume that $\phi^{\alpha}_{n(1)} = 0$ (n=1,2) and $\phi^{\alpha}_{3(1)}$, P^{α}_1 are independent of ξ_3. The equations can be expressed as

$$e^{\alpha}_{113}\phi^{\alpha}_{3(1),kk} = 0; \qquad -\epsilon^{\alpha}_{11} P^{\alpha}_{(1),kk} = 0, \tag{33}$$

and the conditions at the interface Γ are

$$\phi^1_{3(1)}|_{\Gamma} = \phi^2_{3(1)}|_{\Gamma}; \qquad P^1_{(1)}|_{\Gamma} = P^2_{(1)}|_{\Gamma} \tag{34}$$

$$(\sigma^1_{3J(1)} n^1_J + e^1_{113} n^1_1)|_{\Gamma} = -(\sigma^2_{3J(1)} n^2_J + e^2_{113} n^2_1)|_{\Gamma},$$

$$(D^1_{J(1)} n^1_J - \epsilon^1_{11} n^1_1)|_{\Gamma} = -(D^2_{J(1)} n^2_J - \epsilon^2_{11} n^2_1)|_{\Gamma}, \tag{35}$$

with

$$\sigma_{IJ(1)} = 0, \qquad \sigma_{3J(1)} = C_{1313}\phi_{3(1),J} + e_{113} P_{(1),J},$$

$$D_{3(1)} = 0, \qquad D_{J(1)} = e_{113}\phi_{3(1),J} - \epsilon_{11} P_{(1),J},$$

$(I, J, K = 1, 2)$.
The solution of this problem is proposed as

$$\phi^1_{3(1)} = Re\{A\frac{z}{R} + \frac{1}{e^1_{113}} f_1(z)\}, \phi^2_{3(1)} = \frac{1}{e^2_{113}} Re\{f_2(z)\},$$

$$P^1_{(1)} = Re\{B\frac{z}{R} - \frac{1}{\epsilon^1_{11}} g_1(z)\}, P^2_{(1)} = -\frac{1}{\epsilon^2_{11}} Re\{g_2(z)\},$$

where f_1, f_2, g_1 and g_2 have the same expressions as in (21). Following an analogous procedure shown in I_{13} it is possible to prove that the effective coefficient $\overline{\epsilon}_{11}$ has the form

$$\overline{\epsilon}_{11} = \epsilon^1_{11} + 2\pi(a_1 + b_1), \tag{36}$$

where a_1, b_1 are solutions of the following system:

$$\frac{e^1_{113} + e^2_{113}}{e^1_{113}} a_k + \frac{\epsilon^1_{11} + \epsilon^2_{11}}{\epsilon^1_{11}} b_k +$$

$$\frac{e^2_{113}-e^1_{113}}{e^1_{113}}(\sum_{n=1}^{\infty}{}^{*} a_n\eta_{nk}-\pi R^2 a_1\delta_{1k})+$$
$$\frac{\epsilon^2_{11}-\epsilon^1_{11}}{\epsilon^1_{11}}(\sum_{n=1}^{\infty}{}^{*} b_n\eta_{nk}-\pi R^2 b_1\delta_{1k})=$$
$$(\epsilon^1_{11}-\epsilon^2_{11})R\delta_{1k},$$
$$\frac{C^1_{1313}+C^2_{1313}}{e^1_{113}}a_k-\frac{e^1_{113}+e^2_{113}}{\epsilon^1_{11}}b_k+$$
$$\frac{C^2_{1313}-C^1_{1313}}{e^1_{113}}(\sum_{n=1}^{\infty}{}^{*} a_n\eta_{nk}-$$
$$\pi R^2 a_1\delta_{1k})-\frac{e^2_{113}-e^1_{113}}{\epsilon^1_{11}}(\sum_{n=1}^{\infty}{}^{*} b_n\eta_{nk}-\pi R^2 b_1\delta_{1k})=$$
$$(e^1_{113}-e^2_{113})R\delta_{1k},$$

5 The plane problems

In **the problems** $I_{\beta\beta}$ and I_{12} it can be assumed that $N^{\alpha}_{3\beta\beta}=M^{\alpha}_{\beta\beta}=N^{\alpha}_{312}=M^{\alpha}_{12}=0$, and consequently they represent plane strain problems of the theory of elasticity having the form

$$(C^{\alpha}_{1122}+C^{\alpha}_{1212})N^{\alpha}_{kmn,ki}+C^{\alpha}_{1212}N^{\alpha}_{imn,kk}=0, \tag{37}$$

where, α,k,i=1,2; mn=11,22,33,12. Besides they satisfy the following perfect bonding conditions

$$N^1_{kmn}|_{\Gamma}=N^2_{kmn}|_{\Gamma}, \tag{38}$$

$$(\sigma^1_{ij\,(mn)}+C^1_{ij\,mn})n^1_j|_{\Gamma}=-(\sigma^2_{ij\,(mn)}+C^2_{ij\,mn})n^2_j|_{\Gamma}, \tag{39}$$

where

$$\sigma_{ij\,(mn)}=C_{1122}(\xi_1,\xi_2)\delta_{ij}N_{kmn}+C_{1212}(\xi_1,\xi_2)(N_{imn,j}+N_{jmn,i}), \tag{40}$$

i,j=1,2; $\beta=1,2,3$, $\sigma_{i3(mn)}=0$, $\sigma_{33(mn)}=\nu(\xi_1,\xi_2)(\sigma_{11(mn)}+\sigma_{22(mn)})$, $\nu(\xi_1,\xi_2)=\frac{C_{1133}}{C_{1111}+C_{1122}}$, $D_{i(mn)}=0$, $D_{3(mn)}=\frac{e_{311}(\xi_1,\xi_2)}{C_{1133}(\xi_1,\xi_2)}\sigma_{33(mn)}$, and δ_{ij} is the Kroneker's delta.

On the other hand, if we desire to obtain the formulation I_3 it is sufficient to change in equations (37)-(39) the functions N^{α}_{kmn} by ϕ^{α}_{k3} and the components of the tensor C^{α}_{ijmn} in equation (39) by e^{α}_{3ij}.
The analytic solutions of the plane problems for the case of isotropic fibre and matrix have been treated by several authors [see, for instance, [11] , [8] and [9]] by using the complex potencials of Kolosov-Muskhelishvili. Here, upon the basis of the results published in [[11]], it is possible to obtain the analytic expressions for the rest of the effective coefficients.
The functions $N^{\alpha}_{kpq} \equiv N^{\alpha}_{k}$ for the problems $I_{\beta\beta}$, I_3 and I_{12} are sought in the form

$$2C^{\alpha}_{1212}(N^{\alpha}_{1} + iN^{\alpha}_{2}) = \chi_{\alpha}\varphi_{\alpha}(z) - z\overline{\varphi^{'}_{\alpha}} - \overline{\psi_{\alpha}(z)}, \tag{41}$$

where, $\chi_{\alpha} = 3 - 4\nu_{\alpha}$, the functions $\varphi_{\alpha}(z)$ and $\psi_{\alpha}(z)$ are differentiable functions of the complex variable $z = \xi_1 + i\xi_2$ and can be expressed as

$$\varphi_1(z) = \frac{a_0}{R}z + \sum_{k=1}^{\infty}{}^{*} a_k R^k \xi^{(k-1)}(z), \tag{42}$$

$$\psi_1(z) = \frac{b_0}{R}z + \sum_{k=1}^{\infty}{}^{*} b_k R^k \xi^{(k-1)}(z) + \sum_{k=1}^{\infty}{}^{*} a_k R^k \frac{Q^{(k-1)}(z)}{(k-1)!}, \tag{43}$$

$$\varphi_2(z) = \sum_{k=1}^{\infty}{}^{*} c_k \left(\frac{z}{R}\right)^k, \qquad \psi_2(z) = \sum_{k=1}^{\infty}{}^{*} d_k \left(\frac{z}{R}\right)^k. \tag{44}$$

Here

$$Q(z) = \sum_{m,n}{}^{'} \left\{ \frac{\overline{\beta}_{mn}}{(z-\beta_{mn})^2} - 2z\frac{\overline{\beta}_{mn}}{\beta^3_{mn}} - \frac{\overline{\beta}_{mn}}{\beta^2_{mn}} \right\};$$

$a_0, b_0, a_k, b_k, c_k, d_k$ (k=1,3,5...) are certain constants. These constants are real numbers for the problems $I_{\beta\beta}$, I_3 and they are pure imaginary numbers for the problem I_{12}. From equations (41) we can write a_0 and b_0 through a_1 and b_1 respectively, taking into account the periodicity conditions for the functions N^{α}_{1} and N^{α}_{2}. For the determination of a_k, b_k, c_k, d_k it is necessary to use: the perfect bonding conditions, the relations among σ^{α}_{ij} (i,j=1,2) and $\varphi_{\alpha}(z)$, $\psi_{\alpha}(z)$ wich are given by:

$$\sigma^{\alpha}_{11} + \sigma^{\alpha}_{22} = 4Re\{\varphi^{'}_{\alpha}(z)\}$$

$$\sigma^{\alpha}_{22} - \sigma^{\alpha}_{11} + 2i\sigma^{\alpha}_{12} = 2[\overline{z}\varphi^{''}_{\alpha}(z) + \psi^{'}_{\alpha}(z)]$$

and the Laurent series expansions of the functions (42)-(44). Taking into account the above ideas we can obtain a regular system with respect to a_k ([7]). The constants b_k, c_k, d_k (k=1,3,5...) can be expressed through a_k.
Finally, the formulae for the effective coefficients were obtained from each problem as follows:
From $I_{\beta\beta}$:

$$\begin{aligned}
\overline{C}_{11\beta\beta} + \overline{C}_{22\beta\beta} &= \langle C_{11\beta\beta} + C_{22\beta\beta} \rangle - 2\frac{\gamma_2(1)}{C^2_{1212}} \frac{\chi_2 - 1}{R} \gamma c_1, \\
\overline{C}_{22\beta\beta} + \overline{C}_{11\beta\beta} &= \langle C_{22\beta\beta} - C_{11\beta\beta} \rangle + 2\gamma\{(\chi_1 + 1)\frac{a_1}{R} + \gamma_1(\beta)\}, \qquad (45) \\
\overline{C}_{33\beta\beta} &= \langle C_{33\beta\beta} \rangle - \frac{\gamma_2(3)}{C^2_{1212}R} \gamma(\chi_2 - 1)c_1, \qquad (46) \\
\overline{e}_{3\beta\beta} &= \langle e_{3\beta\beta} \rangle + \frac{e^2_{311} - e^1_{311}}{C^2_{1212}R} \gamma(\chi_2 - 1)c_1, \qquad (47)
\end{aligned}$$

From I_3:

$$\overline{\epsilon}_{33} = \langle \epsilon_{33} \rangle - \frac{e^2_{311} - e^1_{311}}{C^2_{1212}R} \gamma(\chi_2 - 1)c_1. \qquad (48)$$

From I_{12}:

$$\overline{C}_{1212} = \langle C_{1212} \rangle - \gamma\{(\chi_1 + 1)\frac{Im\{a_1\}}{R} + C^2_{1212} - C^1_{1212}\}, \qquad (49)$$

where

$$\begin{aligned}
\gamma &= \pi R^2, \\
\gamma_1(\beta) &= \frac{(C^1_{22\beta\beta} - C^1_{11\beta\beta} + C^2_{11\beta\beta} - C^2_{22\beta\beta})}{2}, \\
\gamma_2(\beta) &= \frac{(C^1_{11\beta\beta} + C^1_{22\beta\beta} - C^2_{11\beta\beta} - C^2_{22\beta\beta})}{2}.
\end{aligned}$$

The constants a_1 and c_1 that appear in the above formulae are obtained from the solution of infinite systems of algebraic equations corresponding to each problem.

6 Concluding remarks

In this work, the technique of asymptotic homogenization is applied to the problem of a unidirectional fibre-reinforced piezocomposite material. Analytical expressions for the effective coefficients of binary homogeneous piezoelectric

composite materials with unidirectional cylindrical fibres have been obtained using the theory of complex variable. The formulae obtained for such effective coefficients are a generalization of the relations published by [11] (Chapter 6), for the purely elastic case. The global behaviour of the piezocomposite considered is characterized by: six elastic constants, three piezoelectric constants and two dielectric constants, equations (31), (32), (36), (45)-(49). Such material belongs to the crystalline group $4mm$ and it is called thetragonal system.

References

[1] Bakhalov, N. S. and Panasenko, G. P., Homogenization: Averaging Processes in Periodic Media. Kluwer, Dordrecht, 1989

[2] Benveniste, Y. and Dvorak, G. J., Uniform fields and universal relations in piezoelectric composites. J. Mech. Phys. Solids **40**, 1992, 1295-1312.

[3] Chan, H. L. W. and Unsworth, J., Simple Model for Piezoelectric Ceramic/Polymer 1-3 Composites Used in Ultrasonic Transducer Applications. IEEE Trans. Ultrason. Ferroelectrics and Frequency Control **36**, 1989, 434-441.

[4] Dunn, M. and Taya, M., Micromechanics predictions of effective electro-elastic moduli of piezoelectric composites. Int. J. Solids Structures, **30**, 1993, 161-176.

[5] Grekov, A. A., Kramarov, S. O. and Kuprienko, A. A., Effective properties of a transversely isotropic piezocomposite with cylindrical inclusions. Ferroelectrics **99**, 1989.

[6] Gururaja, T. R., Schulze, W. A., Cross, L. E., Newnhan, R. E., Auld, B. A. and Wang, Y. J., Piezoelectric Composite Materials for Ultrasonic Transducer Applications. Part I: Resonant Modes of Vibration of PZT Rod-Polymer Composites. IEEE Trans. Sonics Ultrason. **SU – 32**, 19985, 481-498.

[7] Kantorovich, L.V. and Krylov, V.I., Approximate methods of higher analysis. Interscience Publishers, INC-New York, P. Noordhoff LTD- Groningen- The Netherlands, 1964.

[8] Malkov, V. A. and Gurgova, O. E., Effective moduli of hybrid piezocomposite with cylindrical inclusions. J. Mech. Composite Materials. (Riga), **6**, 1986, 1017-1020 (in russian).

[9] Meguid, S. A. and Kalamkarov, A. L., Asymptotic homogenization of elastic composite materials with a regular structure. Int. J. Solids Structures. **31**, No.3, 1994, 303-306.

[10] Otero, J. A., Bravo, J. and Ramos, R. R., 1995. (Submitted for publication.)

[11] Pobedria, B. E., Mechanics of composite materials. Moscow State University Press, Moscow (in Rusian), 1984.

[12] Smith, W. A. and Auld, B. A., Modeling 1-3 Composite Applications, IEEE Trans. Ultrason. Ferroelectrics and Frequency Control **38**, 1991, 40-47.

[13] Smith, W. A., Modeling 1-3 Composite Piezoelectrics: Hydrostatic Response. IEEE Trans. Ultrason. Ferroelectrics and Frequency Control **40**, 1993, 41-49.

[14] Taunaumang, H., Guy, I. L. and Chan, H. L. W., Electromechanical properties of 1-3 piezoelectric ceramic/piezoelectric polymer composites. J. Appl. Phys. **76**(**1**),1994, 484-489.

[15] Telega, J. J., Piezoelectricity and homogenization. Application to biomechanics. In Continuum Models and Discrete Systems (ed. by G.A.Maugin). **2**, Longman, London, 1991.

[16] Turbe, N. and Maugin, G. A., On the linear piezoelectricity of composite materials. Math. Meth. in the Appl. Sci. **14**, 1991, 403-412.