Записки научных
семинаров ПОМИ
Том 328, 2005 г.

O. Perrin, M. Zani

LARGE DEVIATIONS FOR SAMPLE PATHS OF GAUSSIAN PROCESSES QUADRATIC VARIATIONS

Abstract. We show a functional large deviations principle for the family of random functions $\left\{ V_n(x) = \sum_{k=1}^{[nx]} (Z_{k/n} - Z_{k-1/n})^2,\ x \in [0,1] \right\}$, where $\{Z_t,\ t \in [0,1]\}$ is a real valued centered Gaussian process.

1. Introduction

Quadratic variations of Gaussian processes were first studied by Lévy [4] who showed that for a standard Wiener process Z on $[0,1]$, almost surely, as $n \longrightarrow +\infty$,

$$\sum_{k=1}^{2^n} [Z_{k/2^n} - Z_{k-1/2^n}]^2 \longrightarrow 1.$$

Further, Baxter [3] showed that for a Gaussian nondifferentiable process Z, some smoothness assumptions on the mean value and on the covariance function outside the diagonal ensure the almost sure convergence of

$$V_n = \sum_{k=1}^{n} (Z_{k/n} - Z_{k-1/n})^2. \tag{1}$$

Later on, Gladyshev [2] improved this result. An important generalization of these variations was considered by Guyon and Léon [17]. They studied so-called H-variations, which generalize the quadratic variations: for Z a Gaussian centered stationary nondifferentiable process, they introduced

$$V_{H,n} = \sum_{k=1}^{n} H\Big(\frac{Z_{k/n} - Z_{k-1/n}}{\sqrt{2(r(0) - r(1/n))}} \Big),$$

where H is a sufficiently smooth real function, and the covariance function r of the process Z has form

$$r(t) = 1 - t^\beta L(t), \quad 0 < \beta < 2,$$

where L is a slowly varying function at 0. Next, Istas and Lang [16] introduced general quadratic forms to estimate the Hölder index of a process. The discretization in this case is no longer regular, and Z is taken at values $Z_{j\Delta(n)}$, where $j = 1, \cdots, n$ and $(\Delta(n), n) \to (0, +\infty)$.

Perrin [5] studied the random process of quadratic variations corresponding to the linear interpolation of the points $(p/n; V_n(p/n))$, where

$$V_n\Big(\frac{p}{n}\Big) = \sum_{k=1}^{p} (Z_{k/n} - Z_{k-1/n})^2 \tag{2}$$

and Z is a Gaussian process with sufficient smoothness of covariance function. He showed the asymptotic normality of this process. He applied this result to study the deformation of a Gaussian stationary process by a function Φ: the estimator $\widehat{\Phi}$ of Φ is built from the quadratic form (2)).

The aim of this paper is to evaluate the associated large deviations, and to derive a LDP for $\widehat{\Phi}$.

Section 2 is devoted to the presentation of the notation and of the main results of the paper. Section 3 is devoted to the proof of these results. Section 4 gathers the proofs of technical lemmas.

2. Main results

2.1. Framework and notation

Let $Z = \{Z_t, t \in [0,1]\}$ be a real valued centered Gaussian process with covariance function $r : (x,y) \in [0,1]^2 \to r(x,y) \in \mathbb{R}^+$ satisfying:

(A1): $r(x,y)$ is continuous on $[0,1]^2$. It has second derivatives uniformly bounded for $x \neq y$.

We define

$$D^-(x) = \lim_{y \to x, y<x} \frac{\partial}{\partial y} r(x,y), \quad x \in]0,1]$$

$$D^+(x) = \lim_{y \to x, x<y} \frac{\partial}{\partial y} r(x,y), \quad x \in [0,1[$$

and the singularity function α of Z by

$$\alpha(x) = D^-(x) - D^+(x), \quad x \in]0,1[. \tag{3}$$

By assumptions on r, α is uniformly continuous on $[0,1]$. We consider quadratic variations of Z built on a regular subdivision of $[0,1]$: for $n \in \mathbb{N}$, $n \geq 1$, define for $k = 1, \cdots, n$

$$\Delta Z_k = Z_{k/n} - Z_{k-1/n},$$

and the random function

$$V_n(x) = \sum_{k=1}^{[nx]} (\Delta Z_k)^2, \quad x \in [0,1], \tag{4}$$

where $[x]$ denotes the largest integer smaller than or equal to x. For $[nx] = 0$, we assume the sum to be null. The function V_n is càdlàg on $[0,1]$. We define a continuous version of V_n which is the linear interpolation of the points $(p/n, V_n(p/n))$ with $p \in \{1, \cdots, n\}$:

$$\begin{cases} v_n(x) = V_n(x) + (nx - [nx])(\Delta Z_{[nx]+1})^2, & x \in [0,1[\\ v_n(1) = V_n(1). \end{cases}$$

Assume

(A2): α has bounded first derivative on $[0,1]$.

We know from Baxter [3] that V_n is a strongly consistent estimator of $\int_0 \alpha(u)du$. Perrin showed the corresponding convergence in distribution:

Theorem 2.1 (Perrin). *Under assumptions* **(A1)**–**(A2)**, *the process*

$$\left\{\sqrt{n}(v_n(x) - \int_0^x \alpha(u)du), \ x \in [0,1]\right\}$$

converges in distribution in $\mathcal{C}[0,1]$ – *the space of continuous functions on* $[0,1]$, *as* n *tends to* ∞, *towards*

$$\left\{\int_0^x \sqrt{2}\alpha(u)dW_u, \ x \in [0,1]\right\},$$

where W *is a standard Wiener process.*

The aim of this paper is to derive a large deviation principle for the random function V_n (or, equally, v_n). For that purpose, we consider the distribution derivative ν_n of V_n:

$$\nu_n = \sum_{k=1}^{n} (\Delta Z_k)^2 \delta_{k/n}. \tag{5}$$

Analogous random measures have been studied by Dembo and Zeitouni [10], Gamboa and Gassiat[12], Gamboa, Rouault, and Zani[13]. In Zani

[15], one can find a version of the present result: large deviations for $\sum_{k=1}^{[nt]} X_k^2$ when $(X_n)_n$ is a stationary Gaussian seqvence with positive spectral density on the torus. When the X_k^2 are i.i.d., it is the famous Mogulskii theorem (see [6] and [7]). In [1], Najim studied LDP for general measures $\sum_{k=1}^{n} Z_i \delta_{x_i^n}$ where (Z_i) are i.i.d and the empirical measure based on the (x_i^n) converges weakly to a probability measure R.

2.2. Large deviations

The Gärtner–Ellis theorem (see [11]), in its abstract version (Baldi), however not valid in this case, will be the main thread of our work. The principal tool is the normalized cumulant generating function (log-Laplace transform).

For any $h \in \mathcal{C}([0,1])$, define

$$\Lambda(h) = \begin{cases} -\dfrac{1}{2} \displaystyle\int_{[0,1]} \log(1 - 2\alpha(t)h(t))\, dt & \text{if} \quad h \in \mathcal{D}_\alpha \\ +\infty \quad \text{otherwise,} \end{cases}$$

where

$$\mathcal{D}_\alpha = \{h \in \mathcal{C}([0,1]) : \quad \text{for any} \quad t \in [0,1], \quad \alpha(t)h(t) < 1/2\}.$$

Let $\mathcal{M}([0,1])$ be the set of positive bounded measures on $[0,1]$, endowed with the weak topology. Define Λ^* the Legendre dual of Λ for any μ in $\mathcal{M}([0,1])$ by

$$\Lambda^*(\mu) = \sup_{h \in \mathcal{C}([0,1])} \left\{ \int_{[0,1]} h\, d\mu - \Lambda(h) \right\}. \tag{6}$$

From Rockafellar ([14, Theorem 5]), we can give an explicit expression of Λ^*. Set for any $\tau > 0$, $\gamma(\tau) = \frac{1}{2}(\tau - 1 - \log \tau)$.

Proposition 2.2. *Let $\mu \in \mathcal{M}([0,1])$ having with respect to the Lebesgue measure λ the decomposition $\mu = l\lambda + \mu^\perp$, where $l \in \mathcal{C}([0,1])$ and $\mu^\perp$ is the singular part. Then whenever $\gamma(\frac{l}{\alpha}) \in L_1([0,1])$ and $\alpha^{-1} \in L_1(d\mu^\perp)$,*

$$\Lambda^*(\mu) = \int_{[0,1]} \gamma\left(\frac{l(t)}{\alpha(t)}\right) dt + \int_{[0,1]} \frac{d\mu^\perp(t)}{2\alpha(t)},$$

and otherwise, $\Lambda^*(\mu) = +\infty$.

We make an additional assumption on α:

(A3): α has a countable number $\{\alpha_k\}_{k\geq 0}$ of zeros in $[0,1]$.

We can now present the main result of this paper.

Theorem 2.3. *Under assumptions* **(A1)–(A3)**, *the family* $\{\nu_n\}$ *satisfies* a LDP *in* $\mathcal{M}([0,1])$ *endowed with the weak topology, with speed* n *and good rate function* Λ^*.

We can give a functional version of this result. Let $D([0,1],\mathbb{R})$ be the space of càdlàg real functions on $[0,1]$ and let $bv([0,1],\mathbb{R}) \subset D([0,1],\mathbb{R})$ be the space of functions with bounded variation. We can identify $bv([0,1],\mathbb{R})$ and $\mathcal{M}([0,1])$ as follows: to φ in $bv([0,1],\mathbb{R})$ corresponds $d\varphi$ in $\mathcal{M}([0,1])$ characterized by $d\varphi([0,t]) = \varphi(t)$. Endow $bv([0,1],\mathbb{R})$ with the weak-$*$ topology denoted by σ. Now, let us define the rate function associated to V_n. Let φ be in $bv([0,1],\mathbb{R})$ and $d\varphi$ the associated measure in $\mathcal{M}([0,1])$. Set

$$J(\varphi) = \int_{[0,1]} \gamma\left(\frac{\dot{\varphi}(t)}{\alpha(t)}\right) dt + \int_{[0,1]} \frac{d\varphi^{\perp}(t)}{2\alpha(t)}.$$

where $\dot{\varphi}(t)dt$ and $d\varphi^{\perp}$ are respectively, the absolute continuous part and the singular part of the measure $d\varphi$ associated to φ. The following theorem is the transcription of Theorem 2.3 in the space $bv([0,1],\mathbb{R})$ (see de Acosta [19]).

Theorem 2.4. *The family of random functions* $\{V_n(t),\ t \in [0,1]\}$ *satisfies* a LDP *on the space* $(bv([0,1]),\sigma)$, *with speed* n *and rate function* J.

Corollary 2.5. *Almost surely,* ν_n *converges weakly towards* ν.

2.3. Application to time deformation

Let Z be our previous Gaussian process satisfying **(A1)** to **(A3)**, and suppose it has form

$$Z_t = \delta(\Phi(t)) \quad t \in [0,1], \tag{7}$$

where δ is a stationary process with known covariance R and Φ is the deformation function satisfying

(B): Φ is bijective and has uniformly bounded second derivatives in $[0,1]$, as well as its inverse.

Expression (7) is equivalent to the following one:

$$r(x,y) = R(\Phi(y) - \Phi(x)).$$

The singularity function α (see [5] for more details) satisfies

$$\Phi(x) = \frac{\int_0^x \alpha(u)du}{\int_0^1 \alpha(u)du},$$

hence an estimator of Φ is given by

$$\widehat{\Phi}_n(x) = \frac{v_n(x)}{v_n(1)}.$$

Perrin showed that $\|\widehat{\Phi}_n - \Phi\|_\infty \to 0$ as $n \to \infty$. We give a large deviation result on $\widehat{\Phi}_n$.

Theorem 2.6. *Under assumptions* **(A1)** *to* **(A3)** *and* **(B)**, *the family* $\{\widehat{\Phi}_n\}_n$ *satifies a large deviations principle in* $(bv([0,1],\sigma)$ *with rate function*

$$H(\psi) := \inf\left\{ J(\phi); \phi \in bv([0,1]); \psi(\cdot) = \frac{\phi(\cdot)}{\phi(1)} \right\}.$$

Proof of Theorem 2.3

3.1. Convergence of ν_n

We first focuse on the asymptotic properties of the measure ν_n.

Proposition 3.1. *Let* $h \in \mathcal{C}([0,1])$, *then* $(\nu_n(h))_n$ *converges in probability to* $\nu(h)$, *where the measure* ν *is defined, for any* h *in* $\mathcal{C}[0,1]$, *by*

$$\nu(h) = \int_{[0,1]} h(t)\alpha(t)\,dt. \tag{8}$$

Proof of Proposition 3.1. We reduce the quadratic form:

$$\nu_n(h) = \sum_{k=1}^n (\Delta Z_k)^2 h\Big(\frac{k}{n}\Big) = \sum_{k=1}^n \lambda_{k,n}\chi_{k,n}, \tag{9}$$

where $\{\chi_{k,n}\}_{k=1,\cdots,n}$ are $\chi^2(1)$–distributed, i.i.d. random variables, and $\{\lambda_{k,n}\}_{k=1,\cdots,n}$ are the eigenvalues of the matrix $\Gamma_n^{1/2} D_n(h) \Gamma_n^{1/2}$. The matrix $\Gamma_n = L_n \Sigma_n^t L_n$ is the covariance matrix of the vector $(\Delta Z_1, \cdots, \Delta Z_n)$ and Σ_n the covariance matrix of the Gaussian vector

$(Z_{0/n}, Z_{1/n}, \cdots, Z_1)$, L_n is the matrix with n rows and $n+1$ columns, defined as follows

$$L_n = \begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & -1 & 1 \end{pmatrix}$$

and $D_n(h)$ is the $n \times n$ diagonal $\mathrm{diag}\{h(i/n),\ i = 1, \cdots, n\}$. The two following lemmas, whose proofs are postponed to the Appendix (see Sec. 4), detail the asymptotic properties of $(\lambda_{k,n})$.

Lemma 3.2. *The family $\{n\lambda_{k,n}\}$ is uniformly bounded in (k, n).*

Lemma 3.3. *For any $p \in \mathbb{N}$, $p \geq 1$,*

$$\lim_{n \to +\infty} \frac{1}{n} \sum_{k=1}^{n} (n\lambda_{k,n})^p = \int_{[0,1]} (\alpha(t)h(t))^p \, dt.$$

Applying Lemma 3.3,

$$\lim_{n \to +\infty} E(\nu_n(h)) = \int_{[0,1]} \alpha(t)h(t) \, dt = \nu(h),$$

and

$$\lim_{n \to +\infty} n\mathrm{Var}\,(\nu_n(h)) = 2 \int_{[0,1]} (\alpha(t)h(t))^2 \, dt,$$

which ends the proof of Proposition 3.1.

Proof of Theorem 2.3

A result analogous to Theorem 2.3 was established in another framework in [12] with both time-invariant discretization and random weights (see also [10] and [9]).

The proof is based on the ideas of Baldi's theorem (see [11, Theorem 4.5.20]). We introduce the normalized cumulant generating function of (ν_n): any $h \in \mathcal{C}([0,1])$, $\Lambda_n(h) := \frac{1}{n} \log E(\exp\{n\nu_n(h)\})$. This function Λ_n will not be defined everywhere but a careful study of the exposed points shows that it does not matter.

From expression (9), we have easily, for any $h \in \mathcal{C}([0,1])$

$$\Lambda_n(h) = \begin{cases} -\dfrac{1}{2n} \displaystyle\sum_{k=1}^{n} \log(1 - 2n\lambda_{k,n}) & \text{if for any} \quad k, \quad n\lambda_{k,n} < 1/2 \\ +\infty & \text{otherwise.} \end{cases}$$

From Lemma 3.3, we can determine the limit of Λ_n in two cases:

- if for any $t \in [0,1]$ $\alpha(t)h(t) < 1/2$, then

$$\lim_{n\to+\infty} \Lambda_n(h) = -\frac{1}{2} \int_{[0,1]} \log(1 - 2\alpha(t)h(t))\, dt = \Lambda(h).$$

- if there exists $t \in [0,1]$ $\alpha(t)h(t) > 1/2$ then for n large enough, $\Lambda_n(h) = +\infty$ and

$$\lim_{n\to+\infty} \Lambda_n(h) = +\infty = \Lambda(h).$$

Whenever h does not belong to any of these two categories, we do not know what happens for the asymptotic behaviour of Λ_n. These two cases do not cover the whole set $\mathcal{C}([0,1])$. Nevertheless, this will be sufficient for the LDP, since they contain a dense subset of exposing hyperplanes of Λ^*. Recall that μ in $\mathcal{M}([0,1])$ is an exposed point of Λ^* if there exists a function h of $\mathcal{C}([0,1])$ such that for any ζ in $\mathcal{M}([0,1])$, $\zeta \neq \mu$,

$$\int h\, d\mu - \Lambda^*(\mu) > \int h\, d\zeta - \Lambda^*(\zeta). \tag{10}$$

The function h is called the exposing hyperplane associated to μ. Let us now prove the upper and lower bound of the Theorem 2.3.

Upper bound

From Theorem 4.5.3 b) of [11], and the following lemma, which proof is postponed to the Appendix, the upper bound holds for compact sets.

Lemma 3.4. *For any $\delta > 0$ and μ in $\mathcal{M}([0,1])$, there exists h_δ in $\mathcal{C}([0,1])$ such that:*

$$\text{for any} \quad t \in [0,1], \quad \alpha(t)h_\delta(t) < 1/2$$

$$\int_{[0,1]} h_\delta(t)\mu(dt) - \Lambda(h_\delta) \geq \Lambda^*_\delta(\mu), \tag{11}$$

where

$$\Lambda^*_\delta(\mu) = \min\left\{\Lambda^*(\mu) - \delta, \frac{1}{\delta}\right\}.$$

Exponential tightness

Note that for a real number a,

$$\left\{ \sup_{\|h\|_\infty \le 1} \nu_n(h) \ge a \right\} \subset \{\nu_n(1) \ge a\}.$$

Denote by $A = \operatorname*{esssup}_{t\in[0,1]}\{\alpha(t)\}$. For any $\theta < 1/2A$,

$$\limsup_n \frac{1}{n} \log P(\nu_n(1) \ge a) \le -\theta a - \frac{1}{2} \int_{[0,1]} \log(1 - 2\theta\alpha(t))\, dt$$

and

$$\lim_{a\to+\infty} \limsup_n \frac{1}{n} \log P(\nu_n(1) \ge a) = -\infty.$$

Hence the sequence (ν_n) is exponentially tight, and the upper bound holds for any closed set of $\mathcal{M}([0,1])$.

Lower bound

We study the set of exposed points of Λ^*. Define $\mathcal{F}$ a subset of $\mathcal{M}([0,1])$ such that $\mu \in \mathcal{F}$ if

- $\mu = l\lambda$ with $l \in \mathcal{C}[0,1]$, $l(t) > 0$ on $[0,1]$.
- For any α_k, a zero of α, $\lim_{t\to\alpha_k} \frac{l(t)-\alpha(t)}{\alpha(t)}$ is finite.

In the two following lemmas, we describe $\mathcal{F}$ as a dense subset of the exposed points of Λ^*.

Lemma 3.5. *Let $\mu = l\lambda$ be in $\mathcal{F}$. Set*

$$f(t) = \frac{1}{2\alpha(t)} \left(1 - \frac{\alpha(t)}{l(t)}\right), \quad t \in [0,1].$$

Then μ is an exposed point of Λ^* with exposing hyperplane f.

Lemma 3.6. *Let μ be in $\mathcal{M}([0,1])$ such that $\Lambda^*(\mu) < +\infty$. There exists a sequence of measures in (μ_n) in $\mathcal{G}$ such that (μ_n) converges weakly to μ and $\lim_{n\to+\infty} \Lambda^*(\mu_n) = \Lambda^*(\mu)$.*

Proof of Corollary 2.5. We apply Theorem 2.3 along with Borel–Cantelli lemma.

4. Appendix

Proof of Lemma 3.2

In order to prove this lemma, we bound the norm of the covariance matrix $n\Gamma_n^{1/2} D_n(h) \Gamma_n^{1/2}$. Let us consider the following norm on operators: if A is a linear operator on a Hilbert space, set

$$\|A\| = \sup_{\|x\|=1} \|Ax\|,$$

where $\langle\ ,\ \rangle$ is the inner product and $\|x\| = \langle x, x\rangle$. When A is hermitian,

$$\|A\| = \sup_{\|x\|=1} |\langle x, Ax\rangle|. \tag{12}$$

This norm $\|.\|$ is unitarily invariant, hence submultiplicative (see [18, Proposition IV 2.4]). Consequently, when A is normal,

$$\|A\| = \max_j \{|\lambda_j|\}, \tag{13}$$

where $\{\lambda_i\}$ is the set of the eigenvalues of A. Now, from Proposition V 1.8 and Theorem X 1.1 of [18] and Lemma 4.1 of [5],

$$\|(n\Gamma_n)^{1/2} D_n(h) (n\Gamma_n)^{1/2}\| \le \|n\Gamma_n\| \, \|D_n(h)\| \le \|\alpha\|_\infty \|h\|_\infty, \tag{14}$$

where $\|.\|_\infty$ is the supremum norm. □

4.2. Proof of Lemma 3.3

Since the eigenvalues are bounded, it is enough to find the limit, for any $p \ge 1$, of $\operatorname{tr}(n\Gamma_n D_n(h))^p / n$, where $\operatorname{tr} A$ denotes the trace of the matrix A. From [5], we have the following approximations

$$\left|(n\Gamma_n)_{k,k} - \alpha\Big(\frac{k}{n}\Big)\right| \le \frac{C_0}{n}, \quad \left|(n\Gamma_n)_{j,k}\right| \le \frac{C_0}{n} \quad \text{if} \quad j \ne k.$$

We detail the computation for $p = 3$, it is easy to see that the other cases follow similarly

$$\begin{aligned}
\frac{1}{n}(n\Gamma_n D_n(h))^3 &= \frac{1}{n}\sum_{j=1}^n [(n\Gamma_n)_{j,j}]^3 \, h\Big(\frac{j}{n}\Big)^3 \\
&+ \frac{3}{n}\sum_{j=1}^n \sum_{k\ne j} (n\Gamma_n)_{j,j} (n\Gamma_n)_{j,k} (n\Gamma_n)_{k,j} h\Big(\frac{j}{n}\Big)^2 h\Big(\frac{k}{n}\Big) \\
&+ \frac{1}{n}\sum_{j=1}^n \sum_{k\ne l\ne j} (n\Gamma_n)_{j,k} (n\Gamma_n)_{k,l} (n\Gamma_n)_{l,j} h\Big(\frac{j}{n}\Big) h\Big(\frac{k}{n}\Big) h\Big(\frac{l}{n}\Big).
\end{aligned}$$

By an argument of Riemann sums, the first term of the RHS above converges to

$$\int_{[0,1]} (\alpha(t)h(t))^3 dt.$$

It is easy to see that the other terms are negligible.

Proof of Lemma 3.4

For any $\delta > 0$, from the definition of Λ^*, there exists h_δ in $\mathcal{C}([0,1])$ such that (11) is statisfied. In case $th_\delta(t) \leq 1/2$, we may add some $\delta_0 < 0$ so that (11) holds with another δ.

Proof of Lemma 3.5

Since γ is strictly convex, for any x, $y \geq 0$ such that $x \neq y$,

$$\gamma(x) - \gamma(y) < (x-y)\gamma'(x) \tag{15}$$

and

$$\gamma'(x) = \frac{1}{2}\Big(1 - \frac{1}{x}\Big).$$

Integrating (15) with $\mu = l\lambda$ and $\zeta \neq \mu$, we obtain (10).

Proof of Lemma 3.6

A version of this lemma can be found in [8] and [13].

Step 1. Suppose $\mu = l\lambda + \mu^\perp$, and $\Lambda^*(\mu) < \infty$. Assume that the function $l \in \mathcal{C}[0,1]$, is nonnegative and the singular part $\mu^\perp$ is such that $1/\alpha(t)$ in $L^1(\mu^\perp)$. Since λ has full support on $[0,1]$, there exists a sequence (h_n) of positive functions in $\mathcal{C}([0,1])$ such that $h_n\, d\lambda \Rightarrow \mu^\perp/\alpha(t)$. From the lower semi-continuity of Λ^*,

$$\liminf_{n\to+\infty} \Lambda^*((l + \alpha(t)h_n)\lambda) \geq \Lambda^*(\mu).$$

Furthermore, one can see that for $\tau > 0$ and $\tau' \geq 0$,

$$\gamma(\tau + \tau') \leq \gamma(\tau) + \frac{\tau'}{2}. \tag{16}$$

Integrating this inequality,

$$\Lambda^*((l + \alpha(t)h_n)\lambda) \leq \Lambda^*(l\lambda) + \frac{1}{2}\int h_n(t)dt.$$

Hence,

$$\liminf_{n\to+\infty} \Lambda^*((l + \alpha(t)h_n)\lambda) \geq \Lambda^*(\mu),$$

which yields the limit.

Step 2. Suppose $\mu = l\lambda$ with l and $\gamma(l/\alpha(t))$ in $L^1([0,1])$.

Fix $\varepsilon > 0$. We approximate l by l_4 in $\mathcal{F}$ by the following construction:

(1) Let l_1 be the function defined by $l_1(x) = 1$ if $x \in [\alpha_k - \beta_k, \alpha_k + \beta_k]$, for any zero α_k of α, and $l_1 = l$ elsewhere. Furthermore, (β_k) are chosen so that $|\Lambda^*(l_1) - \Lambda^*(l)| \leq \varepsilon$ and $||l_1 - l||_1 \leq \varepsilon$.

(2) Set $l_2 = l_1 \mathbb{I}_{l_1 > \eta} + \eta \mathbb{I}_{l_1 \leq \eta}$, where η is chosen such that $|\Lambda^*(l_2) - \Lambda^*(l_1)| \leq \varepsilon$ and $||l_2 - l_1||_1 \leq \varepsilon$. Therefore $l_2 \geq \eta$.

(3) Set $l_3 = 1$ on any $[\alpha_k - \beta_k, \alpha_k + \beta_k]$ and l_3 is positive continuous elsewhere, such that $l_3 \geq \eta/2$ and $||l_3 - l_2||_1 \leq \varepsilon$.

(4) Let l_4 be in $\mathcal{C}([0, M])$, defined by

$$l_4(x) = \begin{cases} 1 \text{ if for any } k \geq 0, x \in [\alpha_k - \beta_k, \alpha_k + \beta_k] \text{ linear on} \\ [\alpha_k - \beta_k - \delta_k, \alpha_k - \beta_k] \cup [\alpha_k + \beta_k, \alpha_k + \beta_k + \delta_k] \\ l_3(x) \text{ elsewhere} \end{cases}$$

with (δ_k) chosen such that $|\Lambda^*(l_4) - \Lambda^*(l_3)| \leq \varepsilon$ and $||l_4 - l_3||_1 \leq \varepsilon$.

Therefore $l_4 \in \mathcal{F}$ and $|\Lambda^*(l_4) - \Lambda^*(l)| \leq 4\varepsilon$ and $||l_4 - l||_1 \leq 4\varepsilon$.

References

1. J. Najim, *A Cramér type theorem for weighted random variables.* — Electr. J. Prob. **7** (2002), 1–32.
2. E. G. Gladyshev, *A new limit theorem for processes with Gaussian increments.* — Theory Prob. Appl. **6** (1961), 52–61.
3. G. Baxter, *A strong limit theorem for Gaussian processes.* — Proc. Amer. Math. Soc. **7** (1956), 522–527.
4. P. Lévy, *Le mouvement Brownien plan.* — Amer. J. Math. **62** (1940), 487–550.
5. O. Perrin, *Quadratic variations for Gaussian processes and application to time deformation.* — Stoch. Proc. Appl. **82** (1999), 293–305.
6. A. A. Mogulskii, *Large deviations for trajectories of multi-dimensional random walks.* — Theory Prob. Appl. **21** (1976), 300–315.
7. A. A. Mogulskii, *Large deviations for processes with independent increments.* — Ann. Prob. **21** (1993), 202–215 .
8. F. Gamboa, E. Gassiat, *Bayesian methods and maximum entropy for ill posed inverse problems.* Preprint Orsay (1995).
9. P. Cattiaux, F. Gamboa, *Large deviations and variational theorems for marginal problems.* — Bernoulli **5** (1999), 81–108.
10. A. Dembo, O. Zeitouni, *Large deviations for subsampling from individual sequences.* — Stat. and Prob. Lett. **27** (1996), 201–205.
11. A. Dembo, O. Zeitouni, *Large Deviations Techniques and Applications*, (second edition). Springer (1998).

12. F. Gamboa, E. Gassiat, *Bayesian methods for ill-posed problems.* — Ann. Stat. **25** (1997), 328–350.
13. F. Gamboa, A. Rouault, M. Zani, *A functional large deviations principle for quadratic forms of Gaussian stationary processes.* — Stat. Prob. Lett. **43** (1999), 299–308.
14. R. T. Rockafellar, *Integrals which are convex functionals. II.* — Pac. J. Math. **39** (1971), 439–469.
15. M. Zani, *Grandes déviations pour des fonctionnelles issues de la statistique des processus.* Thèse, Orsay (2000).
16. J. Istas, G. Lang, *Quadratic variations and estimation of the local Hölder index of a gaussian process.* — Ann. Inst. H. Poincaré **33** (1997), 407–437.
17. X. Guyon, J. Léon, *Convergence en loi des H-variations d'un processus gaussien stationnaire.* — Ann. Inst. H. Poincaré **25** (1989), 265–282.
18. R. Bhatia, *Matrix Analysis.* — Graduate Texts in Mathematics, Springer, New-York (1996).
19. A. de Acosta, *Large deviations for vector-valued Lévy processes.* — Stoch. Proc. Appl. **51** (1994), 75–115.

Université des Sciences
Sociales, Toulouse, France
E-mail: perrin@cict.fr

Université Paris12 Val-de-Marne,
Créteil, France
E-mail: zani@univ-paris12.fr

Поступило 7 октября 2005 г.