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Turbulent Diffusion in Lower Atmospheric Layers

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New turbulent diffusion model has been considered for the passive impurities transfer in the turbulent air flow applying the mean field method. Stochastic differential equations have been obtained for the mean and fluctuating terms of the impurity concentration. They contain velocity pulsation which is a random function of the spatial coordinates and time. Effective turbulent diffusion coefficient has been obtained containing both molecular and turbulent diffusion coefficients. This nonstationary statistical characteristic includes the arbitrary correlation function of the velocity pulsations, longitudinal and transverse diffusion coefficients potential energy of impurity particles and time. Analytical calculations are carried out for the anisotropic spatial-temporal Gaussian correlation function. Nonstationary effective turbulent diffusion coefficient has been obtained and a calm case is considered. Numerical calculations are carried out using the experimental data. Normalized concentration distribution of a passive impurities in a turbulent wind flow is analyzed at different distances from a pollutant source and a velocity of an air flow. Globules of an impurity distribution near the pollutant source are computed. The isolines of a normalized concentration distribution is analyzed numerically in a culm case. New peculiarities have been revealed in a nonstationary case. Isolines of globules containing passive impurities are elongated along a wind flow. Over time globules are combined and separate containing different concentrations and linear scales. Globules with low concentration have big linear scales and short live time.

Keywords: Statistical moments, stochastic integro-differential equation, turbulence, concentration distribution.

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1. Introduction

Statistical characteristics of scattered electromagnetic waves in different random media are widely studied [1, 2] particularly in the ionospheric plasma [3–8]. Investigation of the diffusion processes caused by turbulence in the lower atmospheric layers by calculating second order statistical moments is one of the urgent problems.

Currently, there is no single physical and mathematical model that can explain and take into account all the numerous aspects of the problem of atmospheric diffusion. There are two main approaches to solving the problem of scattering matter in a moving liquid or gaseous medium depending on certain factors characterizing the medium and source - this is the theory of gradient transfer (or semi-empirical theory of diffusion) and statistical theory of diffusion. Semi-empirical theory is based on the properties of impurity motion relative to coordinate systems fixed in

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space; that is, it is Eulerian. Statistical theory considers diffusion as turbulence in Lagrange variables. There is a close relationship between these approaches, they describe the same phenomenon, but their areas of application do not always overlap. There are a number of problems of atmospheric diffusion, where consideration is possible only on the basis of one of these theories. V.V. Kadomtsev's turbulent diffusion models were reviewed [9].

From the point of view of practical application, the possibility of comparing the results of two different approaches to the description of turbulent diffusion is very useful. It allows to reasonably select the coefficients of the semi-empirical equation to determine specific problems, to determine in conveying cases the scope of application of a particular approach, since each of them has advantages and disadvantages. In particular, in some cases it is rational to apply a combination of these two approaches.

In view of the emphasis placed on the protection of the atmosphere from pollution by harmful impurities, methods for calculating the scattering of impurities in the atmosphere are becoming increasingly important, with the help of which it is possible to assess the level of air pollution by various sources under certain meteorological conditions [10, 11]. The solution of this problem is due to the consideration of many factors affecting scattering of impurities in the atmosphere. These include source type, impurity properties, etc.

It is known that mass transfer processes in liquids and gases are carried out by molecular and convective diffusions. It has been experimentally established that in turbulent flows, unlike laminar flows, whole groups of molecules characterized by mixing paths and turbulent viscosity participate in mixing processes. Turbulence contributes to the mixing of different impurities present in the media [12]. The interaction of impurities leads to a temporary evolution of the irregular structure of the medium. The latter passes in a metastable state by various types of relaxation processes taking place in it. Inhomogeneities can lead to both velocity pulsations and a change in the direction of diffusion processes, to the appearance of local instability, keeping the remaining areas of the system in equilibrium. The term "impurities" will hereinafter refer to both particles and clusters of molecules forming a liquid element.

The interaction of impurities occurs in two opposite directions: the desire for segregation and for normal diffusion [13, 14]. With small Reynolds numbers between particles making a Brownian movement in a viscous liquid, a weak slowly decreasing interaction occurs with a distance. The relative displacement of the particle in the environment of the remaining particles is characterized along with a chaotic, defined regular part. The joint movement of particles occurs as if there are deterministic forces between them, depending in general on the shape and distance between particles. On average, such movement has a selected direction. On the other hand, small Re numbers correspond to a large region of molecule localization in a paired interaction. As the Re number increases, the mixing intensity of the molecules decreases, conditions are created for the accumulation of inhomogeneities. The alignment of the latter occurs not with the individual movement of individual molecules, but with the chaotic movement of groups of molecules. Therefore, the description of turbulent flows is possible taking into account the movement of whole groups of molecules.

This paper proposes a model describing molecular and turbulent diffusion taking into account the interaction of diffusing particles. The evolution of impurity concentration is described by the integro-differential equation. The solution uses the Picard iteration method. Based on the expression of the perturbed impurity concentration, the effect of unstable negative diffusion is explained. The limit value of the concentration of impurities is determined from which this effect occurs. A significant role in this is played by the type of potential energy of interaction between impurity particles. Based on the mean field method, similar to the work of [15, 16], a general expression of an effective diffusion coefficient for an arbitrary tensor correlation function of a solenoidal vector velocity field is obtained. Analysis shows that at large values of the Prandtl diffusion number, in a non-stationary incompressible liquid, the coefficient of turbulent diffusion is anisotropic and in a complex way depends on the coefficients of molecular diffusion. Expressions of longitudinal and transverse, relative to flow motion, coefficients of turbulent diffusion are obtained. An expression of effective potential energy is used describing the interaction of spherical diffusing particles having a potential hole in spatial-wavy winds at small Reynolds numbers [17].

2. Materials and Methods

Let us consider the passive impurities concentration distribution using the mean field method [17]. Concentration $N(\mathbf{r}, t)$ and velocity $\mathbf{V}(\mathbf{r}, t)$ of an incompressible medium satisfy equation

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_\alpha}(NV_\alpha) - D\ \Delta N = \Sigma(\mathbf{r}, t).$$

The source of impurity $\Sigma(\mathbf{r}, t)$ is an arbitrary deterministic function of the coordinate and time, D is the coefficient of the molecular diffusion. Submit functions N and $\mathbf{V}$ as a sum of slowly varying and fluctuating terms, which are random functions of the coordinates and time

$$N(\mathbf{r}, t) = N_0(\mathbf{r}, t) + n(\mathbf{r}, t),\ \ \mathbf{V}(\mathbf{r}, t) = \mathbf{V}_0 + \mathbf{u}(\mathbf{r}, t),\ \ V_0 = \text{const}.$$

Ensemble average of these random functions are taken to be zero $< n >=< u >= 0$. The closed set of equations for the mean and fluctuating concentrations $(N_0 \gg n)$ is

$$\frac{\partial N_0}{\partial t} + V_{0\alpha}\frac{\partial N_0}{\partial x_\alpha} - D\frac{\partial^2 N_0}{\partial x_\alpha^2} = -\frac{\partial}{\partial x_\alpha} < nu_\alpha > +\Sigma(\mathbf{r}, t),$$
$$\frac{\partial n}{\partial t} + V_{0\alpha}\frac{\partial n}{\partial x_\alpha} - D\frac{\partial^2 n}{\partial x_\alpha^2} = -\frac{\partial}{\partial x_\alpha} < u_\alpha N_0 > .$$

Angular brackets denote an ensemble average.Taking into account the initial condition $n(\mathbf{k}, 0) = 0$ we have

$$n(\mathbf{k}, t) = ik_\beta \exp[-g(k)t] \int\limits_{-\infty}^{\infty} d\mathbf{k}' u_\beta(\mathbf{k} - \mathbf{k}', t')N_0(\mathbf{k}', t') \int\limits_0^t dt' \exp[g(k)t'],$$

where $g(k) = ik_\alpha V_{0\alpha} + Dk^2$; $\alpha, \beta = x, y, z$. $3D$ spectral density of the mean concentration satisfies the intego-differential equation

$$\begin{aligned} &\frac{\partial N_0(\mathbf{k},t)}{\partial t} + g(k)N_0(\mathbf{k},t) \\ &= - < u^2 > k_m \int\limits_0^t d\tau \int\limits_{-\infty}^{\infty} d\mathbf{k}' k'_\beta W_{m\beta}(\mathbf{k}-\mathbf{k}',\tau) \exp[-g(k')\tau] \\ &\times N_0(\mathbf{k}, t-\tau) + \Sigma(\mathbf{k},t). \end{aligned} \tag{1}$$

Is the arbitrary second rank correlation tensor $W_{m\beta}(\boldsymbol{\rho},\tau)$ of a random velocity field describing homogeneous and stationary stochastic process? Impurity concentration substantially depends on a turbulent velocity of an air flow having statistical nature. Represent this tensor as a product of a spatial spectral amplitude and a fast-decreased temporal function $W_{m\beta}(æ,\tau) = W_{m\beta}(æ)f(\tau), æ = \mathbf{k} - \mathbf{k}'$. If the mean concentration satisfies the initial condition $N_0(\mathbf{k}, 0) = 0$, the equation (1) can be rewritten as

$$\frac{\partial N_0(\mathbf{k},t)}{\partial t} + \xi(k)N_0(\mathbf{k},t) = \Sigma(\mathbf{k},t), \tag{2}$$

where $\xi(k) = ik_\alpha V_{0\alpha} + k^2 D_{\text{eff}}(\mathbf{k})$,

$$D_{\text{eff}}(\mathbf{k}) = D + < u^2 > \frac{k_m}{k^2} \int\limits_{-\infty}^{\infty} d\mathbf{k}' k'_\beta \int\limits_0^{\infty} d\tau W_{m\beta}(æ,\tau) \exp[-g(k')\tau].$$

Turbulence has been caused by: concentration gradient of impurities, convectional transfer of inhomogeneities and potential gradient of the field, induced at the given point by all particles. Mainly wind field determines impurity concentration distribution. Evolution of the impurity concentration distribution satisfies the linear intego-differential equation

$$\begin{aligned} &\frac{\partial N(\mathbf{r},t)}{\partial t} + \frac{\partial}{\partial x_\alpha}(NV_\alpha) - D\frac{\partial^2 N}{\partial x_\alpha^2} \\ &= \frac{\mu}{\Lambda}\nabla\left\{1 + \sum_{n=1}^{\infty}\frac{1}{n!}\lambda^n[N^{(n-1)} - 1]\nabla\right\} \int\limits_{-\infty}^{\infty} d\mathbf{r}'\Pi(\mathbf{r}-\mathbf{r}')N(\mathbf{r}',t) + \Sigma(\mathbf{r},t), \end{aligned} \tag{3}$$

where μ is a particle mobility, $\Pi(\mathbf{r}-\mathbf{r}')$ is the potential energy between diffusible particles locating at points $\mathbf{r}$ and $\mathbf{r}'$, Λ is a power of a pollutant source.

We will seek the solution of the equation (3) using the Picard iteration method $N(\mathbf{r},t) = N^{(0)}(\mathbf{r},t) + \lambda N^{(1)}(\mathbf{r},t) + \lambda^2 N^{(2)}(\mathbf{r},t) + \cdots$ As a result we obtain the set of differential equations for the mean and fluctuation concentration:

$$\frac{\partial N_0}{\partial t} + V_{0\alpha}\frac{\partial N_0}{\partial x_\alpha} - D\frac{\partial^2 N_0}{\partial x_\alpha^2} - \frac{\mu}{\Lambda}\frac{\partial^2}{\partial x_\alpha^2}\int d\mathbf{r}'\Pi(\mathbf{r}-\mathbf{r}')N_0(\mathbf{r}',t)$$
$$= -\frac{\partial}{\partial X_\alpha} < nU_\alpha > + \Upsilon(\mathbf{r},t),$$

$$\frac{\partial n}{\partial t} + V_{0\alpha}\frac{\partial n}{\partial x_\alpha} - D\frac{\partial^2 n}{\partial x_\alpha^2} - \frac{\mu}{\Lambda}\frac{\partial^2}{\partial x_\alpha^2}\int d\mathbf{r}'\Pi(\mathbf{r}-\mathbf{r}')n(\mathbf{r}',t)$$
$$= -\frac{\partial}{\partial X_\alpha} < N_0 u_\alpha > \frac{\partial}{\partial x_\alpha}(nu_\alpha - < nu_\alpha >).$$

Using the incompressibility condition $k_\alpha u_\alpha(\mathbf{k},\omega) = 0$, the spatial-temporal Fourier transform for the effective turbulent diffusion coefficient we obtain

$$D_{\text{eff}}^{\text{turb}}(\mathbf{k},\omega) = \frac{1}{(2\pi)^4}\frac{k_m k_\beta}{k^2}\int_{-\infty}^{\infty} d\mathbf{k}' d\omega' d\boldsymbol{\rho} d\tau L^{-1}(\mathbf{k}',\omega')W_{m\beta}(\boldsymbol{\rho},\tau)$$
$$\times \exp[-i(\mathbf{k}-\mathbf{k}')\rho + i(\omega-\omega')\tau], \tag{4}$$

In the general case spatial-temporal second rank spectral correlation tensor is [18]

$$W_{\alpha\beta}(\mathbf{k},\omega) = F(k,\omega)\cdot\delta_{\alpha\beta} + G(k,\omega)\cdot k_\alpha k_\beta + iH(k,\omega)\cdot\varepsilon_{\alpha\beta\nu}k_\nu.$$

F, G and H are functions of the wavenumber and frequency:

$$F(k,\omega) = \widetilde{A} - \frac{1}{k}\frac{\partial\widetilde{B}}{\partial k}, \quad G(k,\omega) = -\frac{1}{k^2}\left(\frac{\partial^2\widetilde{B}}{\partial^2 k} - \frac{1}{k}\frac{\partial\widetilde{B}}{\partial k}\right), \quad H(k,\omega) = \frac{1}{k}\frac{\partial\widetilde{C}}{\partial k}.$$

Bochner's theorem imposes the restrictions on these functions. The third term does not contribute to the solenoidal velocity vector field. The pole of the integrand (4) is $\omega' = (\mathbf{k}'\mathbf{V}_0) - ik'^2 D_{\text{eff}}^{\text{turb}}(\mathbf{k}')$. Using the residue theory integration yields

$$D_{\text{eff}}^{\text{turb}}(\mathbf{k},\omega) = 2\pi\frac{k_m k_\beta}{k^2}\int_{-\infty}^{\infty} d\mathbf{k}' W_{m\beta}(\text{\ae},\Omega),$$

where $\text{\ae} = \mathbf{k}-\mathbf{k}',\ \Omega = \omega - (\mathbf{k}'\mathbf{V}_0) + ik'^2 D_{\text{eff}}^{\text{turb}}(\mathbf{k}')$. We choose the spherical coordinate system and the spatial-temporal spectral correlation tensor of the velocity pulsation submit in the form

$$W_{m\beta}(\text{\ae},\Omega) = < u^2 > (\text{\ae}^2\delta_{m\beta} - \text{\ae}_m\text{\ae}_\beta)\cdot P(\text{\ae},\Omega,l,T).$$

Transversal and longitudinal (with respect to the vector $\mathbf{V_0}$ ($\mathbf{V_0} \parallel Z$)) components of the effective turbulent diffusion for arbitrary function $P(\text{\ae},\Omega,l,T)$ can be written as:

$$D_{\perp}^{\text{turb}}(k_{x,y},\omega)$$
$$= 2\pi < u^2 > \int\limits_0^\infty dk' k'^4 \int\limits_0^\pi d\theta \sin\theta \int\limits_0^{2\pi} d\varphi \left[1 - \sin^2\theta \begin{pmatrix} \cos^2\varphi \\ \sin^2\varphi \end{pmatrix}\right] P(æ,\Omega,l,T),$$

$$D_{\|}^{\text{turb}}(k_z,\omega) = 2\pi < u^2 > \int\limits_0^\infty dk' k'^4 \int\limits_0^\pi d\theta \sin^3\theta \int\limits_0^{2\pi} d\varphi P(æ,\Omega,l,T) \,|_{k=k_z},$$

where φ and θ are azimuthal and polar angles, respectively; $< u^2 >$ is the variance of the turbulent velocity, $< u^2 >^{1/2} \cong l/T$ is the root-mean-square velocity.

Now consider the model of the passive impurities' propagation and distribution in the atmosphere using the modify mean field method [16]. Concentration $N(\mathbf{r},t)$ and the velocity $V(\mathbf{r},t)$ the satisfy stochastic Fokker-Planck equation:

$$\frac{\partial N}{\partial t} + \frac{\partial}{\partial x_\alpha}(NV_\alpha) = (D\delta_{\alpha\beta} + D_{\alpha\beta})\frac{\partial^2 N}{\partial x_\alpha \partial x_\beta} + \Sigma(\mathbf{r},t),$$

where $D_{\alpha\beta}$ is the second rank tensor of the turbulent diffusion.

If a polutant source having power Λ is located at an altitude H above the Earth surface $S(\mathbf{r},t) = q(\mathbf{r}) \cdot Q(t) = \Lambda Q(-t/T)\delta(x-H)\delta(y)\delta(z)$ and the wind velocity is directed along the Z axis, using the initial and solenoidal $(k_\alpha W_{\alpha\beta}(\mathbf{k},t) = k_\beta W_{\alpha\beta}(\mathbf{k},t) = 0)$ conditions, solution of equation (2) describing the distribution of the mean concentration of a passive impurity can be written as:

$$N_0(\mathbf{r},t) = 2Q\Lambda \int\limits_{-\infty}^{\infty} d\mathbf{k} \frac{\cos(k_x H)}{\xi(k)} \{1 - \exp[-\xi(k)t]\} \exp(i\mathbf{kr}), \tag{5}$$

$$D_{\text{eff}}(\mathbf{k}) = D + \frac{1}{4\pi} \frac{< u^2 >}{D_\perp D_\|^{1/2}} \frac{k_m k_\beta}{k^2} \int\limits_{-\infty}^{\infty} d\boldsymbol{\rho} G_{m\beta}(\boldsymbol{\rho}) \left(\frac{\rho_\perp^2}{D_\perp} + \frac{\rho_z^2}{D_\|}\right)^{-1/2}$$
$$\times \exp\left[-\sqrt{q}\left(\frac{\rho_\perp^2}{D_\perp} + \frac{\rho_z^2}{D_\|}\right)^{-1/2} + \frac{V_0}{2D_\|}\rho_z - i\mathbf{k}\boldsymbol{\rho}\right].$$

Here: $q = \frac{1}{T} + \frac{V_0^2}{4D_\|}$, $\boldsymbol{\rho}_\perp$ is the transversal wave vector with respect to the V_0 air flow velocity, $D_\perp$ and $D_\|$ are the transversal and longitudinal diffusion coefficients of the velocity pulsations. Using the Gaussian correlation function

$$G_{m\beta}(\boldsymbol{\rho},\tau) =< u^2 > \left[\left(1 - \frac{\rho^2}{l^2}\right)\delta_{m\beta} + \frac{\rho_m \rho_\beta}{l^2}\right] \exp\left(-\frac{\rho^2}{l^2} - \frac{\tau^2}{T^2}\right)$$

in the polar coordinate system, we obtain

$$D_{\text{eff}}(\mathbf{k}) = D + \frac{1}{4\pi}\frac{< u^2 >}{D_\perp D_\parallel}\int\limits_{-\infty}^{\infty} d\rho_z \int\limits_{0}^{\infty} d\rho_\perp \int\limits_{0}^{2\pi} d\varphi \left(\frac{\rho_\perp^2}{D_\perp} + \frac{\rho_z^2}{D_\parallel}\right)^{-1/2}$$

$$\times \left\{ \left(1 - \frac{\rho_\perp^2}{l_\perp^2} - \frac{\rho_z^2}{l_\parallel^2}\right) - \frac{1}{k^2}\left[\sqrt{B}(k_x \cos\varphi + k_y \sin\varphi) + \frac{\rho_z}{l_\parallel}k_z\right]^2 \right\}$$

$$\times \exp\Bigg[- B - \frac{\rho_z^2}{l_\parallel^2} - \sqrt{q}\left(\frac{\rho_\perp^2}{D_\perp} + \frac{\rho_z^2}{D_\parallel}\right)^{1/2} + \frac{V_0}{2D_\parallel}\rho_z$$

$$-i(k_x\rho_\perp\cos\varphi + k_y\rho_\perp\sin\varphi + k_z\rho_z)\Bigg],$$

where $B = \rho_\perp^2/l_\perp^2$, $l_\parallel$ and $l_\perp$ are longitudinal and transversal characteristic spatial scales of the velocity pulsations along and perpendicular to the air flow, respectively. Using the Jacobi-Anger formulae, taking into account a recurrence relation of the Bessel function and applying the saddle-point method at

$$\left(1 + \frac{V_0^2 T}{4D_\parallel}\right)^{1/2} \gg 1 \quad \text{and} \quad V_0\left(\frac{T}{D_\parallel}\right)^{1/2} \gg 1$$

the effective diffusion coefficient can be rewritten as

$$D_{\text{eff}}(\mathbf{k}) = D + \frac{1}{2}\frac{< u^2 >}{k^2}\sqrt{\frac{\pi}{p}}\frac{T^{1/4}}{D_\perp^{3/4}}\sum_{n=-\infty}^{\infty} i^n \int\limits_{-\infty}^{\infty} d\rho_\perp \rho_\perp^{1/2}\big\{(1-B)k^2 J_n(t_1)J_n(t_2)$$

$$+ \frac{B}{4}k_x^2 J_n(\beta)\big[J_{n-2}(t_1) - 2J_n(t_1) + J_{n+2}(t_1)\big] + \frac{B}{4}k_y^2 J_n(t_1)$$

$$\times \big[J_{n-2}(t_2) - 2J_n(t_2) + J_{n+2}(t_2)\big]\big\}\exp\left(-B - \frac{p}{\sqrt{TD_\perp}}\rho_\perp\right). \quad (6)$$

Here: $p = \left(1 + \frac{V_0^2 T}{4D_\parallel}\right)^{1/2}$, $B = \rho_\perp^2/l_\perp^2$, $t_1 = k_x\rho_\perp$, $t_2 = k_y\rho_\perp$. Using the well-known relation for the Bessel function $J_{-n}(x) = (-1)^n J_n(x)$ and the designation $\eta = (V_0 l_\perp)^2/(8D_\perp D_\parallel)$ consider two limit cases.

a) $t_1 \ll 1$ and $t_2 \ll 1$.

$$D_{\text{eff}} = D + R\eta^{1/2}\exp(\eta)\big[K_{3/4}(\eta) - K_{1/4}(\eta)\big],$$

where $K_\nu(x)$ is the McDonald function, $R = \sqrt{2\pi} < u^2 > l_\perp^2/(8D_\perp)$. If $\zeta < 1$ we yield:

$$D_{\text{eff}} = D + R\eta^{-1/4} = D + D_{\text{turb}}. \quad (7)$$

Velocity pulsations increase the effective turbulent diffusion coefficient D_{eff}. It depends on both transversal and longitudinal diffusion coefficients and transversal characteristic spatial scale $\sim l_{\perp}^{3/2}$; D_{turb} is inversely proportional to the wind velocity V_0. If the velocity of an air flow substantially increases, velocity pulsations do not give the contribution. If $\eta > 1$, $D_{\text{eff}} = D$.

b) $t_1 \gg 1$ and $t_2 \gg 1$. Using the asymptotic expression for the Bessel function, from (6) we obtain

$$D_{\text{eff}}(\mathbf{k}) = D + \frac{1}{2\sqrt{\pi}} \frac{< u^2 >}{(kl_{\perp})^2} \left(\frac{D_{\parallel}}{V_0^2 D_{\perp}^3} \right)^{1/4} \frac{1}{(k_x k_y)^{1/2}} \int\limits_0^{\infty} d\rho_{\perp} \big[2(kl_{\perp})^2$$
$$- \rho_{\perp}^2 (4k_x^2 + 4k_y^2 + 4k_z^2) \big] \rho_{\perp}^{-1/2} \big[\cos(k_- \rho_{\perp}) \tag{8}$$
$$+ \sin(k_+ \rho_{\perp}) \big] \exp\left(-B - \frac{V_0}{\sqrt{D_{\parallel} D_{\perp}}} \rho_{\perp} \right),$$

and at $\eta > 1$

$$D_{\text{eff}}(\mathbf{k}) = D + \frac{< u^2 >}{V_0} \left(\frac{D_{\parallel}}{D_{\perp}} \right)^{1/2} \frac{1}{(k_x k_y)^{1/2}},$$

where $k_{\pm} = k_x \pm k_y$. At small-scale velocity pulsations the effective diffusion coefficient contains only parameters of a turbulent flow. In the case of large-scale velocity fluctuations, D_{eff} depends also on the wavelength of wave propagating in a nonstationary medium i.e., it is inversely proportional to the trasversal wave number, i.e. $k_{\perp}^{-1}$. Knowledge of D_{eff} allows to calculate the integral (5).

A particular interest, in terms of protection of the atmosphere from pollution, represents studying propagation of impurity in the atmosphere at abnormal weather conditions to which, in particular, the calm belongs. In this case, there is no wind transfer of the impurity, and very high concentrations can be observed near the source. In areas with a sharply continental climate, there are pollutant emissions with a vertical length of up to several hundred meters or more [10]. Find the passive impurity concentration distribution when the turbulent flow rate is zero, $V_0 = 0$. Using (8) at $\eta_1 = l_{\perp}^2/(8TD_{\perp}) > 1$ applying the saddle-point method, after some manipulations for the effective diffusion coefficient and the mean concentration of a passive impurity we obtain

$$D_{\text{eff}} = D + G_0 \sqrt{T} \eta_1 \exp(\eta_1) \left[K_{3/4}(\eta_1) - K_{1/4}(\eta_1) \right],$$

$$N_0(\mathbf{r}, t) = 2Q\Lambda\pi^2 D_{\text{eff}}^{-1} \left[(x-H)^2 + y^2 + z^2 \right]^{-1/2}$$
$$\times \left\{ 1 - erf\left[(D_{\text{eff}} t)^{-1/2} \right] \left((x-H)^2 + y^2 + z^2 \right) \right\}$$
$$+ (H \to -H), \quad \xi(k) = k^2 D_{\text{eff}}.$$

Here $G_0 = 2\sqrt{\pi} < u^2 > l_{\perp}/\sqrt{D_{\perp}}$.

For a small-scale inhomogeneities ($k_\perp l_\perp \ll 1$), at big Peclets number $V_0 l/D_\parallel \gg 1$ we obtain:

$$D_* =< u^2 > \frac{l_\perp^2}{D_\perp}\eta^{1/2}\left[K_{3/4}(\eta) - K_{1/4}(\eta)\right]. \tag{9}$$

In a calm case ($V_0 = 0$) we have:

$$D_*^0 = G_0\eta_1 \exp\left(\eta_1\right)\left[K_{3/4}(\eta_1) - K_{1/4}(\eta_1)\right]. \tag{10}$$

Consider different cases of the parameter η. Using the asymptotic formulas of the Macdonald function from equation (7) it follows, that: at $\eta < 1$ velocity pulsations lead to the increase of the turbulent diffusion coefficient D_{eff} depending on as the characteristic transverse spatial scale of the velocity pulsation $l_\perp^{3/2}$, as well as the horizontal ($D_\parallel$) and transversal ($D_\perp$) diffusion coefficients. Increasing the wind velocity V_0 parameter D_{eff} decreases. If V_0 substantially increases, velocity pulsations do not give the contribution. At $\eta > 1$ turbulent diffusion coefficient is determined mainly by the velocity of a turbulent stream.

At calm meteorological conditions (no impurity transfer by the wind) high concentration can be observed near the source. In areas with sharply continental climate there are calms to vertical extent up to several hundred meters and more [19].

Substituting (9) and (10) equations into (5) for the normalized turbulent diffusion coefficient of the passive impurities and in the calm case (V_0) we obtain:

$$\begin{aligned}\Upsilon = {}& \frac{2\pi^2}{D_* z}\exp\left\{-\frac{1}{4}\frac{V_0}{D_* z}\left[(x+H)^2+y^2\right]\right\}\left\{1+\frac{1}{4z^2}\left[(x+H)^2+y^2\right]\right\}\\ &-\frac{\sqrt{\pi}}{2}\frac{\Gamma}{\sqrt{G}}\left[\exp\left(2\sqrt{G\gamma}\right)\cdot erf\left(\sqrt{\frac{G}{t}}+\sqrt{\gamma t}\right)+\exp\left(2\sqrt{G\gamma}\right)\cdot erf\left(\sqrt{\frac{G}{t}}+\sqrt{\gamma t}\right)\right.\\ &\left.-\exp\left(2\sqrt{G\gamma}\right)+\exp\left(-2\sqrt{G\gamma}\right)\right]+(H\to -H),\end{aligned} \tag{11}$$

$$\begin{aligned}\Upsilon_0 = {}& \frac{2\pi^2}{D_*^0}\left[(x+H)^2+y^2+z^2\right]^{-1/2}\\ &\times\left\{1 - erf\left[(D_*^0 t)^{-1/2} left((x+H)^2+y^2+z^2\right)\right]+(H\to -H),\end{aligned} \tag{12}$$

where $erf(x) = \frac{2}{\sqrt{\pi}}\int\limits_0^x dt\exp(-t^2)$ is the Gaussian error function,

$$\Gamma = \frac{\pi^{3/2}}{2}\frac{1}{D_*^{3/2}}\exp\left(\frac{zV_0}{2D_*}\right),\ G = \frac{1}{4D_*}\left[(x+H)^2+y^2+z^2\right],$$

$$\gamma = \frac{V_0^2}{4D_*},\quad \Upsilon \equiv \frac{N_0(\mathbf{r},t)}{Q\Lambda},\quad \Upsilon_0 \equiv \frac{N_0^0(\mathbf{r},t)}{Q\Lambda}.$$

The inequality $\left[(x+H)^2+y^2\right] \leq z^2$ imposes the restriction on a distance z in equations (11) and (12). The condition $N_0^0(\mathbf{r},t)=0$ is satisfied for arbitrary distances z in a calm case. In the absence of impurity transfer by wind, very high concentrations of impurity can be observed near the source.

3. Results

Numerical simulations of the statistical characteristics of pollutant transfer in a turbulent air flow are carried out using the experimental data [10, 19, 20]. The height of the source of the impurities eruption is located at an altitude of 100-200 meters. RMS pulsation of the longitudinal velocity component is of the order of $\sqrt{<u^2>} = 0.18 \div 0.66\ m/sec$, the Lagrangian spatial and temporal scales are in the interval $l_\perp = (100 \div 180)\ m$, $T = (125 \div 300)\ sec$, respectively. The mean velocity of a wind is in the interval $V_0 = (1 \div 7)\ m/sec$. Transversal turbulent diffusion coefficient is of the order of $D_\perp = (11; 30)\ m^2/sec$, longitudinal diffusion coefficient can be calculated taking into account the data of the experimental measurements $D_\parallel = nD_\perp, n = 15 \div 30$.

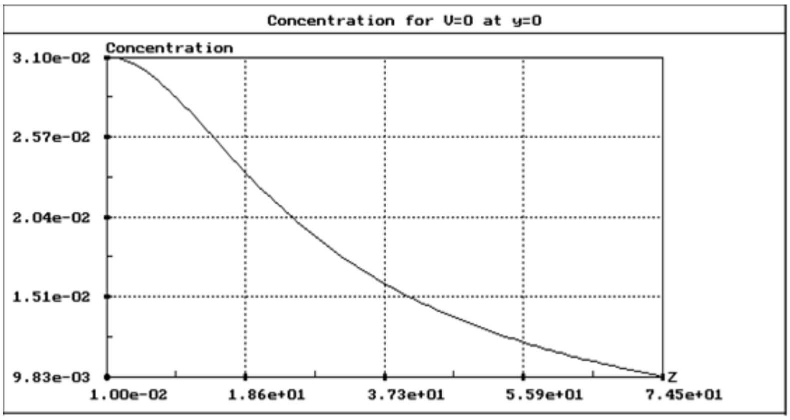

Figure 1. Dependence of the distribution of the normalized concentration of a passive impurity Υ_0 versus distance Z in stationary $(t \to \infty)$ calm case V_0.

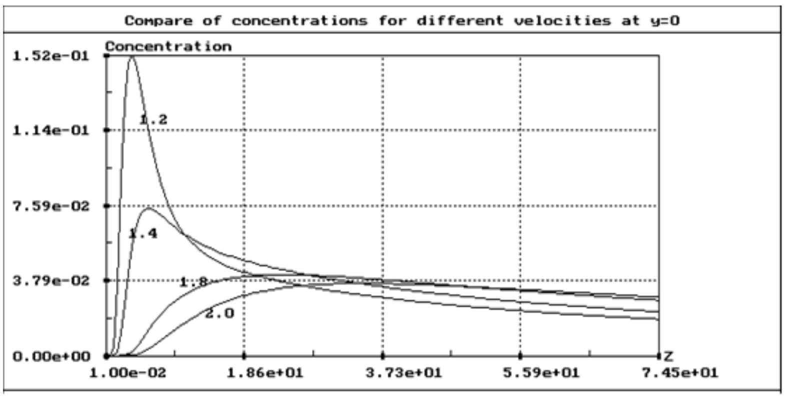

Figure 2. As in Figure 1, at different velocities of a turbulent air flow (see equation (11)).

Figure 1 shows the normalized passive impurities distribution Υ_0 at different distances Z from a pollutant source in calm ($V_0 = 0$) and stationary ($t \to \infty$) cases. Parameters: $H = 100\ m, D_\perp = 11\ m^2/sec, n = 15, T = 120\ sec, l_\perp = 150\ m$. Concentration slowly decreases, at a distance 80 m reaches minimum $\sim 10^{-4}$ and after remains constant.

Figure 2 represents concentration distribution at different velocities of a wind: $V_0 = (1.2; 1.4; 1.8; 2.0)\ m/sec$. In both cases concentration of a passive impurity decreases with distance. Parameters: $H = 180\ m, D_\perp = 30\ m^2/sec, n = 15, T = 180\ sec, l_\perp = 150\ m, t = 42\ hour$. 1) At $V_0 = 1.2\ m/sec$ concentration reaches maximum 0.06 at $Z = 26.45\ m$ and is zero at $Z = 130\ m$; 2) At $V_0 = 1.4\ m/sec, \Upsilon_{\max} = 0.01$ at $Z = 17.65\ m$ and is zero at $Z = 180\ m$; 3) At $V_0 = 1.6\ m/sec, \Upsilon_{\max} = 0.02$ at $Z = 12.85\ m$ and is zero at $Z = 200\ m$. Impurity concentration decreases exponentially in all cases. The obtained results are in an agreement with [21] in the presence and absence of a fog at an altitude of $H = 100\ m$.

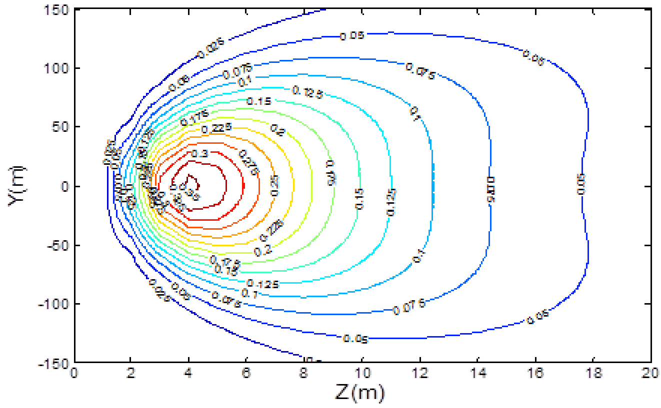

Figure 3. Globule near the source at $H = 120\ m, V_0 = 1.0\ m/sec, t = (100 \div 500)\ sec$.

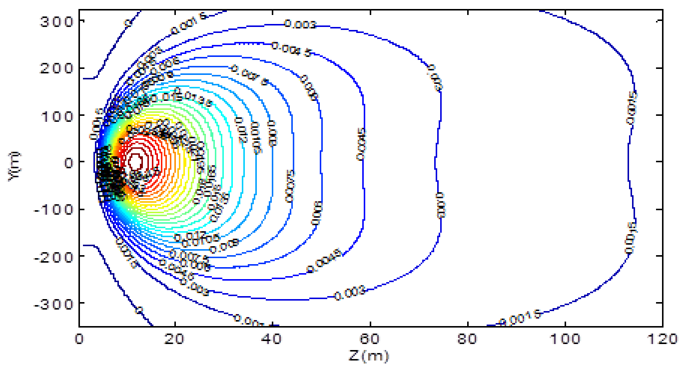

Figure 4. Globule near the source at $H = 180\ m, V_0 = 1.2\ m/sec, t = (100 \div 500)\ sec$.

Figures 3 and 4 represents the evaluation of an impurity globules near the source in a time interval $t = (100 \div 500)\ sec$. The source of pollution is at an altitude $H = 120$ meters from the surface of the Earth (see Figure 3) at wind velocity $V_0 = 1.0\ m/sec$ and $H = 180\ m$ (see Figure 4) at $V_0 = 1.2\ m/sec$. In all figures impurity concentrations are indicated on the curves.

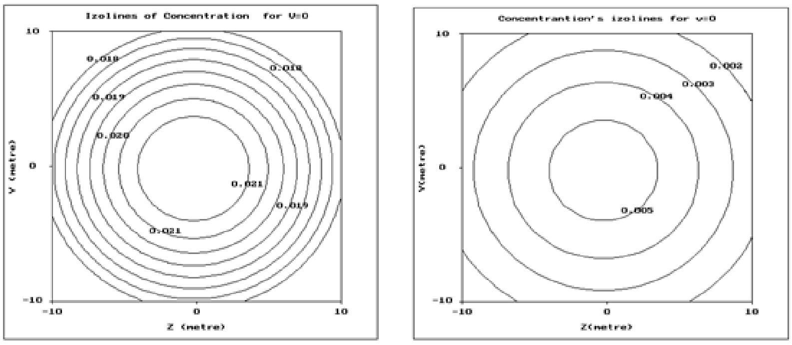

Figure 5. Isolines of the impurity concentrations in a calm case.

Figure 5 illustrates isolines of a normalized impurity concentration distribution Σ_0 for a calm case (V_0) at stationary ($t \to \infty$) (left figure) and nonstationary ($t = 10$ sec) (right figure) cases in the YOZ plane. Parameters: $H = 180$ $m, D_\perp = 22$ $m^2/sec, n = 15, T = 110$ sec. In a stationary case (no wind), the so-called calm case, the isolines of globules have a form of concentration circles. Globules with lower impurity concentrations correspond to large radius circles. Increasing concentrations of impurities, radius of isolines circles decreases.

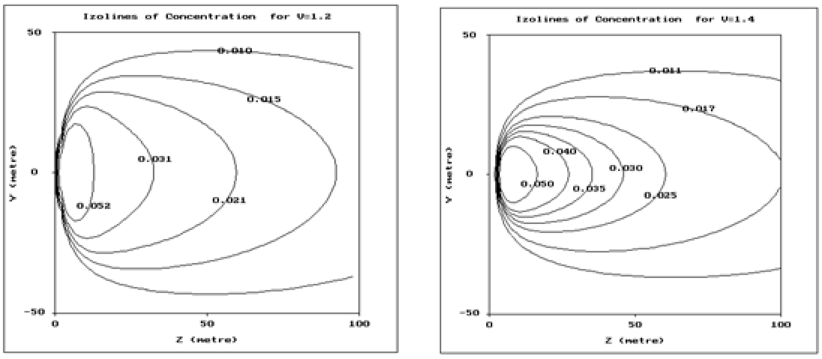

Figure 6. Evaluation of impurity globules at wind velocities of $V_0 = 1.2$ m/sec (left figure) and for $V_0 = 1.9$ m/sec (right figure).

Figure 6 illustrates the globules deformation of an impurity concentration distribution using equation (12) from source to 100 meters at different wind speeds $V_0 = 1.2, 1.9$ m/sec Increasing a velocity of an airflow, the isolines of globules deform and stretch along the wind direction. Parameters of the Figures 7–8: $H = 200$ $m, T = 18$ $sec, n = 20, D_\perp = 35$ m^2/sec.

Table 1. Globules formation parameters

Concentration Υ	Time of creation sec	Time of disappearance sec	Lifetime sec	Initial coordinates of the globule formation m	Diameter of the globules m
0.004	700	800	100	900	100
0.003	650	900	250	850	130
0.002	590	870	280	680	180
0.001	475	500	25	480	200

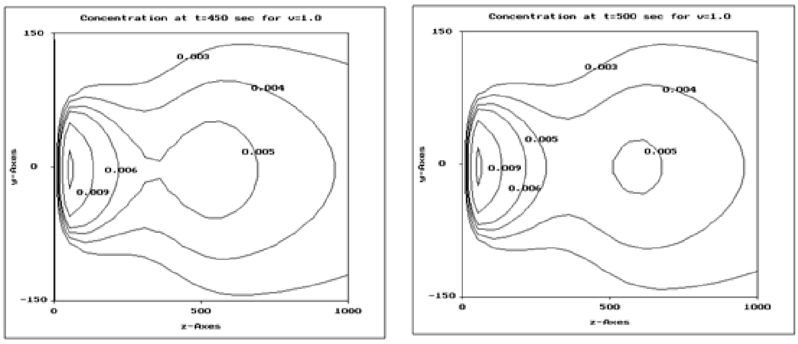

Figure 7. Evolution of an isolines corresponding to the passive impurity concentration in a nonstationary case at $V_0 = 1\ m/sec$ in a time interval $t = 100 \div 450\ sec$.

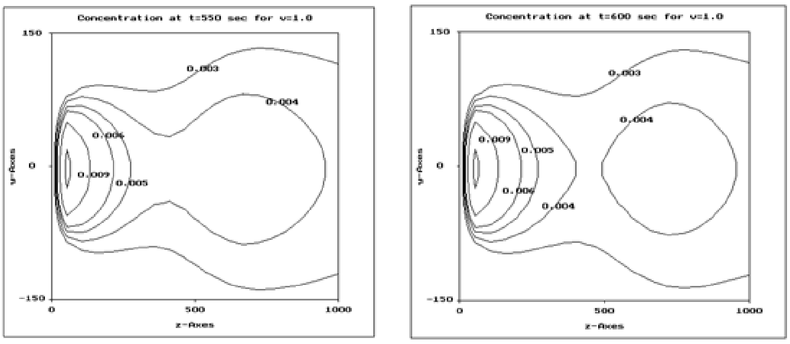

Figure 8. As on Figures 7 at time interval $t = 500 \div 600\ sec$.

Numerical simulation of the normalized turbulent diffusion coefficient is carried out in a non-stationary case. Globules of the passive impurities with different concentrations and sizes carried by wind are constantly changing. In the vicinity of a source, these globules stretch along the wind direction. At some distance from a source, some globules with a large concentration are so elongated that new globules appear inside them. As a result, their concentration increases, impurities break off from the main stream and continue to move independently along the stream, changing their original shape. After some time, globules are disappeared due to diffusion. The lifetime of globules depends on the concentration of impurities and on the size. Figures defining the evolution of isolines of globules with different concentrations of impurities and sizes are given as illustrations.

4. Conclusions

The paper proposes a new statistical model of the turbulent diffusion in the lower atmospheric layers, based on the Picard and the mean field method. An anisotropic effective turbulent diffusion coefficient is obtained for an arbitrary second rank correlation tensor of a solenoidal velocity vector field which is a random function of spatial coordinates and time. Longitudinal and transverse effective diffusion coefficients contain molecular diffusion, interaction between impurities and diffusion coefficients in two mutually perpendicular directions. The relationship between these coefficients has been established experimentally. Particular attention is paid to the effective potential energy between interactive impurities. Its spectral function has a hole at small Reynolds numbers that can lead to a new effect, so-called turbulent diffusion instability [22].

Numerical calculations are carried out to study the formation and evolution of globules of different concentrations and linear sizes. In a calm case isolines of globules having form of concentric circles, globules with higher concentrations are located near the source of impurities eruption than globules with lower concentrations. In a non-stationary case, globules with various concentrations move towards the direction of a wind. Globules with a small concentration of impurities have large linear dimensions and a short lifetime. Over time, shapes of the globules isolines are distorted, they stretch towards the wind speed, sometimes the globules merge and they begin to move independently along the flow. The lifetime of globules with a low concentration of impurities is less than a globule with a higher concentration.

Investigation of these processes is an urgent problem in modern ecology aimed at combating atmospheric pollution, meteorology and atmospheric physics.

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References

[1] A. Ishimaru. *Wave Propagation and Scattering in Random Media*, Multiple Scattering, Turbulence, Rough Surfaces and Remote Sensing, IEEE Press, Piscataway, New Jersey, USA, vol. 2, 1997

[2] S.M. Rytov, Yu.A. Kravtsov and V.I. Tatarskii. *Principles of Statistical Radiophysics*, Waves Propagation Through Random Media. Berlin, New York, Springer, vol. 4. 1989

[3] G. Jandieri and N. Tugushi. *Transformation of the Spatial Spectrum of Scattered Radio Waves in the Conductive Equatorial Ionosphere*, Electronics, **12** (2023), 2759-2770, DOI: 103390/electronics12132759

[4] G. Jandieri and N. Tugushi. *Statistical characteristics of the temporal spectrum of scattered radiation in the equatorial ionosphere*, J. of Environmental & Earth Sciences., **5** (2023), 85-94, DOI: 10.30564/jees.v5il.5442

[5] G. Jandieri, A. Ishimaru, B. Rawat and N. Tugushi. *Temporal Spectrum of a Scattered Electromagnetic Waves in the Conductive Collision Turbulent Magnetized Plasma*, Advanced Electromagnetics, **11** (2022), 1-8, DOI: 10.7716/aem.v11i1.1859

[6] G. Jandieri, A. Ishimaru, B. Rawat, et al. *Statistical Moments and Scintillation Level of Scattered Electromagnetic Waves in the Magnetized Plasma*, Progress In Electromagnetics Research C, **84** (2018), 11-22, DOI: 10.2528/PIERC18030602

[7] G. Jandieri, A. Ishimaru, V. Jandieri, et al. *Model computations of angular power spectra for anisotropic absorptive turbulent magnetized plasma.* Progress in Electromagnetics Research, **70** (2007), 307-328, DOI: 10.2528/PIER070113103

[8] G.V. Jandieri, V.G. Gavrilenko, A.A. Semerikov. *On the effect of absorption on multiple waves scattering in a magnetized turbulent plasma*, Waves in Random Media, **9** (1999), 427-440, DOI: 10.2528/PIER070113103

[9] O.G. Bakunin. *The role of B.B. Kadomtsevs ideas in shaping the current understanding of turbulent transport* (Russian), Uspekhi Phys. Nauk (UFN), **183**, 11 (2013), 1237-1254, DOI: 10.3367/UFNr.0183.201311f.1237

[10] M.E. Berliand.*Modern Problems of Atmospheric Diffusion and Pollution of the Atmosphere*, Hydrometeoizdat, Leningrad, 1975

[11] J.O. Hinze.*Turbulence*, New York: Mc Graw-Hill, 1959

[12] F. Lane. *Frequency Effects in the Radar Return from Turbulent, Weakly Ionized Missiles Wakes*, AIAA 5th Aerospace Sciences Meeting, **5**, 12 (1967), 2193-2200

[13] A.S. Monin, A.M. Iaglom. *Statistical Hydromechanics* (Russian), Nauka, Moscow, 1965

[14] A.S. Monin. *The Spread of Smoke in a Surface layer of the Atmosphere*, Atmospheric diffusion and air pollution, Nauka, Moscow, 1962

[15] G.V. Jandieri, Zh.M. Diasamidze, V.G. Jandieri. *To the theory of electromagnetic waves scattering in media with spatial-temporal flux* (Russian), Izv. VUZov, Radiophysics, **39**, 9 (1996), 1091-1095

[16] G. Jandieri, Zh. Diasamidze, V. Jandieri. *Electromagnetic wave propagation in a randomly crystalline medium*, European Phys. Journal, Applied Physics, **6** (1999), 115-119

[17] V.A. Namiot. *Long range interaction between Brownian particles*, JETP, **90**, 4 (1986), 1212-1214

[18] F. Krause, K.H. Radler. *Mean-field Magnetohydrodynamics and Dynamo Theory*, Academie-Verlag, Berlin, 1980

[19] N.L. Byzova, E.K. Garger, V.N. Ivanov. *On the experimental estimation of the Lagrangian turbulence time scale*, Russian Atmospheric and Ocean Physics, **6**, 6 (1970), 547-555

[20] V.N. Ivanov. *The Utilization of Eulerian Turbulence Parameters for Estimation of Lagrangian Characteristics* (Russian), Meteorological Aspects of Air Pollution, Edit by Berliand M.E., Hydrometeorological Publishing House, Leningrad, 1971

[21] F.A. Gisina. *Distribution of impurity from continuous point source in case of fog*, News of the USSR Academy of Sciences, Series of geophysics, **7** (1964), 1116-1120

[22] G. Jandieri, Z. Sanikidze. *To the Problem of Turbulent Diffusion Instability*, Bulletin of TICMI, **28**, 1 (2024) 33-40