EXTRACT FROM LECTURES

VARIATIONAL FORMULATION FOR REFINED THEORIES, GENERALIZED HELLINGER-REISSNER VARIATIONAL PRINCIPLE

T. Vashakmadze

I. Javakhishvili Tbilisi State University

The variational formulation for construction of refined theories has (see Chien Wei-zang [2], Ciarlet [3], Hellinger [4], Lukasievicz [5], Reissner [8], Washizu [11]) very important meaning. There are also given historical notes, commentaries, comparisions of different theories with richest bibliography. As follow from these works, variational formulation has also other convenient properties (see e.g. [11], part A), but I call readers attention to following well-known destinguishing féatures such as the possibility of investigation and approximate solution of initial boundary value problems in most general Sobolev spaces.

For simplicity and definiteness we consider linear theory of elasticity, when Ω_h is isotropic homogeneous elastic media with constant thickness. But the consideration of more general cases (such as, anisotropy and non-linearity) doesn't represent any difficulties.

Let the boundary conditions be such:

$$u\Big|_{S_1} = \sigma_n\Big|_{S_2} = 0, \qquad S_1 \cup S_2 = S, \quad \sigma_3\Big|_{S^\pm} = g^\pm. \tag{1}$$

Here and below are used notations of Vashakmadze [9].

Following Mikhlin [6], let us consider the following functional:

$$\Phi[u] = \int\limits_{\Omega_h} (\frac{1}{2}\sigma e + uf)dv - \int\limits_{S^+} u\sigma_n ds - \int\limits_{S^-} u\sigma_n ds, \tag{2}$$

where n is outward normal.

If we use the third equality of Hooke's law, we have

$$\Phi[u] = \frac{1}{2}\int\limits_{\Omega_h} \Big[((\lambda^* + 2\mu)u_{\alpha,\alpha} + \lambda^* u_{3-\alpha}, 3-\alpha)u_{\alpha,\alpha} + \mu(u_{1,2}+u_{2,1})^2 +$$

$$+\sigma_{\alpha,3}(u_{\alpha,3}+u_{3,\alpha}) + \frac{\lambda}{\lambda+2\mu}\sigma_{33}u_{\alpha,\alpha} + \sigma_{33}u_{3,3}\Big] dv + \int\limits_{\Omega_h} uf dv - \int\limits_{S^\pm} u\sigma_n ds.$$

Assuming (see e.g. Berdichevski [1]):

$$u_i(x_1,x_2,t) = v_i(x_1,x_2) + tw_i(x_1,x_2) + r_i[u], \tag{3}$$

for $\Phi[u]$ immediately follows:

$$\Phi[u] = \frac{1}{2}\int\limits_{D}\Bigg[(\lambda^* + 2\mu)[2hv_{\alpha,\alpha}^2 + \frac{2h^3}{3}w_{\alpha,\alpha}] + \lambda^*\,[2hv_{\alpha,\alpha}v_{3-\alpha,3-\alpha}+$$

$$+\frac{2h^3}{3}w_{\alpha,\alpha}w_{3-\alpha,3-\alpha}\Bigg] + \mu\left(2hv_{\alpha,3-\alpha}^2 + \frac{2h^3}{3}w_{\alpha,3-\alpha}^2\right) + \tag{4}$$

$$+(v_{3,\alpha} + w_\alpha)\int\limits_{-h}^{h}\sigma_{\alpha,3}dt + w_{3,\alpha}\int\limits_{-h}^{h}t\sigma_{\alpha,3}dt + \frac{\lambda}{\lambda+2\mu}\left(v_{\alpha,\alpha}\int\limits_{-h}^{h}\sigma_{3,3}dt+\right.$$

$$\left.+w_{\alpha,\alpha}\int\limits_{-h}^{h}t\sigma_{3,3}dt\right) + w_3\int\limits_{-h}^{h}\sigma_{3,3}dt\Bigg]\,dw + \int\limits_{D}\Bigg[v_i\int\limits_{-h}^{h}f_i dt+$$

$$+w_i\int\limits_{-h}^{h}tf_i dt\Bigg]\,dw - \int\limits_{S^\pm}(v_i \pm hw_i)\sigma_{3n}ds + R[u],$$

where $r_{ij}[u]$ is defined from (3) evidently and

$$R[u] = \int\limits_{\Omega_h}(\frac{1}{2}\sigma_{ij}r_{ij}[u] + f_i r_i[u])dv - \int\limits_{S^\pm}\sigma_{3n}r_i[u]ds.$$

Now let us consider the difference

$$\Phi[v,w] = \Phi[u] - R[u]. \tag{5}$$

This expression will be called **Hellinger-Reissner generalized two-dimensional functional.** Euler-Lagrange conditions give us the following two-dimensional system of partial differential equations:

$$
\begin{aligned}
&2h[\mu\Delta v_\alpha+(\lambda^*+\mu)\partial_\alpha(v_{\alpha,\alpha}+v_{3-\alpha,3-\alpha})]+\frac{\lambda}{2(\lambda+2\mu)}\int_{-h}^{h}\sigma_{33,\alpha}dt=\\
&=\int_{-h}^{h}f_\alpha dt-(g_\alpha^+-g_\alpha^-),\quad \alpha=1,2,\\
&\frac{1}{2}\int_{-h}^{h}\sigma_{\alpha 3,\alpha}dt=\int_{-h}^{h}f_3dt-(g_3^+-g_3^-),\\
&\frac{2h^3}{3}[\mu\Delta w_\alpha+(\lambda^*+\mu)\partial_\alpha(w_{\alpha,\alpha}+w_{3-\alpha,3-\alpha})]-\frac{1}{2}\int_{-h}^{h}\sigma_{\alpha 3}dt+\\
&+\frac{\lambda}{2(\lambda+2\mu)}\int_{-h}^{h}t\sigma_{33,\alpha}dt=\int_{-h}^{h}tf_\alpha dt-h(g_\alpha^++g_\alpha^-),\quad \alpha=1,2.\\
&\frac{2h^3}{3}\Delta w_3-\frac{1}{2}\int_{-h}^{h}\sigma_{33}dt=\int_{-h}^{h}tf_3dt-h(g_3^++g_3^-).
\end{aligned}
\tag{6}
$$

If now we use formulae of type (2.15) and (2.16) (see [9]), assume also $w_3\equiv 0$, from (5) follows parametrical representation of refined theories full identical to (P_h) schemes (see [9]).

Thus, the following conclusions are true.

The functional $\Phi[u]-R[u]$ corresponds to the refined theories (in wide sense) and scheme (P_h) for this functional represents Euler's equations.

In particular, from (6_{3-5}) equations follow:

$$
\begin{aligned}
&\frac{2h^3}{3}[\mu\Delta w_\alpha+(\lambda^*+\mu)graddivw_+]-\frac{\mu h}{(1+2\gamma)}(w_\alpha+v_{3,\alpha})=\\
&=\int_{-h}^{h}tf_\alpha dt-h(g_\alpha^++g_\alpha^-)-\frac{\lambda}{2(\lambda+2\mu)}\int_{-h}^{h}t\sigma_{33,\alpha}dt=F_\alpha,\\
&\frac{\mu h}{(1+2\gamma}[\Delta v_3+w_{\alpha,\alpha}]=\int_{-h}^{h}f_3dt-(g_3^++g_3^-)=F_3.
\end{aligned}
\tag{7}
$$

This system, of course, gives (see e.g. [8]) when $\gamma=0,1$, Mindlin's theory [7] for $\gamma=\dfrac{12-\pi^2}{2\pi^2}$, for theory of [11] - Vekua [10] $\gamma=0$ (compare with [11], ch.8, §8.8. or [5], ch.2, section 2.1; here it's discussed physical aspects of such differences).

The form of system of differential equations of refined theories for anisotropic case, as follows from [9], with respect to averaged deflection v_3 and components of normals rotation w_α is such:

$$\frac{2h^3}{3}\left[(c_{1\alpha}\partial_1+c_{\alpha 6}\partial_2)w_{1,\alpha}+(c_{\alpha 6}\partial_1+c_{26}\partial_2)w_{2,\alpha}+(2c_{16}\partial_1+c_{66}\partial_2)w_{1,3-\alpha}+\right.$$

$$\left.+(c_{66}\partial_1+2c_{26}\partial_2)w_{2,3-\alpha}\right]-\frac{h}{(1+2\gamma)\delta}\left[a_{3+\alpha,3+\alpha}(w_\alpha+v_{3,\alpha})-\right.$$

$$\left.-a_{3+\alpha,6-\alpha}(w_{3-\alpha}+v_{3,3-\alpha})\right]=F_\alpha,$$

$$\frac{h}{(1+2\gamma)\delta}\left[(a_{44}\partial_{11}-2a_{45}\partial_{12}+a_{55}\partial_{22})v_3+(a_{3+\alpha,3+\alpha}\partial_\alpha-a_{3+\alpha,6-\alpha}\partial_{3-\alpha})w_\alpha\right]=F_3.$$

where, a, b are elasticity and rigidity numbers defined from Hooke's law (about coefficient c see [9]) $\delta=a_{44}a_{55}-a_{45}^2$.

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