ON NONLINEAR AND NON-SHALLOW SHELLS

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In the present paper shell-like bodies will be called thin and shallow if the following assumptions are fulfilled

$$a_\alpha^\beta - x_3 b_\alpha^\beta \cong a_\alpha^\beta, \quad -h(x_1,x_2) \leq x_3 \leq h(x_1,x_2), \quad (\alpha,\beta=1,2)$$

where a_α^β and b_α^β are mixed components of the metric tensor and tensor of the curvature of the shell middle surface, x_3 is a thickness coordinate, h is a semi-thickness, x_1, x_2 are curvilinear coordinates of the middle surface. And if

$$a_\alpha^\beta - x_3 b_\alpha^\beta \neq a_\alpha^\beta, \quad \textit{more exactly} \;\; |x_3 b_\alpha^\beta| < 1,$$

then the shell will be called non-shallow.

For linear theory of thin and shallow shells I. Vekua [1] developed the method of reduction of three-dimensional problems of elasticity to two-dimensional ones.

By means of the method of I.Vekua we obtained the system of differential equations for non-shallow shell with boundary conditions in case of geometrical and physical non-linearity.

1. Complete system of equations of the three-dimensional non-linear theory of elasticity may be written as [2],[3]:

a) equilibrium aquations

$$\partial_i(\sqrt{g}\boldsymbol{\sigma}^i) + \sqrt{g}\boldsymbol{\Phi} = 0, \; \boldsymbol{\sigma}^i = \sigma^{ij}(\boldsymbol{R}_j + \partial_j \boldsymbol{u}) \; \left(\partial_i = \frac{\partial}{\partial x_i}\right), \tag{1}$$

where g is discriminant of the metric form of the space,$\boldsymbol{\sigma}^i$ are contravariant vector of stress,$\boldsymbol{\Phi}$ is an external force, σ^{ij} are contravariant components of the stress tensor, $\boldsymbol{R}_i$ are covariant base vectors of the spase, $\boldsymbol{U}$ is a displacement vector.

b) equations of the state, there exists variety of approach about the relations of tensors of stress and strain (Seth B.R., Signorini A., Murnaghan F.O., John F., Blatz P.J. and Ko W.L.,Novozhilow B.B., ect)

The material is cold hyper-elastic, if the stress is obtained from the energy function of the strain with the relation

$$\sigma^{ij} = \frac{\partial \ni}{\partial \varepsilon_{ij}},$$

where $\ni$ is an energy function of the strain, ε_{ij} are covariant components of the strain tensor.

The theory of second order hyper-elasticity has the form [4]:

$$\ni = \frac{1}{g}E^{ijpq}\varepsilon_{ij}\varepsilon_{pq} + \frac{1}{3}E^{ijpqsk}\varepsilon_{ij}\varepsilon_{pq}\varepsilon_{sk}\sigma^{ij} = E^{ijpq}\varepsilon_{pq} + E^{ijpqsk}\varepsilon_{pq}\varepsilon_{sk}, \qquad (2)$$

where E^{ijpq}, E^{ijpqsk}, are coefficients of elasticity of the first and second orders.

For isotropic materials coefficients of elasticity of the first and second orders are expressed only by two coefficients of Lamé

$$E^{ijpq} = \lambda g^{ij}g^{pq} + \mu(g^{ip}g^{jq} + g^{iq}g^{jp}), \qquad (3)$$

and coefficients of elasticity of second order are defined by formula [4]

$$\begin{gathered} E^{ijpqsk} = E_1 g^{ij}g^{pq}g^{sk} + E_2(g^{ij}g^{pq}g^{sk} - g^{ij}g^{pk}g^{qs}) + \\ E_3 g^{ip}g^{jq}g^{sk} + E_4 g^{is}g^{pq}g^{jk}, \end{gathered} \qquad (4)$$

here E_1, E_2, E_3 andE_4 are modules of elasticity of secod order for isotropic and elastic materials.

c) boundary conditions for the stress vectors, acting on the area with the normal $\overset{*}{\boldsymbol{l}}$, has the form

$$\boldsymbol{\sigma}_{(\overset{*}{l})} = \boldsymbol{\sigma}^i \overset{*}{l}_i, \quad (\overset{*}{l}_i = \overset{*}{\boldsymbol{l}}\boldsymbol{R}_i) \qquad (5)$$

For reducing three-dimensional problems to two-dimensional ones it is necessary to write(1-5) in bases of the middle surface, therefore it is expedient to consider the coordinate system, normally connected with the middle surface, i. e.

$$\boldsymbol{R}(x_1, x_2, x_3) = \boldsymbol{r}(x_1, x_2) + x_3\boldsymbol{n}(x_1, x_2),$$

where $\boldsymbol{R}$ and $\boldsymbol{r}$ are radius-vectors of the space and middle surface, $\boldsymbol{n}$ is a normal of the surface. The relations between covariant and contravariant base vectors of the space and the surface are expressed as [1]:

$$\boldsymbol{R}_i = A_{ij}\boldsymbol{r}^j = A_{i.}^{.j}\boldsymbol{r}_j; \ \boldsymbol{R}^i = A^{ij}\boldsymbol{r}_j = A_{.j}^{i.}\boldsymbol{r}^j, \boldsymbol{R}_3 = \boldsymbol{R}^3 = \boldsymbol{r}_3 = \boldsymbol{r}^3 = \boldsymbol{n}, \qquad (6)$$

where

$$\begin{gathered} A_{\alpha\beta} = a_{\alpha\beta} - x_3 b_{\alpha\beta}, \ \sqrt{g}A^{\alpha\beta} = \sqrt{a}[a^{\alpha\beta} + x_3(b^{\alpha\beta} - 2Ha^{\alpha\beta})], \\ A_{\alpha 3} = A^{\alpha 3} = A_{.3}^{\alpha.} = A_{\alpha.}^{.3} = 0, \ A_{33} = A^{33} = A_{.3}^{3.} = A_{3.}^{.3} = 1, \\ A_{\alpha.}^{.\beta} = a_{\alpha}^{\beta} - x_3 b_{\alpha}^{\beta}, \ \sqrt{g}A_{.\beta}^{\alpha.} = \sqrt{a}[a_{\beta}^{\alpha} + x_3(b_{\beta}^{\alpha} - 2Ha_{\beta}^{\alpha})], \\ \sqrt{g} = \sqrt{a}(1 - 2Hx_3 + Kx_3^2, \ \sqrt{a} = a_{11}a_{22} - a_{12}^2, \ (a_{\alpha\beta} = \boldsymbol{r}_\alpha\boldsymbol{r}_\beta) \end{gathered} \qquad (7)$$

H and K are middle and Gaussian curvatures of the surface, a is a discriminant of the metric form of the surfase.

From (2),(3) and (4) we get

$$\begin{gathered} \sigma^i = A_{i_1}^i A_{p_1}^p \left\{ \left[M^{i_1 j_1 p_1 q_1} + A_{k_1}^k M^{i_1 j_1 p_1 q_1 s_1 k_1}(\boldsymbol{r}_{s_1}\partial_k\boldsymbol{u} + \tfrac{1}{2}A_{s_1}^s\partial_s\boldsymbol{u}\partial_k\boldsymbol{u}) \right] \times \right. \\ \left. \times(\boldsymbol{r}_{q_1}\partial_p\boldsymbol{u} + \frac{1}{2}A_{q_1}^q\partial_p\boldsymbol{u}\partial_q\boldsymbol{u}) \right\} \boldsymbol{r}_{j_1} + A_{j_1}^j\partial_j\boldsymbol{u}), \end{gathered} \qquad (8)$$

where

$$\begin{gathered} M^{i_1j_1p_1q_1} = \lambda a^{i_1j_1}a^{p_1q_1} + \mu(a^{i_1p_1}a^{j_1q_1} + a^{i_1q_1}a^{j_1p_1}), \\ M^{i_1j_1p_1q_1s_1k_1} = (E_1+E_2)a^{i_1j_1}a^{p_1q_1}a^{s_1k_1} - E_2a^{i_1j_1}a^{p_1s_1}a^{q_1k_1} + \\ E_3a^{i_1p_1}a^{j_1q_1}a^{s_1k_1} + E_4a^{i_1k_1}a^{j_1s_1}a^{p_1q_1}. \end{gathered} \tag{9}$$

The stress tensor of the lateral surfase $d\widehat{S} = d\widehat{s}dx_3$ with the normal

$$\widehat{\boldsymbol{l}} = [(1-x_3k_s)\boldsymbol{l} - x_3\tau_s\boldsymbol{s}]\frac{ds}{d\widehat{s}}, \ \ (\widehat{\boldsymbol{s}} = \frac{d\boldsymbol{R}}{d\widehat{s}} = [(1-x_3k_s)\boldsymbol{s} + x_3\tau_s\boldsymbol{l}]\frac{ds}{d\widehat{s}}),$$

has the form [5]

$$\boldsymbol{\sigma}_{(\widehat{l})} = \boldsymbol{\sigma}^\alpha(\widehat{\boldsymbol{l}}\boldsymbol{R}_\alpha) = \sqrt{\frac{g}{a}}\boldsymbol{\sigma}^\alpha(\boldsymbol{l}\boldsymbol{r}_\alpha)\frac{ds}{d\widehat{s}}, \ (\widehat{\boldsymbol{l}} \times \widehat{\boldsymbol{s}} = \boldsymbol{n}), \tag{10}$$

where $\boldsymbol{l}$ and $\boldsymbol{s}$ are unit vectors of the tangential normal and tangent lateral curve of the middle surface ($\boldsymbol{l} \times \boldsymbol{s} = \boldsymbol{n}$),$k_s$ and τ_s is the normal curvature and geodesic forsion of this curve, $d\widehat{s}$ and ds are linear elements of the surface $x_3 = const$ and middle surface, moreover

$$d\widehat{s} = \sqrt{1 - 2x_3k_s + x_3^2(k_s^2 + \tau_s^2)}ds.$$

2. Now, following I. Vekua [1], assume the validity of the expansions

$$\left(\sqrt{\frac{g}{a}}\boldsymbol{\sigma}^i, \boldsymbol{u}, \sqrt{\frac{g}{a}}\boldsymbol{\Phi}\right) = \sum_{m=0}^{N}(\overset{(m)}{\boldsymbol{\sigma}}{}_i, \overset{(m)}{\boldsymbol{u}}, \overset{(m)}{\boldsymbol{\Phi}})P_m\left(\frac{x_3}{h}\right),$$

where

$$(\overset{(m)}{\boldsymbol{\sigma}}{}^i, \overset{(m)}{\boldsymbol{u}}, \overset{(m)}{\boldsymbol{\Phi}}) = \frac{2m+1}{2h}\int_{-h}^{h}\left(\sqrt{\frac{g}{a}}\boldsymbol{\sigma}^i, \boldsymbol{u}, \sqrt{\frac{g}{a}}\boldsymbol{\Phi}\right)P_m\left(\frac{x_3}{h}\right)dx_3, \tag{11}$$

here P_m are Legendre's polynomials of order m.

By substituting these expansions into (1),(8),(10) satisfying beforehand the boundary conditions of face surface $x_3 = \pm h$

$$\boldsymbol{\sigma}_3(x_1, x_2, \pm h) = \overset{(\pm)}{\boldsymbol{\sigma}}_3,$$

we get the system of two-dimensional equations.For example, equilibrium equations have the form [5]:

$$\nabla_\alpha \overset{(m)}{\boldsymbol{\sigma}}{}^\alpha - \frac{2m+1}{h}(\overset{(m-1)}{\boldsymbol{\sigma}_3} + \overset{(m-3)}{\boldsymbol{\sigma}_3} + \cdots) + \overset{(m)}{F} = 0,$$

where ∇_α are covariant derivatives on the surface, and

$$\begin{gathered} \overset{(m)}{\boldsymbol{F}} = \overset{(m)}{\boldsymbol{\Phi}} + \frac{2m+1}{2h}\left(\sqrt{\frac{g_+}{a}}\overset{(+)}{\boldsymbol{\sigma}}_3 - (-1)^m\sqrt{\frac{g_-}{a}}\overset{(-)}{\boldsymbol{\sigma}}_3\right), \\ \left(\sqrt{g_\pm} = \sqrt{a}(1 \pm 2Hh + Kh^2)\right). \end{gathered}$$

References

[1] Vekua I.N., Shell Theory: General Methods of Constraction. M. Nauka, 1982. 288p (Russian, English Translation - Pitman Advanced Publishing Program, Boston-London-Melburne 1985, 1-287.).

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[4] Oden J.T., Finite Elements of Non-Linear Continua. M., "Mir", 1976, 464p (in Russian).

[5] Meunargia T.V., On Two-Dimensional Equation of the Linear Theory of Shells. Trud. Inst.Prikl. math. Inst.I.N.Vekua, Tbilisi, State Univer., 1990, v.38, p.5-43.