SOME PROBLEMS OF ELASTICITY FOR THE TWO-COMPONENT ISOTROPIC MIXTURE

G. Khatiashvili

N.I. Muskhelishvili Institute of Computational Mathematics
of Georgian Academy of Sciences

The linearized theory of elasticity for the indicated medium was given by T.R. Steel [1]. The two-dimensional problems for the isotropic mixture are considered by T.R. Steel [2] and M.O. Basheleishvili [3]. Some three-dimensional basic problems for indicated medium are considered by D.G. Natroshvili, A.J. Jagmaidze and M.J.Svanadze [4].

We consider some problems of Saint-Venant's tyipe and antiplane deformation for the medium bounded by cylindrical surface.

Let us consider the rectangular system of Cartesian coordinates $0x_1x_2x_3$ and the body, made from the mixture, occupying the domain Ω, bounded by cylindrical surface $f(x_1,x_2)=0$ and planes $x_3=0$ and $x_3=l$ $(l>0)$.

It is asssumed that the components of displacements, stress and strains $u_j^{(\gamma)}$ $\tau_{ij}^{(\gamma)}$ and $e_{ij}^{(\gamma)}$ of the two different substances (with number 1 and 2) constituent the mixture, obey the Steel's law [1]

$$\begin{aligned}\tau_{ij}^{(\gamma)}&=[(-1)^\gamma\alpha_2+\lambda_{3\gamma-2}divu^{(1)}+\lambda_{4-\gamma}divu^{(2)}]\delta_{jk}-\\&-2\mu_{2\gamma-1}e_{jk}^{(1)}+2\mu_{4-\gamma}e_{jk}^{(2)}+(-1)^\gamma 2\lambda_5 h_{jk}\ \ (j,k=1,2,3;\ \ \gamma=1,2),\end{aligned}\tag{1}$$

where δ_{ij} is Kroneker's symbol; α_2, λ_m and μ_k are the constants of elasticity ($\alpha_2==\lambda_3-\lambda_4$) and the values $e_{jk}^{(\gamma)}$ and h_{jk} are given by the equalities

$$2e_{jk}^{(\gamma)}=D_ju_k^{(\gamma)}+D_ku_j^{(\gamma)},\ \ 2h_{jk}=D_j(u_k^{(1)}+u_k^{(2)})-D_k(u_j^{(1)}+u_j^{(2)}),\tag{2}$$

where $D_j=\partial/\partial x_j$.

In [4] the inequalities

$$\begin{aligned}&\lambda_5<5;\ \ \mu_1,\ \mu_2>0,\ g=\mu_1\mu_2-\mu_3^2>0,\ \lambda_1-\frac{\alpha_2}{\rho}\rho^{(2)}+\frac{2}{3}\mu_1>0,\\&G=\left(\lambda_2+\frac{2}{3}\mu_2+\frac{\alpha_2}{\rho}\rho^{(2)}\right)\left(\lambda_1+\frac{2}{3}\mu_1-\frac{\alpha_2}{\rho}\rho^{(1)}\right)>0,\end{aligned}\tag{3}$$

are established, where $\rho^{(\gamma)}$ are the densities of substances with number 1 and 2 $(\gamma=1,2)$.

It is assumed that γ henceworth always will receive only meanings 1 and 2.

The equations of the elastic equilibrium of the isotropic mixture may be represented in the form [1,4,2]

$$\Delta(a_\gamma u^{(\gamma)}+Cu^{(3-\gamma)})+graddiv(bu^{(\gamma)}+du^{(3-\gamma)})=-\rho^{(\gamma)}\Phi_\star^{(\gamma)},\tag{4}$$

where $u^{(\gamma)}(u_1^{(\gamma)},\ u_2^{(\gamma)},\ u_3^{(\gamma)})$ are vectors of the displacements, $\Phi_\star^{(\gamma)}(\Phi_1^{(\gamma)},\ \Phi_2^{(\gamma)}\ \Phi_3^{(\gamma)})$ are vectors of the mass-forces and constants $a_\gamma,\ b_\gamma,\ c$ and d are given by the equalities

$$\begin{aligned} &a_\gamma = \mu_\gamma - \lambda_5,\ \ b_\gamma = \mu_\gamma + \lambda_\gamma + \lambda_5 + (-1)^\gamma \frac{\alpha_2}{\rho}\rho^{(3-\gamma)}, \\ &C = \mu_3 + \lambda_5,\ \ d = \mu_3 + \lambda_3 - \lambda_5 - \frac{\alpha_2}{\rho}\rho^{(1)} = \mu_\gamma + \lambda_4 - \lambda_5 + \frac{\alpha_2}{\rho}\rho^{(2)}. \end{aligned} \tag{5}$$

The components of the stress $\tau_{ij}^{(\gamma)}$. determining the elastic equilibrium of the cylindrical beam due the exstension by longitudonal force $\tau_{33}^{(\gamma)} = \tau^{(\gamma)}$ (parallel to $0x_3$ axis) are given on the form

$$\tau_{jj}^{(\gamma)} = (-1)^\gamma \alpha_2 + \lambda_{3\gamma-2}B_1 + \lambda_{4-\gamma}B_2 + 2\mu_{2\gamma-1}A_j^{(1)} + 2\mu_{4-\gamma}A_j^{(2)},$$

$$\tau_{\beta k}^{(\gamma)} = 0\ (\beta \neq k),\ (j,\beta,k = 1,2,3),$$

where

$$B_\gamma = \frac{1}{3G}\left[a_\gamma^*\left(\lambda_{3-\gamma} + \frac{2}{3}\mu_{3-\gamma} - (-1)^\gamma \frac{\alpha_2}{\rho}\rho^{(\gamma)}\right) - \right.$$

$$\left. -a_{3-\gamma}^*\left(\lambda_{5-2\gamma} + \frac{2}{3}\mu_{5-2\gamma} - \frac{\alpha_2}{\rho}\rho^{(\gamma)}\right)\right],\ \ a_\gamma^* = \frac{1}{\omega_0}\tau^{(\gamma)},$$

ω_0 is the area of crossection of the beam and the constants $A_j^{(\gamma)}$ are expressed by constants of elasticity and B_k, which easily will be easily obtained from the end conditions of the beam.

The components of the stress in the problem of torsion due to torque (couple forces) will be given in the form

$$\tau_{13}^{(\gamma)} = (-1)^\gamma \tau_\star(\mu_3 - \mu_\gamma)(D_1 f_\star - x_2),\ \tau_{23}^{(\gamma)} = (-1)^\gamma \tau_\star(\mu_3 - \mu_\gamma)(D_2 f_\star + x_1),$$

where the function $f_\star(x_1,x_2)$ is the solution of the following boundary-value problem: $D_1^2 f_\star + D_2^2 f_\star = 0$ in the domain ω of the crossection of beam, $D_n^2 f_\star \equiv \equiv n_1 D_1 f_\star + n_2 D_2 f_\star = x_2 n_1 - x_1 n_2$ in each point of the closed curve S of the boundary of domain ω where $n(n_1,n_2)$ is the unit vector of the outward normal S and the constants $\tau_\star$ is determined from the end conditions.

Other components of stress are equal to zero.

Now we consider the antiplane deformation of the medium, made from the isotropic mixture, when the components of the displacements $u_1^{(\gamma)}$ and $u_2^{(\gamma)}$ are equal to zero and $u_3^{(\gamma)} = F^{(\gamma)}(x_1,x_2)$, where the functions $F^{(\gamma)}$ will be determined in the particular problem.

Introduce for the functions $T^{(\gamma)}(x_1,x_2)$ and $V^{(\gamma)}(x_1,x_2)$ the linear algebraic operators

$$\begin{aligned} &M_\gamma T^{(1,2)} \equiv (\mu_{4-\gamma} + (-1)^\gamma \lambda_5)T^{(2)} + (\mu_{2\gamma-1} + (-1)^\gamma \lambda_5)T^{(1)}, \\ &N_\gamma V^{(1,2)} \equiv (-1)^\gamma g^{-1}[(\mu_{5-2\gamma} - \lambda_5)V^{(2)} - (\mu_{\gamma+1} + \lambda_5)V^{(1)}]. \end{aligned} \tag{6}$$

$$If\ \ T^{(\gamma)} = N_\gamma V^{(1,2)},\ \ then\ \ M_j T^{(1,2)} = V^{(j)}\ \ (j = 1, 2). \tag{7}$$

Conforming stress to displacements $u_3^{(\gamma)} = F^{(\gamma)}(x_1, x_2)$ will be given in the form

$$\begin{aligned} &\tau_{jj}^{(\gamma)} = (-1)^\gamma \alpha_2,\ \ \tau_{m3}^{(\gamma)} = M_\gamma(D_m F^{(1,2)}), \\ &\tau_{12}^{(\gamma)} = \tau_{21}^{(\gamma)} = 0,\ \ (m = 1, 2;\ j = 1, 2, 3). \end{aligned} \tag{8}$$

Consider the medium (finite or infinite), bounded by cylindrical surface $f(x_1, x_2) = 0$ and propose that in its each point external forces are acting uniformly (along the axis) $\tau^{(\gamma)}(s)$, where s is the arc of the boundary S of the domain ω, of the crossection of cylindrical surface.

In this case in the medium the displacements $u_3^{(\gamma)} = F^{(\gamma)}(x_1, x_2)$ must satisfay the equations $D_1^2 F^{(\gamma)} + D_2^2 F^{(\gamma)} = 0$ in the domain ω and the boundary conditions $D_n F^{(\gamma)} = N_\gamma \tau^{(1,\epsilon)}$ on the S. Then, the indicated displacements $u_j^{(\gamma)}$ will satisfy the equations (4) and corresponding components of the stress on the cylindrical surface will satisfy the boundary conditions

$$\tau_{jn}^{(\gamma)} = \tau_{1j}^{(\gamma)} n_1 + \tau_{2j}^{(\gamma)} n_2 = \frac{1}{2}(j-1)(j-2)\tau^{(\gamma)}\ \ (j = 1, 2, 3). \tag{9}$$

Let us consider some particular cases exercises

1. We consider the infinite elliptic cylinder, the crossection of which by plane $0x_1x_2$ is the elliptic domain, bounded by the ellipse

$$x_1 = acos\theta,\ \ x_2 = bsin\theta\ \ (0 \le \theta \le 2\pi;\ a > b,\ a^2 - b^2 = c^2). \tag{10}$$

The displacements $u_3^{(\gamma)}$, determining the elastic state will be represented as the series with respect to Faber's polynomials

$$u_3^{(\gamma)} = \frac{1}{\pi} Re \sum_{k=1}^{\infty} \frac{1}{k} \frac{t_1^k + t_2^k}{1 - \lambda^{2k}} \int_0^{2\pi} \varphi_k(\theta) N_\gamma \tau^{(1,2)} d\theta,\ \ (\lambda = \frac{a-b}{a+b} < 1),$$

where

$$t_1 = z + \sqrt{z^2 - c^2},\ \ t_2 = z - \sqrt{z^2 - c^2},\ \ z = x_1 + ix_2,\ \ (i^2 = -1),$$

$$\varphi_k(\theta) = (a^2 sin^2\theta + b^2 cos^2\theta)^{1/2}[(1 + \lambda^k)cosk\theta - i(1 - \lambda^k)sink\theta].$$

2. Now we consider the problem of the antiplane deformation of the infinite medium with the elliptic hole (10), in each point of which the exterior forces (9) are acting uniformly (along the axis $0x_3$).

Coresponding displacements, determining the solution of this problem will be given in the form

$$u_3^{(\gamma)} = \frac{1}{\pi} Re \left[\sum_{k=1}^{\infty} (kt_1^k)^{-1} \int_0^{2\pi} (a^2 sin^2\theta + b^2 cos^2\theta)^{1/2} e^{ik\theta} N_\gamma \tau^{(1,2)} d\theta + \right.$$

$$\left. + \left(\int_s N_\gamma \tau^{(1,2)} ds \right) lnt_1 \right].$$

3. Now we consider the case,when along the axis $0x_3$ of the infinite medium of the space $0x_1x_2x_3$ are acting the concentrated forces $P^{(\gamma)} = const$, parallel to axis $0x_3$ and concentrated couple forces by moments $m_1^{(\gamma)}$ and $m_2^{(\gamma)}$ where the vectors $\vec{m}_1^{(\gamma)}$ and $\vec{m}_2^{(\gamma)}$ are parallel to $0x_1$ and $0x_2$ respectively.

The displacements coresponding to indicated loading may be represented in the form $(m_j^{(\gamma)} = const)$

$$(u_3^{(\gamma)})_p = \frac{1}{2\pi}(N_\gamma P^{(1,2)})lnr, \quad (x_1^2 + x_2^2 = r^2), \quad (u_3^{(\gamma)})_m = \frac{(-1)^\gamma}{\pi r^2 g}\Big[(\mu_{(5-2\gamma)}-$$

$$-\lambda_5)(m_1^{(2)}x_2 - m_2^{(2)}x_1) - (\mu_{\gamma+1} + \lambda_5)(m_1^{(1)}x_2 - m_1^{(1)}x_1)\Big].$$

4. The stress state of the infinite medium with circular cylindrical hole $x_1^2+x_2^2 = R^2$ is considered when in its each point jumping forces of the following kind are acting uniformly (along the $0x_3$ axis)

$$(\tau^{(\gamma)})_1 = \begin{cases} p_1^{(\gamma)}, & 0 \le \theta \le \pi; \\ -p_1^{(\gamma)}, & \pi \le \theta \le 2\pi. \end{cases} \quad (\tau^{(\gamma)})_2 = \begin{cases} -p_2^{(\gamma)}, & -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}; \\ p_2^{(\gamma)}, & \dfrac{\pi}{2} \le \theta \le \dfrac{3}{2}\pi. \end{cases}$$

$$(p_j^{(\gamma)} = const).$$

We represent the corresponding displacements in the form $u_3^{(\gamma)} = ReH_\gamma(z)$, where $H_\gamma(z)$ are the analytical functions of the complex variable $z = x_1 + ix_2$ in the exterior (with respect to circular hole) domain ω. If the $H'_\gamma(z)$ vanish at the infinity for the indicated jumping forces, the introduced complex potential obtains the following form

$$iz[H_\gamma(z)]_j = \frac{2R}{\pi}(N_\gamma P_j^{(1,2)})ln\frac{z + z_0^{(j)}}{z - z_0^{(j)}}, \quad (j = 1, 2;\ z_0^{(1)} = R,\ z_0^{(2)} = Ri)$$

From these expressions we may obtain the values (meanings) of the complex potentials for the hydrodynamic dipole of the plane, incompressible , stationary and irrotational movement of fluids.

References

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