ON SOME GENERALISATION OF NON-LOCAL BOUNDARY VALUE PROBLEMS FOR ELLIPTIC EQUATIONS

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In the present work there are considered the following problems:

Problem 1. Consider a bounded domain Ω, with the boundary Γ. Let $\Omega_i, (i=\overline{1,m})$ be domains with boundaries Γ_i, such that each of the following domains is placed strictly inside of the proceeding ones. In addition, Γ_i represents a diffeomorfic image of $\Gamma, X^{(i)} \in \Gamma_i, X \in \Gamma, I_i(X) = X^{(i)}$. Γ and Γ_i are Liapunov surfaces.

Let L be uniform elliptic operator of the following type:

$$L \equiv \sum_{i,k=1}^{n} a_{i,k}(x)\frac{\partial^2}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x)\frac{\partial}{\partial x_i} + c(x),$$

where $a_{i,k}(x), b_i(x), c(x)(c(x) \leq 0)$ are prescribed functions.

We have to find the regular solution of the following equation:

$$Lu(x) = F(x), \quad x \in \Omega, \tag{1}$$

satisfying the following non-local conditions

$$u(X) = \sum_{i=1}^{m} q_i u(X^{(i)}) + \Phi(X), \tag{2}$$

where Φ, F are prescribed functions, F is defined on Ω, Φ - on $\Gamma, q_i(i=\overline{1,m})$ are prescribed constants, $\sum\limits_{i=1}^{m} q_i = q_0$.

We assume, that $a_{i,k}, b_i, c, F$ and Φ are such, that there exists the regular solution of Dirichlet problem for the equation (1).

In order to solve the problem there was suggested the following iteration procedure:

$$\begin{aligned} &Lu^{(k+1)}(x) = F(x), \\ &u^{k+1}(X) = \sum_{i=1}^{m} q_i u(k)(X^{(i)}) + \Phi(X), \quad k = 0,1,... \\ &u^{(0)}(X^{(0)}) = 0. \end{aligned} \tag{3}$$

Theorem 1. *If* $\left|\sum\limits_{i=1}^{m} q_i\right| = q_0 < 1$, *in addition, either all* $q_i \leq 0$ *or* $q_i > 0$, *then there exists the unique regular solution of the problem (1)-(2), (3) iteration procedure converges with speed of geometrical progression and the following estimation is valid:* $\max\limits_{\bar{\Omega}} |(x) - u^{(k)}(x)| \leq cq_0^k, \forall k$, *where* c *is the constant, which does not depend on* $u(x)$ *and* $u^{(k)}(x)$.

Problem 2. In this case domain Ω is kernel with Γ boundary and a radius. Respectively, $\Omega_i(i=\overline{0,m})$ are concentrated kernels with Γ_i boundaries and a_i radiuses, $a > a_0 > a_1 > ... > a_m > 0$. The centre of coordinate system is placed in the centre of kernels. Here we bring in the prescribed functions: $F(r,\theta,\phi) \in C^1(\Omega)$ and $f(\theta,\phi) \in C^2(\Gamma)$.

We have to find the regular solution of the equation

$$\Delta u(x) = 0, \quad x \in \Omega, \tag{4}$$

satisfying the followig generalisation of before known non-local boundary conditions,

$$\alpha \frac{\partial u}{\partial r}\bigg|_{r=a} + \beta u\bigg|_{r=a} = \sum_{i-0}^{m} \alpha_i u\bigg|_{r=a_i} + \frac{\gamma}{a-a_0}\int_{a_0}^{a} u dr + f, \tag{5}$$

where $u(x) \in C^2(\Omega)\bigcap C^1(\bar{\Omega})$, Δ is the Laplace operator written in spherical coordinates, α, β, α_i and γ are prescribed constants.

Theorem 2. *If $\beta \geq 0$ and $\beta > \sum\limits_{i=0}^{m} \alpha_i + \gamma$, then there exists the unique regular solution of the problem (4)-(5), it is unique and can be written in spherical functions:*

$$u(x) = \frac{1}{\beta - \sum\limits_{0}^{m} \alpha_i - \gamma} Y_0 + \sum_{n=1}^{\infty} \left(\tfrac{r}{a}\right)^n \left[\alpha \tfrac{n}{a} + \beta - \sum_{0}^{m} \alpha_i \left(\tfrac{a_i}{a}\right)^n - \right.$$
$$\left. - \tfrac{\gamma}{(n+1)(a-a_0)}\left[a - a_0\left(\tfrac{a_0}{a}\right)^n\right]\right]^{-1} Y_n,$$

if $\beta = \sum\limits_{n=1}^{m} \alpha_i - \gamma$ and $\int\limits_0^{2\pi}\int\limits_0^{\pi} f \sin\theta d\theta d\varphi = 0$, solution is defined with precision of any constsnt,

$$u(x) = \sum_{n=1}^{\infty} \left(\tfrac{r}{a}\right)^n \left[\alpha \tfrac{n}{a} + \beta - \sum_{0}^{m} \alpha_i \left(\tfrac{a_i}{a}\right)^n - \right.$$
$$\left. \tfrac{\gamma}{(n+1)(a-a_0)}\left[a - a_0\left(\tfrac{a_0}{a}\right)^n\right]\right]^{-1} Y_n + c,$$

where c is an arbitrary constant, $Y_n(\theta,\varphi)$ is a spherical harmonic which can be expressed by Legandre's goint polynomials and Fourier coefficients.

Problem 3. In this case domain Ω is cylinder, $\Gamma_0^{(1)}$ and $\Gamma_0^{(2)}$ are upper side and bottom boundaries, Y is a side boundary, with a radius. Respectively, Ω_i are consentrated cylinders, with a_i radiuses Γ_i side boundaries; The heights of these cylinders equal to each other, it is l.

We have to find the regular solution of Laplace equation (4), satisfying (5) non-local condition, in addition

$$u\bigg|_{\Gamma_0^{(1)}} = u\bigg|_{\Gamma_0^{(2)}} = 0. \tag{6}$$

Theorem 3. *If $\beta \geq 0$ and $\beta > \sum\limits_{0}^{m} \alpha_i + \gamma$, then there exists the unique regular solution of the problem (4)-(5)-(6) and it can be expressed by Bessell* *functions:*

$$u(x)=\sum_{n=0}^{\infty}\sum_{k=1}^{\infty}\left[\frac{\alpha}{2}\frac{I_{n-1}\left(\frac{\pi k}{l}a\right)-I_2 n+1\left(\frac{\pi k}{l}a\right)}{I_n\left(\frac{\pi k}{l}a\right)}+\beta-\right.$$

$$\left.-\sum_{i=0}^{m}\alpha_i\frac{I_n(a_i)}{I_n(a)}-\gamma\frac{I_n^0\left(\frac{\pi k}{l}(a-a_0)\right)}{I_n(a)}\right]^{-1}Y_{nk},$$

where

$$I_n^0\left(\frac{\pi k}{l}(a-a_0)\right)=\sum_{\nu=0}^{\infty}\frac{1}{\gamma(\nu+1)\gamma(\nu+n+1)(2\nu+n+1)}\left(\frac{\pi k}{l}(a-a_0)\right)^{2\nu+n},$$

Y_{nk} *are expressed by Fourier coeficients,* I_n *are modified functions of Bessell.*