ON A COUPLING OF DIFFERENTIAL EQUATIONS VIA INTERFACE BOUNDARY CONDITIONS

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It is well known that different mathematical models can be used for the description of the same phenomena. Of course each of these models has own range of validity and the relationships between dependent variables of different models are known. As usual, the simple models are easier to solve but they have restricted areas of validity. More comlicated models can be successfuly applied to broader areas but they require more efforts to be solved. From computational point of view, naturally, it is expedient to make adaptation by mathematical models with the purpose to use simpler models where it is possible. So, if we study some problem in Ω then there may exist different sub domains of Ω where different mathematical models are most suitable. In this case one can make adaptation by mathematical models, i.e. use different models in different sub domains and set relationships between dependent variables of different mathematical models as interface boundary conditions. With account of the above the problem can be formally formulated as follows:

$$\alpha_i \frac{\partial^{k_i} u_i}{\partial t^{k_i}} + A_i u_i = f_i \quad in \ \ (0,T)\times\Omega_i, \tag{1}$$

$$\frac{\partial^\ell u_0}{\partial t^\ell} = u_{i\ell} \quad in \ \ \Omega_i \ \ at \ \ t=0, \quad \ell = 0,1,...,k-1; \tag{2}$$

$$L_i u_i = \varphi_i \quad on \ \ (0,T)\times\gamma_i; \tag{3}$$

$$L_{ij} u_i = L_{ji} u_j + \varphi_i \quad on \ \ (0,T)\times\gamma_{ij}, \tag{4}$$

where $i=1,2,...,m$, $j=1,2,...,m$, $k_i = const$, $\alpha_i = const$,

$$\begin{gathered} \overline{\Omega} = \bigcup_{i=1}^{m} \overline{\Omega}_i, \quad \gamma = \partial\overline{\Omega}, \quad \gamma_i = \gamma\cap\overline{\Omega}_i, \\ \Omega_i\cap\Omega_j = \emptyset \ \ if \ \ i\neq j, \quad \gamma_{ij} = \overline{\Omega}_i\cap\overline{\Omega}_j \ \ when \ \ i\neq j, \end{gathered} \tag{5}$$

f_i, $u_{i\ell}$, φ_i, φ_{ij} are given functions, A_i, L_i, L_{ij}, L_{ji} are given operators. Solution of the problem u is defined as u_i on $(0,t)\times\Omega_i$. In the formulaes given above (1) correspond to different equations, (2) are initial conditions, (3) are boundary conditions, (4) are interface boundary conditions.

The following proposition shows that (1)-(4) is not a correct problem.

Proposition 1. if

$$L_{ij} u_i = \tilde{\varphi}_{ij} \quad on \ \ j\in I_i, \quad I_i = \{j, \ \gamma_{ij}\neq\emptyset\} \tag{6}$$

and the problem (1)-(3), (6) is correct for any i fixed then the coupled problem (1)-(5) has an infinite number of solutions.

Of cource we suppose that $\tilde{\varphi}_{ij}$ are taken from suitable function spaces. Note that updating $\tilde{\varphi}_{ij}$ on $\gamma_{1j,\, j\in I_1}$, yields correct problems for u_1 and boundary conditions for j-th equations that can be calculated from (4). Clearly, on γ_{2j}, $j \in I_2$, we already have boundary conditions on γ_2 and γ_{12}, if $\gamma_2 \neq 0$, $\gamma_{12} \neq \emptyset$. For other boundaries of Ω_2 arbitrary functions $\tilde{\varphi}_{2j}$ can be updated. Repeating the same procedure for other sub domains we obtain correct problems for all $i = 1, 2, ..., m$. Since some $\tilde{\varphi}_{ij}$ functions were updated arbitrarily, validity of the proposition is evident.

Thus with account of the above proposition it is necesarry to modify the coupled problem in such a way that it be correct. We suppose that for this sake the following approaches can be used:

(i) To overlap a part of domains Ω_i and Ω_j, $\Omega_{ij} = \Omega_i \cap \Omega_j$, $i \neq j$, meas $\Omega_{ij} \neq 0$; to consider each of problems (1)-(3) in $\Omega_i \cup \Omega_{ij}$ and $\Omega_j \cup \Omega_{ij}$ correspondingly and to set additional requirements

$$\int\limits_{\Omega ij} |L_{ij}u_i - L_{ji}u_j - \varphi_{ij}| d\Omega \to min.$$

Such an approach was considered in [1] for coupling viscous and inviscid models of incompressible fluid.

(ii) To add or derive somehow additional conditions on interface boundaries γ_{ij} as for example it was made in [2] for coupling various models for transonic flow computation.

(iii) To overlap domains as in (i) and to set interface boundary conditions (4) on $\partial\Omega_{ij}$.

(iv) To select interface boundary in special way within $(0, T) \times \Omega_{ij}$.

Below the last two approaches are demonstrated for the following simple problem: coupling of second order hyperbolic and elliptic equations in one space dimension. Namely, coupled according to (1)-(4) problem is the following:

$$\frac{\partial^2 u_1}{\partial t^2} = a^2 \frac{\partial^2 u_1}{\partial x^2} \ \ in \ \ (0, T) \times \Omega_1, \ \ \Omega_1 = (0, 1), \tag{7}$$

$$u_1(0, x) = u_{10}(x), \ u_{1t}(0, x) = u_{11}(x), \ x \in \Omega_1, \tag{8}$$

$$u_1(t, 0) = 0; \tag{9}$$

$$\frac{\partial^2 u_2(t, x)}{\partial x^2} = 0 \ \ in \ \ (0, T) \times \Omega_2, \ \ \Omega_2 = (1, 2), \tag{10}$$

$$u_2(t, 2) = 0, \tag{11}$$

$$u_2(t, 1) = \frac{\partial u_1(t, 1)}{\partial x}. \tag{12}$$

According to approach (iii) let us introduce domain of overlapping as $\Omega_{12} = (1, 1 + b)$ and set the corresponding problem:

$$\frac{\partial^2 u_1}{\partial t^2} = a^2 \frac{\partial^2 u_1}{\partial x^2} \ \ on \ \ (0, T) \times (\Omega_1 \cup \Omega_{12}), \tag{13}$$

$$u_1(0,x) = \tilde{u}_{10}(x),\ u_{it}(0,x) = \tilde{u}_{11}(x)\ \ in\ \Omega_1 \cup \Omega_{12}, \tag{14}$$

$$u_2(t,1) = \frac{\partial u_1(t,1)}{\partial x},\ \ u_2(t,1+b) = \frac{\partial u_1(t,1+b)}{\partial x}. \tag{15}$$

Note that (10)-(11), (13)-(15) correspond to coupled problem according to the approach (iii) given above,

$$\tilde{u}_{10}(x) = \begin{cases} u_{10}(x),\ x \in \Omega_1, \\ u_2(0,x),\ x \in \Omega_2, \end{cases} \tag{16}$$

$$\tilde{u}_{11}(x) = \begin{cases} u_{11}(x),\ x \in \Omega_1, \\ u_{2t}(0,x),\ x \in \Omega_2, \end{cases} \tag{17}$$

where $u_2(0,x), u_{2t}(0,x)$ are solutions of (9) with boundary conditions (10) and

$$u_2(0,1) = u_{10}(1)\ \ or\ \ u_{2t}(0,1) = u_{11}(1) \tag{18}$$

correspondingly.

The following proposition hold true:

Proposition 2. In case of sufficiently smooth initial data satisfying consistency conditions classical solution of coupled problem (10)-(11), (13)-(18) exists and is unique.

In order to demonstrate approach (iv) let us consider coupled problem of the type (7)-(12) but with interface boundary curve γ_{12}, where γ_{12} lies in $(0,T) \times \overline{\Omega}_{12}$. This means that on the left of γ_{12} one has problem similar with (7)-(9), on the right of γ_{12} one has problem similar with (10),(11) and on γ_{12} one has interface boundary condition similar with (12). For such a problem the following proposition holds true:

Proposition 3. In $(O,T) \times \Omega_{12}$ there exists an infinite number of interface boundary curves γ_{12} for which solution of the corresponding coupled problem exists and is unique.

References

[1] Dinh Q.V., Glowinski R., Periaux J., Terrasson G., On the Coupling of Viscous and Inviscid Models for Incompressible Fluid Flows Via Domain Decomposition. SIAM, Philadelphia, pp. 350-369.

[2] Berger H., Warnecke G., Wendland W.L., Analysis of a FEM/BEM Coupling Method for Transonic Flow Computations. Preprint 93-9, Mathematisches Institut A der Universitat stuttgart, 1993, pp. 1-44.