ON A MODEL OF A BAR WITH VARIABLE THICKNESS

G. Jaiani

I. Vekua Institute of Applied Mathematics
I. Javakhishvili Tbilisi State University

Like I. Vekua's [1] method, when in order to construct theory of plates and shells all mechanical quantities of three-dimensional theory of elasticity have been expanded in the Fourier-series of Legendre polynomials with respect to the variable along thickness, and then only first $N+1,\ N=0,1,\cdots,$ terms have been left, in the bar model under consideration all mechanical quantities have been expanded in double Fourier-series of Legendre polynomials with respect to the variables along thickness, and width of the bar, and then first $(N_3+1)(N_2+1),\ N_3,\ N_2=0,1,\cdots,$ terms have been left. This case will be called (N_3, N_2) approximation. In this approximation we have the system of $3(N_3+1)(N_2+1)$ equations as follows

$$\Lambda_j(h_2^{2n_2+1}h_3^{2n_3+1}\overset{n_3,n_2}{v}_{j,1})_{,1}(x_1,t)+\overset{n_3,n_2}{E_j}(\cdots\overset{r,s}{v}_{k,1}\cdots)+\overset{n_3,n_2}{M_j}(\cdots\overset{r,s}{v_k}\cdots)+\overset{n_3,n_2}{X_j^0}=$$

$$=\rho h_2^{2n_2+1}h_3^{2n_3+1}\frac{\partial^2 \overset{n_3,n_2}{v_j}}{\partial t^2},\quad j=1,2,3,\quad n_i=\overline{0,N_i},\quad i=2,3,\quad \rho\ \textit{is density},$$

$$\Lambda_j=\begin{cases}\lambda+2\mu, & j=1,\\ \mu, & j=2,3,\end{cases}\qquad \overset{r,s}{v_k}=\frac{\overset{r,s}{u_k}}{h_2^{r+1}h_3^{s+1}} \tag{1}$$

λ and μ are Lamé constants, $\overset{n_3,n_2}{E_j},\ \overset{n_3,n_2}{M_j}$ are certain linear functions of their arguments where $r=\overline{0,N_3},\ s=\overline{0,N_2},\ k=1,2,3,$ in general, with unbounded coeficients,

$$\overset{n_3,n_2}{X_j^0}:=\sqrt{1+\left(\overset{(+)}{h}_{3,1}\right)^2}\int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\overset{(+)}{Q_j^{h_3}}(x_1,x_2,\overset{(+)}{h_3})P_{n_2}(a_2x_2-b_2)dx_2+$$

$$+(-1)^{n_3}\sqrt{1+\left(\overset{(-)}{h}_{3,1}\right)^2}\int\limits_{\overset{(-)}{h_2}}^{\overset{(+)}{h_2}}\overset{(-)}{Q_j^{h_3}}(x_1,x_2,\overset{(-)}{h_3})P_{n_2}(a_2x_2-b_2)dx_2+$$

$$+\sqrt{1+\left(\overset{(+)}{h}_{2,1}\right)^2}\int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}}\overset{(+)}{Q_j^{h_2}}(x_1,\overset{(+)}{h_2},x_3)P_{n_3}(a_3x_3-b_3)dx_3+$$

$$+(-1)^{n_3}\sqrt{1+\left(\overset{(-)}{h}_{2,1}\right)^2}\int\limits_{\overset{(-)}{h_3}}^{\overset{(+)}{h_3}}\overset{(-)}{Q_j^{h_3}}(x_1,\overset{(-)}{h_2},x_3)P_{n_3}(a_3x_3-b_3)dx_3+\overset{n_3,n_2}{X_j},\quad i=1,2,3,$$

$\overset{(+)}{Q}{}_j^{h_3}$, $\overset{(-)}{Q}{}_j^{h_3}$, $\overset{(+)}{Q}{}_j^{h_2}$, $\overset{(-)}{Q}{}_j^{h_2}$ are components of surface forces acting on surfaces $\overset{(+)}{h}_3$, $\overset{(-)}{h}_3$, $\overset{(+)}{h}_2$, $\overset{(-)}{h}_2$ respectively; the bar occupies the following domain

$$\{(x_1,x_2,x_3):\ \ 0<x_1<l,\ \ \overset{(-)}{h}_i(x_1)<x_i<\overset{(+)}{h}_i(x_1),\ \ i=2,3,\ \ l=const\};$$

$$2h_i(x_1):=\overset{(+)}{h}_i-\overset{(-)}{h}_i\geq 0,\ \ i=2,3;\ \ h_i\in C^1;$$

$$\overset{n_3,n_2}{X_j}:=\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3}P_{n_2}(a_2x_2-b_2)P_{n_3}(a_3x_3-b_3)X_j dx_2dx_3,\ \ j=1,2,3,$$

X_j are volume forces; $P_n(\cdot)$ are Legendre Polynomials;

$$a_i:=\frac{2}{\overset{(+)}{h}_i-\overset{(-)}{h}_i},\quad b_i:=\frac{\overset{(+)}{h}_i+\overset{(-)}{h}_i}{\overset{(+)}{h}_i-\overset{(-)}{h}_i},\quad i=2,3;$$

$$\overset{n_3,n_2}{u_j}:=\int\limits_{\overset{(-)}{h}_2}^{\overset{(+)}{h}_2}\int\limits_{\overset{(-)}{h}_3}^{\overset{(+)}{h}_3}u_j(x_1,x_2,x_3)P_{n_2}(a_2x_2-b_2)P_{n_3}(a_3x_3-b_3)dx_2dx_3,\ \ j=1,2,3,$$

$u_j(x_1,x_2,x_3)$ are components of displacement vector.

In $(0,0)$ approximation the system (1) will have the form

$$(h_2h_3v_{j,1}(x_1,t))_{,1}+\overset{0,0}{Y_j}=\Lambda_j^{-1}\rho h_2h_3\frac{\partial^2 v_1(x_1,t)}{\partial t^2},\ j=1,2,3, \tag{2}$$

$$v_j(x_1,t):=\frac{\overset{0,0}{u_j}(x_1)}{h_2(x_1)h_3(x_1)},\qquad \overset{0,0}{Y_1}=\frac{\overset{0,0}{X_1^0}}{\lambda+2\mu,}\qquad \overset{0,0}{Y_i}=\frac{\overset{0,0}{X_i^0}}{\mu,}\quad i=2,3.$$

Let us consider statical case of (2). Obviously

$$v_j(x_1)=-\int\limits_{x_1^0}^{x_1}\frac{d\tau}{h_2(\tau)h_3(\tau)}\int\limits_{x_1^0}^{\tau}\overset{0,0}{Y_j}(t)dt+c_1^j\int\limits_{x_1^0}^{x_1}\frac{d\tau}{h_2(\tau)h_3(\tau)}+c_2^j, \tag{$*$}$$

$$x_1^0=const\in]0,l[,\quad c_\alpha^j=const,\ \ \alpha=1,2,\ \ j=1,2,3.$$

Let further

$$\overset{N_3,N_2}{I_0}:=\int\limits_{o}^{\varepsilon}h_2^{-2N_2-1}(\tau)h_3^{-2N_3-1}(\tau)d\tau,\ \ \varepsilon=const>0,$$

$$\overset{N_3,N_2}{I_\ell}:=\int\limits_{\ell-\varepsilon}^{\ell}h_2^{-2N_2-1}(\tau)h_3^{-2N_3-1}(\tau)d\tau,\ \ \varepsilon=const>0,$$

$$v_j(0)=\varphi_j^0 \quad if \quad \overset{0,0}{I_0}<+\infty, \quad j=1,2,3, \tag{3}$$

$$v_j(l)=\varphi_j^l \quad if \quad \overset{0,0}{I_\ell}<+\infty, \; j=1,2,3, \tag{4}$$

$$v_j(x_1)=O(1), \quad x_1\to 0_+ \quad if \; \overset{0,0}{I_0}=+\infty, \; j=1,2,3, \tag{5}$$

$$v_j(x_1)=O(1), \quad x_1\to l_- \quad if \; \overset{0,0}{I_\ell}=+\infty, \; j=1,2,3, \tag{6}$$

$$\overset{0,0}{X}_{1j}(0)=\Lambda_j h_2h_3v_{j,1}|_{x_1=0}=\psi_j^0, \quad if \; \overset{0,0}{I_0}\le+\infty, \quad j=1,2,3, \tag{7}$$

$$\overset{0,0}{X}_{1j}(l)=\Lambda_j h_2h_3v_{j,1}|_{x_1=l}=\psi_j^l, \quad if \; \overset{0,0}{I_\ell}\le+\infty, \quad j=1,2,3, \tag{8}$$

φ_j^0, ψ_j^0, φ_j^l, ψ_j^l, $j=1,2,3$, are given constants.

Let $\overset{0,0}{Y_j}\in L([0,l])$, and if $\overset{0,0}{I_0}=+\infty$ ($\overset{0,0}{I_\ell}=+\infty$), it is such that iterated integral in $(*)$ is bounded by $x_1\to 0_+$ (l_-).

If $\dfrac{1}{h_2\cdot h_3}$ is locally summable in $]0,l[$ then for regular solutions ($v_j\in$ $\in C^2(]0,l[)$) only the following problems are correct: (2), (3), (4) ($v_j\in$ $\in C([0,l])$); (2), (5), (4) ($v_j\in C(]0,l])$); (2), (3), (6) ($v_j\in C([0,l[)$); (2), (5), (6) (v_j is bounded); (2), (3), (8) ($v_j\in C([0,l[)$, $h_2h_3v_{j,1}\in C([0,l])$; (2), (7), (4) ($v_j\in C(]0,l])$, $h_2h_3v_{j,1}\in C([0,l])$; the mixed problem when on the one end of the bar for some displacement components the Dirichlet and for others either the Neumann or (5), (6) type conditions are given but on the other end for first components either the Neumann or the Dirichlet or (5), (6) type conditions and for the second components the Dirichlet conditions (the Neumann and (5), (6) type conditions are not admissible) are given, are correct.

In the case (2), (5), (6) we have the solution up to the rigid transfer.

If $\dfrac{1}{h_2\cdot h_3}$ has nonsummable singularities in $]0,l[$ then only arbitrary three (but one for any fixed j) out of six conditions of (3), (4) can be given (in particular, it contains cases when either only (3) or only (4) can be given).

The following boundary conditions can be similary set in the (N_3,N_2) approximation:

$$\overset{n_3,n_2}{v_j}(0)=\overset{n_3,n_2}{\varphi_j^0}, \;\; j=1,2,3, \;\; n_i=\overline{0,N_i}, \;\; i=2,3, \;\; if \quad \overset{N_3,N_2}{I_0}<+\infty; \tag{9}$$

$$\overset{n_3,n_2}{v_j}(\ell)=\overset{n_3,n_2}{\varphi_j^\ell}, \;\; j=1,2,3, \;\; n_i=\overline{0,N_i}, \;\; i=2,3, \;\; if \quad \overset{N_3,N_2}{I_\ell}<+\infty; \tag{10}$$

$$\overset{n_3,n_2}{v_j}(x_1)=0(1), \; x_1\to 0_+, \; j=1,2,3, \; n_i=\overline{0,N_i}, \; i=2,3, \; if \;\; \overset{N_3,N_2}{I_0}=+\infty; \quad (11)_0$$

$$\overset{n_3,n_2}{v_j}(x_1)=0(1), \; x_1\to \ell_-, \; j=1,2,3, \; n_i=\overline{0,N_i}, \;\; i=2,3, \; if \;\; \overset{N_3,N_2}{I_\ell}=+\infty; \quad (11)_\ell$$

$$\lim_{x_1\to 0_+} h_2^{n_2}h_3^{n_3}\overset{n_3,n_2}{X}_{1j}(x_1)=\lim_{x_1\to 0_+}\Bigg\{\Lambda_j h_2^{2n_2+1}h_3^{2n_3+1}\overset{n_3,n_2}{v}_{j,1}-\Lambda_j\sum_{k=2}^{3}\sum_{s=n_k+1}^{N_k}h_2^{\delta_{k2}s+(\delta_{k3}+1)n_2+1}.$$

$$\cdot h_3^{\delta_{k3}s+(\delta_{k2}+1)n_3+1} h_k^{-1} \overset{n_k}{b_s^k} \overset{\delta_{k2}n_3+\delta_{k3}s,\delta_{k2}s+\delta_{k3}n_2}{v_j} - \tag{12}$$

$$-\left[\delta_{j1}\lambda\sum_{i=2}^{3}+\delta_{ij}(\delta_{j2}+\delta_{j3})\mu\right]\sum_{s=n_i+1}^{N_i} h_2^{\delta_{i2}s+(\delta_{i3}+1)n_2+1} h_3^{(\delta_{i2}+1)n_3+\delta_{i3}s+1} h_i^{-1} \overset{n_i}{b}_{is}\cdot$$

$$\left.\cdot \overset{\delta_{i2}n_3+\delta_{i3}s,\delta_{i2}s+\delta_{i3}n_2}{v_{\delta_{j1}i+\delta_{j2}+\delta_{j3}}}\right\} = \overset{n_3,n_2}{\psi_j^0},\ j=1,2,3,\ n_i=\overline{0,N_i}\ (i=2,3),\ if\ \overset{N_3,N_2}{I_0} \le +\infty;$$

$$\lim_{x_1\to l_-} h_2^{n_2}h_3^{n_3}\overset{n_3,n_2}{X_{1j}}(x_1) = \overset{n_3,n_2}{\psi_j^\ell},\ j=1,2,3,\ n_i=\overline{0,N_i}\ (i=2,3),\ if\ \overset{N_3,N_2}{I_\ell} \le +\infty; \tag{13}$$

where $\overset{n_3,n_2}{\varphi_j^0}$, $\overset{n_3,n_2}{\varphi_j^\ell}$, $\overset{n_3,n_2}{\psi_j^0}$, $\overset{n_3,n_2}{\psi_j^\ell}$ are given constants,

$$\overset{n_k}{b_s^k} := \left(s+\frac{1}{2}\right)\left[\overset{(+)}{h}_{k,1} - (-1)^{s+n_k}\overset{(-)}{h}_{k,1}\right],\quad \overset{n_i}{b}_{is} := \left(s+\frac{1}{2}\right)\left[1-(-1)^{n_i+s}\right].$$

The boundary value problems (1),(9),(10); (1),(9),(13); (1),(10),(12); (1),(9),$(11)_\ell$; (1),(10),$(11)_o$ are uniquely solvable. The problem (1),$(11)_o$,$(11)_\ell$ is solvable up to the rigid motion.

If the bar has cusped end (i.e. at this end at least one of h_i, $i=2,3$, vanishes), and $N_2, N_3 \to +\infty$, in limit case (which obviously coincides with three-dimensional case) the boundary conditions (9), or (10), or both disappear, and will be replaced by boundedness of $\overset{n_3,n_2}{v_j}$, $j=1,2,3$, $n_2, n_3 = 0,1,...$, i.e. by boundedness of displacement vector in a neighbourhood of the corresponding end of the bar.

If in a neighbourhood of a cusped end stresses are bounded then at the above end all moments of stress vector will be equal to zero. Non-zero stress vector moments given at a cusped bar end mean that this end is loaded by concentrated surface force, and concentrated moments of coresponding order.

Remark. The (N_3, N_2) approximation is equavalent to the assumption

$$u_j(x_1,x_2,x_3) \cong \sum_{n_2=0}^{N_2}\sum_{n_3=0}^{N_3}\left(n_2+\frac{1}{2}\right)\left(n_3+\frac{1}{2}\right)\frac{1}{2^{n_2+n_3}n_2!n_3!}\cdot$$

$$\cdot\frac{d^{n_2}[(x_2-\overline{h}_2)^2-h_2^2]^{n_2}}{dx_2^{n_2}}\frac{d^{n_3}[(x_3-\overline{h}_3)^2-h_3^2]^{n_3}}{dx_3^{n_3}}\overset{n_3,n_2}{v_j}(x_1),\quad j=1,2,3,$$

where

$$2\overline{h}_i(x_1) := \overset{(+)}{h_i}(x_1) + \overset{(-)}{h_i}(x_1),\quad i=2,3.$$

References

[1] Vekua, I.N. Shell Theory: General Methods of Construction. Pitman Advanced Publishing Program, Boston-London-Melburne, 1985, pp. 1-287.