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NOTES ON EFFECTIVE DESCENT AND PROJECTIVITY IN QUASIVARIETIES OF UNIVERSAL ALGEBRAS

Dedicated to Walter Tholen on the occasion of his sixtieth birthday

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Abstract. We present sufficient conditions under which effective descent morphisms in a quasivariety of universal algebras are the same as regular epimorphisms and examples for which they are the same as regular epimorphisms satisfying projectivity.

1. Preliminaries

A variety is a full subcategory of the category of structures for a first order (one sorted) language, closed under substructures, products and homomorphic images. It is a regular category not necessarily exact for which effective descent morphisms are exactly the regular epimorphisms (strong surjective homomorphisms). The same is true of "prevarieties" (full subcategories of the category of structures, closed under substructures, products and strong homomorphic images) [4]. Any quasivariety is the subcategory of a variety orthogonal to a set of epimorphisms which are either strong surjective homomorphisms or bijective homomorphisms. Projectivity of the domain of such a bijective homomorphism w.r.t. a regular epimorphism p was shown in [2] to be a necessary and sufficient condition for p to be an effective descent morphism in models of Preorder.

In the case of universal algebras varieties are exact categories and consequently their effective descent morphisms are the regular epimorphisms i.e., the surjective homomorphisms. A quasivariety of universal algebras as the subcategory of a variety of universal algebras orthogonal to a set of regular epimorphisms is a regular category whose regular epimorphisms are again the surjective homomorphisms. A quasivariety of universal algebras is (a full subcategory of the category of structures for a first order (one sorted) algebraic language) axiomatizable by quasi-identities [3], that is, by sentences of the form

$$\forall x_1 ... \forall x_n \ \bigwedge_{i=1}^{k} \theta_i \to \delta$$

where each θ_i and δ are identities.

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Proposition 2.2 below, is an equivalent to the following well known criterion for effective descent found in [1]:

1.1. PROPOSITION. *Let $\mathbb{C}$ be a category with pullbacks and $\mathbb{D}$ be a full subcategory of $\mathbb{C}$, closed under pullbacks. Let $p : E \to B$ be a morphism in $\mathbb{D}$. Then:*

If p is an effective descent morphism in $\mathbb{C}$, then p is an effective descent morphism in $\mathbb{D}$ if and only if for each pullback

$$\begin{tikzcd} D \arrow[r, "s"] \arrow[d, "\delta"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B \end{tikzcd}$$

in $\mathbb{C}$, A is in $\mathbb{D}$ whenever D is in $\mathbb{D}$.

2. Effective descent and Projectivity

Let $\mathbb{C}$ be a category with pullbacks in which the classes of regular epimorphisms and effective descent morphisms coincide and let $\mathcal{S}$ be a set of regular epimorphisms in $\mathbb{C}$.

2.1. DEFINITION. Given a morphism $p : E \to B$, we say that (A, α) in $(\mathbb{C} \downarrow B)$ has the *factorization property* with respect to p, if for each r in $\mathcal{S}$ and each commutative square

$$\begin{tikzcd} \operatorname{dom} r \arrow[rr, "u"] \arrow[dr, "r"] \arrow[dd] & & A \arrow[dd, "\alpha"] \\ & \operatorname{cod} r \arrow[ur, dashed] & \\ E \arrow[rr, "p"'] & & B \end{tikzcd}$$

u factors through r.

2.2. PROPOSITION. *Let $p : E \to B$ be a morphism in $\mathcal{S}^{\perp}$ which is a regular epimorphism in $\mathbb{C}$; p is an effective descent morphism in $\mathcal{S}^{\perp}$ if and only if for any (A, α) in $(\mathbb{C} \downarrow B)$ with the factorization property w.r.t. p, A is in $\mathcal{S}^{\perp}$.*

PROOF. Since $\mathcal{S}^{\perp}$ is closed under pullbacks and any regular epimorphism in $\mathbb{C}$ is an effective descent morphism in $\mathbb{C}$, from Proposition 1.1 we only need to show that the pullback condition in there is equivalent to the condition stated here. In fact, we only have to show that (A, α) in $(\mathbb{C} \downarrow B)$ has the factorization property w.r.t. p if and only if $E \times_B A$ is in $\mathcal{S}^{\perp}$.

Let (A, α) be in $(\mathbb{C} \downarrow B)$ and consider the pullback

$$\begin{tikzcd} E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\ E \arrow[r, "p"'] & B. \end{tikzcd}$$

Note that, since E is in $\mathcal{S}^{\perp}$, for each r in $\mathcal{S}$, any morphism from dom r into E factors through r.

Given r in $\mathcal{S}$ and $u : \operatorname{dom} r \to A$ satisfying $\alpha \circ u = p \circ h$ for some h, then $h = v \circ r$ for some v, and if $E \times_B A$ is in $\mathcal{S}^{\perp}$, the unique morphism $f : \operatorname{dom} r \to E \times_B A$ induced by h and u (which must satisfy $\pi_2 \circ f = u$), is of the form $f = g \circ r$, for some g. Therefore, $u = \pi_2 \circ f = \pi_2 \circ g \circ r$.

$$\begin{tikzcd}
\operatorname{dom} r \arrow[dr, "r"] \arrow[ddrrr, bend left, "u"] \arrow[dddrr, bend right, "h"'] & & & \\
& \operatorname{cod} r \arrow[dr, dashed, "g"] \arrow[drr, bend left, "w"] \arrow[ddr, bend right, "v"'] & & \\
& & E \times_B A \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & A \arrow[d, "\alpha"] \\
& & E \arrow[r, "p"'] & B
\end{tikzcd}$$

Conversely, if (A, α) has the factorization property w.r.t. p and $f : \operatorname{dom} r \to E \times_B A$, then $\pi_1 \circ f = v \circ r$ for some v, and since $\alpha \circ \pi_2 \circ f$ factors through p, $\pi_2 \circ f = w \circ r$ for some morphism w. Let $g : \operatorname{cod} r \to E \times_B A$ be the morphism induced by v and w. Then, $\pi_1 \circ g \circ r = v \circ r = \pi_1 \circ f$ and $\pi_2 \circ g \circ r = w \circ r = \pi_2 \circ f$, that is, $f = g \circ r$. ∎

2.3. Corollary. *Let $\mathcal{S}$ be contained in the class of epimorphisms of $\mathbb{C}$, and $p : E \to B$ be a morphism in $\mathcal{S}^{\perp}$ which is a regular epimorphism in $\mathbb{C}$; p is an effective descent morphism in $\mathcal{S}^{\perp}$ if for each r in $\mathcal{S}$,* dom r *(or* cod r*) is projective w.r.t. p (i.e. for each r in $\mathcal{S}$ every $f : \operatorname{dom} r \to B$ factors through p).*

Proof. Let (A, α) in $(\mathbb{C} \downarrow B)$ have the factorization property w.r.t. p, and let r be in $\mathcal{S}$ and $u : \operatorname{dom} r \to A$. Then, by projectivity, $\alpha \circ u = p \circ h$ for some h, and by the factorization property, u factors through r; therefore A is in $\mathcal{S}^{\perp}$.

Notice that since each r in $\mathcal{S}$ is an epimorphism and both E and B are in $\mathcal{S}^{\perp}$, projectivity of dom r (w.r.t. p) is equivalent to projectivity of cod r (w.r.t. p). ∎

Suppose that $\mathbb{Q}$ is the full subcategory of a variety $\mathbb{V}$ determined by a set Σ of quasi-identities.

Each conjunction θ of identities induces a congruence C_θ on the free object $F_{\mathbb{V}}(V)$ in $\mathbb{V}$ over the set V of variables of the language. A quasi-identity $\theta \to \delta$ induces then a canonical surjective homomorphism $F_{\mathbb{V}}(V)/_{C_\theta} \to F_{\mathbb{V}}(V)/_{C_\theta \vee C_\delta}$. Let $\mathcal{S}$ be the set of these homomorphisms, one for each quasi-identity in Σ. Then $\mathbb{Q}$ is the subcategory of $\mathbb{V}$ orthogonal to $\mathcal{S}$, i.e., $\mathcal{S}^{\perp} = \mathbb{Q}$ in $\mathbb{V}$.

Corollary 2.3 reads then

2.4. COROLLARY. *Let $p : E \to B$ be a surjective homomorphism in $\mathbb{Q}$. Then 1. implies 2.:*

1. *for each $\theta \to \delta \in \Sigma$ and each assignment to the variables s into B, if $B \models \theta[s]$ then there is an assignment s_1 into E such that $p \circ s_1 = s$ and $E \models \theta[s_1]$.*

2. *p is an effective descent morphism in $\mathbb{Q}$.*

and Proposition 2.2 reads

2.5. PROPOSITION. *Let $p : E \to B$ in $\mathbb{Q}$ be a surjective homomorphism; p is an effective descent morphism in $\mathbb{Q}$ if and only if for each algebra A in $\mathbb{V}$ and each homomorphism $\alpha : A \to B$, if A is not a model of Σ there exist $\theta \to \delta \in \Sigma$ and assignments s_1 and s_2 into E and A respectively, such that $p \circ s_1 = \alpha \circ s_2$, $E \models \theta[s_1]$ and $A \models \theta \wedge \neg\delta[s_2]$.*

Not all regular epimorphisms in $\mathbb{Q}$ are effective descent morphisms as shown in the following examples borrowed from [1]:

2.6. EXAMPLE.

1. A binary operation $\cdot$ with at most one idempotent that is, models of

$$\forall x \forall y \quad x \cdot x \approx x \wedge y \cdot y \approx y \to x \approx y.$$

   Take E with no idempotents, B the terminal object, A with two idempotents (not a model) and $p : E \to B$ and $\alpha : A \to B$ the only possible morphisms; dom r is the free object on two idempotents and for a morphism $u : \mathrm{dom}\, r \to A$, $\alpha \circ u$ cannot factor through p for if it did, E had at least one idempotent. Therefore, by Proposition 2.2, p is not an effective descent morphism.

2. Models of $\{x \cdot c \approx x,\ c \cdot x \approx x,\ x \cdot (x \cdot (x \cdot x)) \approx c \to x \cdot x \approx c\}$ ($\cdot$ a binary operation and c a constant).

   Take the Abelian groups $E = \mathbb{Z}$, $B = \mathbb{Z}_2$, $A = \mathbb{Z}_4$ (not a model) and $p : E \to B$ and $\alpha : A \to B$ the canonical homomorphisms. Suppose that $\alpha \circ u = p \circ h$, for some h. Then, since $\mathbb{Z}$ is a model, $h(x + x) = 0$, and because 0 is the only nilpotent in $\mathbb{Z}$, $h(x) = 0$. Then, $\alpha \circ u(x) = p \circ h(x) = 0$ so that $u(x)$ is either 0 or 2 in $\mathbb{Z}_4$, and in any case, $u(x) + u(x) = 0$. Hence u factors through r and therefore, by Proposition 2.2, p is not an effective descent morphism.

In particular not all regular epimorphisms in $\mathbb{Q}$ satisfy projectivity. But they do in some common quasivarieties:

Let $\mathbb{Q}$ denote the category of models of a theory T whose axioms are sentences either of the form:

1. $\forall x_1 ... \forall x_n\ \theta$, or of the form

2. $\forall w_1 ... \forall w_n\ \theta_1 \wedge ... \wedge \theta_k \rightarrow \bigwedge_{i=1}^m x_i \approx y_i \wedge \bigwedge_{j=1}^l z_j \approx t_j$ where θ and θ_i are identities, all variables z_j are distinct from all the variables x_i and y_i and each z_j does not occur in any of the terms t_j

with the additional condition that

$$T \vdash \bigwedge_{i=1}^{m} x_i \approx y_i \wedge \bigwedge_{j=1}^{l} z_j \approx t_j \rightarrow \theta_1 \wedge ... \wedge \theta_k$$

for each formula of type 2..

2.7. PROPOSITION. *If $p : E \rightarrow B$ is a regular epimorphism in $\mathbb{Q}$, then for each formula of form 2. and for each $s : V \rightarrow B$ (V the set of variables) such that*

$$B \vDash \theta_1 \wedge ... \wedge \theta_k[s]$$

there exists $s' : V \rightarrow E$ with $p \circ s' = s$ and $E \vDash \theta_1 \wedge ... \wedge \theta_k[s']$, i.e., projectivity is satisfied w.r.t. p.

PROOF. Let $p : E \rightarrow B$ be a regular epimorphism in $\mathbb{Q}$. Then, p is a surjective homomorphism. If $B \vDash \theta_1 \wedge ... \wedge \theta_k[s]$, since B is a model of T, $B \vDash \bigwedge_{i=1}^m x_i \approx y_i \wedge \bigwedge_{j=1}^l z_j \approx t_j[s]$, that is $s(x_i) = s(y_i)$ and $s(z_j) = \bar{s}(t_j)$. As p is surjective, we may consider $s' : V \rightarrow E$ with $s'(x_i) = s'(y_i) \in p^{-1}(s(x_i)) = p^{-1}(s(y_i))$ and for the other variables choose $s'(v) \in p^{-1}(s(v))$. Then, $p \circ s' = s$.

Let $s'' : V \rightarrow E$ be such that $s''(z_j) = \bar{s'}(t_j)$ and equal to s' for variables other than the z_j's. Then, $p \circ s'' = s$. Also, s' and s'' coincide in the variables occurring in the t_j's (since the z_i do not occur there) and so

$$\bar{s''}(t_j) = \bar{s'}(t_j) = s''(z_j) \quad \text{and} \quad s''(x_i) = s'(x_i) = s'(y_i) = s''(y_i)$$

that is, $E \vDash \bigwedge_{i=1}^m x_i \approx y_i \wedge \bigwedge_{j=1}^l z_j \approx t_j[s'']$. As E is a model of T, from the additional condition it follows that $E \vDash \theta_1 \wedge ... \wedge \theta_k[s'']$. ∎

2.8. COROLLARY. *In $\mathbb{Q} = Mod(T)$ with T as above, the classes of regular epimorphisms and of effective descent morphisms coincide.*

PROOF. Follows from Corollary 2.3. ∎

The following are such quasivarieties:

2.9. EXAMPLE. The categories of

1. Models of ("joint") injectivity:

$$\forall x_1 ... \forall x_n \forall y_1 ... \forall y_n \quad f x_1 ... x_n \approx f y_1 ... y_n \wedge g x_1 ... x_k \approx g y_1 ... y_k \rightarrow \wedge_{i=1}^n x_i \approx y_i,\ k \leq n$$

2. Models of cancellation law: $\forall x \forall y \forall z \quad fxz \approx fyz \rightarrow x \approx y$

3. Torsion free abelian groups:

   Axioms of abelian group $+ ... \forall x \quad f...fxx...x \approx c \rightarrow x \approx c$

4. Models of

$$\{\forall x \quad x \wedge x \approx x, \forall x \forall y \quad x \wedge y \approx x \& (x \wedge z \approx y \wedge z) \& (x \vee z \approx y \vee z) \rightarrow x \approx y\}$$

5. Models of

$$\{\forall x \quad x \wedge x \approx x, \forall x \quad x \vee x \approx x, \forall x \quad x \vee \bar{x} \approx 1, \forall x \forall y \quad (x \wedge y) \vee (\overline{x \vee y}) \approx 1 \rightarrow x \approx y\}$$

where the axioms in 4. (respectively 5.) were picked among the axioms of ortholattices (respectively orthomodular lattices).

2.10. Observation. Let T be the theory of monoids with the identity denoted by 1, satisfying $x^4 \approx 1 \rightarrow x^2 \approx 1$ and let $\mathbb{Q} = Mod(T)$. Then $T \vdash x^2 \approx 1 \rightarrow x^4 \approx 1$. E, B and p as in Example 2.6.2. are in $\mathbb{Q}$ but the surjective homomorphism p is not an effective descent morphism in $\mathbb{Q}$. This shows the necessity of "linearity" of the identities $z_i \approx t_i$ of Proposition 2.7.

In the cases of Example 2.6, the given regular epimorphisms are not effective descent morphisms but also do not satisfy projectivity. In fact, in both cases, being an effective descent morphism is equivalent to being a regular epimorphism satisfying projectivity. Note that a morphism for which projectivity holds need not be a regular epimorphism.

2.11. Proposition.

1. *For $\mathbb{Q} = Mod(x \cdot x \approx x \wedge y \cdot y \approx y \rightarrow x \approx y)$ with $\cdot$ a binary operation, a morphism is an effective descent morphism in $\mathbb{Q}$ if and only if it is a regular epimorphism and satisfies projectivity.*

2. *For $\mathbb{Q} = Mod(\{x \cdot c \approx x,\ c \cdot x \approx x,\ x \cdot (x \cdot (x \cdot x)) \approx c \rightarrow x \cdot x \approx c\})$ with $\cdot$ a binary operation and c a constant, a morphism is an effective descent morphism in $\mathbb{Q}$ if and only if it is a regular epimorphism and satisfies projectivity.*

Proof. Since surjective homomorphisms in $\mathbb{Q}$ which satisfy projectivity are effective descent morphisms it is enough to show that an effective descent morphism in $\mathbb{Q}$ satisfies projectivity. Let $p : E \rightarrow B$ in $\mathbb{Q}$ be an effective descent morphism.

1. Suppose that $B \models x \cdot x \approx x \wedge y \cdot y \approx y[s]$. Then, since B is in $\mathbb{Q}$, $s(x) = s(y)$. Call this element b_0. Take $a \notin |B|$ and let A be the algebra defined by $|A| = |B| \cup \{a\}$ the underlying set, and operation $a \cdot a = a$ and $c \cdot d = \alpha(c) \cdot \alpha(d)$ otherwise, where $\alpha : |A| \rightarrow |B|$ is the function given by $\alpha(a) = b_0$ and $\alpha(b) = b$ for $b \in |B|$; $\alpha : A \rightarrow B$ is a homomorphism since $b_0 \cdot b_0 = b_0$.

A is not in $\mathbb{Q}$ because $a \cdot a = a$ and $b_0 \cdot b_0 = b_0$ but $a \neq b_0$. Moreover, since α is a homomorphism and B is in $\mathbb{Q}$ satisfying $b_0 \cdot b_0 = b_0$, the only assignments s' for which

$A \models x \cdot x \approx x \wedge y \cdot y \approx y \wedge x \not\approx y[s']$ are such that $\{s'(x), s'(y)\} \subseteq \{a, b_0\}$. Since for any such assignment $\alpha \circ s' = s$, by Proposition 2.5 there exists an assignment s_1 into E such that $E \models x \cdot x \approx x \wedge y \cdot y \approx y[s_1]$ and $p \circ s_1 = s$. This says that projectivity w.r.t. p is satisfied.

2. $\mathbb{Q}$ is a subcategory of the variety $\mathbb{V} = Mod(\{x \cdot c \approx x,\ c \cdot x \approx x\})$. Suppose that $B \models x \cdot (x \cdot (x \cdot x)) \approx c[s]$. Let $s(x) = b_0$ and $c^B = 1$. Since B is in $\mathbb{Q}$, $b_0 \cdot b_0 = 1$. Take $a_0, a_1 \notin |B|$ distinct and let A be the algebra defined by $|A| = |B| \cup \{a_0, a_1\}$ with $c^A = 1$, $a_i \cdot 1 = a_i = 1 \cdot a_i$, $i = 0, 1$, $a_0 \cdot a_0 = a_1$ and $b \cdot d = \alpha(b) \cdot \alpha(d)$ otherwise, where $\alpha : |A| \to |B|$ is the function defined by $\alpha(a_0) = b_0$, $\alpha(a_1) = 1$ and $\alpha(b) = b$ for $b \in |B|$; $\alpha : A \to B$ is a homomorphism and A is in $\mathbb{V}$.

But A is not in $\mathbb{Q}$ since

$$\begin{aligned} a_0 \cdot (a_0 \cdot (a_0 \cdot a_0)) &= a_0 \cdot (a_0 \cdot a_1) = a_0 \cdot (\alpha(a_0) \cdot \alpha(a_1)) \\ &= a_0 \cdot (b_0 \cdot 1) = a_0 \cdot b_0 = \alpha(a_0) \cdot \alpha(b_0) = b_0 \cdot b_0 = 1 \end{aligned}$$

and $a_0 \cdot a_0 = a_1 \neq 1$. In fact the only assignment s' for which $A \models x \cdot (x \cdot (x \cdot x)) \approx c \wedge x \not\approx c[s']$ is the above one i.e., $s'(x) = a_0$. But $\alpha \circ s'(x) = \alpha(a_0) = b_0$, that is $\alpha \circ s' = s$. Hence, by Proposition 2.5 there exists an assignment s_1 into E such that $E \models x \cdot (x \cdot (x \cdot x)) \approx c[s_1]$ and $p \circ s_1 = s$. This says that projectivity w.r.t. p is satisfied. ∎

For not all effective descent morphisms in a quasivariety of universal algebras projectivity is satisfied as shown in the following

2.12. Example.

1. Consider the quasivariety $\mathbb{Q}$ of (multiplicative) semigroups satisfying the quasi-identity
$$x^4 \approx x^2 \wedge y^4 \approx y^2 \to x^2 \approx y^2,$$
and the following objects of $\mathbb{Q}$:

   - the algebra $E = \{u^i : i \geq 1\} * \{v\}$ of non-empty finite strings of u's and v's, multiplication being concatenation, with the property that any substring of v's may be replaced by a single v (in particular, $v^i = v$, for any $i \geq 2$);
   - the algebra $B = \{u, v\}$ such that the multiplication of any two elements is always v.

   Sending u to u and everything else to v defines a regular epimorphism $p : E \to B$. Note that p does not satisfy projectivity as in B both elements satisfy $x^4 \approx x^2$, while v is the sole element of E that satisfies this identity. On the other hand, p is an effective descent morphism: indeed if A is an associative magma such that elements $a, b \in A$ exist that satisfy
$$a^4 = a^2,\ b^4 = b^2 \textit{ and } a^2 \neq b^2$$

then also

$$(a^2)^4 = (a^2)^2,\ (b^2)^4 = (b^2)^2 \text{ and } (a^2)^2 \neq (b^2)^2$$

and it follows that for any morphism of magmas $\alpha : A \to B$, sending x and y to v in E, on the one hand, and x to a^2 and y to b^2 in A, on the other hand, gives the required assignments.

2. Let $\mathbb{Q}$ be the same quasivariety as in example 1. above, $E = \{e^i : i \geq 0\}$ of possibly empty finite strings of e's with multiplication given by $e^i \cdot e^j = e^{i+j}$, $B = \{-1, 1\}$ with the usual multiplication and $p : E \to B$ the unique morphism such that $p(e^0) = 1$ and $p(e) = -1$. In particular, let p from additive natural numbers onto B be defined by $p(0) = 1$ and $p(n) = (-1)^n$ for $n \neq 0$.

3. Take $\mathbb{Q}$ to be the quasivariety of monoids, with the identity represented by 1, which satisfy $x^4 \approx x^2 \to x^2 \approx 1$. Let E, B and p be as in example 2. with 1 interpreted as e^0 and 1, respectively.

3. Necessary conditions and Sufficient conditions

We end these notes with necessary conditions and sufficient conditions for a regular epimorphism to be an effective descent morphism in a quasivariety $\mathbb{Q}$ of universal algebras.

Suppose that $\mathbb{Q}$ is the full subcategory of a variety $\mathbb{V}$ determined by a set Σ of quasi-identities. Each element of Σ is of the form $\theta \to \delta$ with $\theta = \bigwedge_i t_1^i \approx t_2^i$ and $\delta = u_1 \approx u_2$. Take $\mathbf{\Theta} = C_\theta =< (t_1^i, t_2^i) >$ to be the congruence on the algebra of terms $T(V)$, generated by the pairs (t_1^i, t_2^i) with $t_1^i \approx t_2^i$ subformulas of θ.

Below we will denote by $\pi_{\mathbf{\Theta}}$ both the canonical projection $T(V) \to T(V)/\mathbf{\Theta}$ and its restriction to V. Also, for any substitution $\varphi : V \to T(V)$, $\varphi(\theta)$ will denote the formula $\bigwedge_i \bar{\varphi}(t_1^i) \approx \bar{\varphi}(t_2^i)$ and analogously for $\varphi(\delta)$.

The following sufficient condition reflects what happens in Example 2.12:

3.1. Proposition. *Let $p : E \to B$ be a surjective homomorphism in $\mathbb{Q}$. Then 1. implies 2.:*

1. *For each $\theta \to \delta$ in Σ and each assignment $s : V \to B$ such that $B \models \theta[s]$ there exist an assignment s_1 into E and a substitution $\varphi : V \to T(V)$ such that $p \circ s_1 = \bar{s} \circ \varphi$, $E \models \theta[s_1]$ and*
$$Th\mathbb{V}, \theta \vdash \varphi(\theta) \quad \text{and} \quad Th\mathbb{V}, \theta, \varphi(\delta) \vdash \delta.$$
2. *p is an effective descent morphism in $\mathbb{Q}$.*

Proof. Let A be in $\mathbb{V}$, $\alpha : A \to B$ be a homomorphism and suppose that A is not in $\mathbb{Q}$. Then, there exist $\theta \to \delta$ in Σ and an assignment $s : V \to A$ such that $A \models \theta \wedge \neg\delta[s]$, so that $B \models \theta[\alpha \circ s]$. By assumption, for some assignment s_1 into E and some substitution $\varphi : V \to T(V)$ we have that $p \circ s_1 = \overline{\alpha \circ s} \circ \varphi = \alpha \circ \bar{s} \circ \varphi$, $E \models \theta[s_1]$ and $A \models \varphi(\theta) \wedge \neg\varphi(\delta)[s]$, that is $A \models \theta \wedge \neg\delta[\bar{s} \circ \varphi]$. By Proposition 2.5, p is an effective descent morphism in $\mathbb{Q}$. ∎

3.2. Example. In Example 2.12.1. take $s_1(x) = s_1(y) = v$, $\varphi(x) = x^2$ and $\varphi(y) = y^2$.

3.3. Corollary. *Let $p : E \to B$ be a surjective homomorphism in $\mathbb{Q}$. Then 1. implies 2.:*

1. *For each $\theta \to \delta$ in Σ and each assignment $s : V \to B$ if $B \models \theta[s]$ then there exist an assignment s_1 into E and a substitution $\varphi : V \to T(V)$ which is the identity on the variables of δ such that $p \circ s_1 = \bar{s} \circ \varphi$, $E \models \theta[s_1]$ and $Th\mathbb{V}, \theta \vdash \varphi(\theta)$.*
2. *p is an effective descent morphism in $\mathbb{Q}$.*

Proof. It follows from Proposition 3.1 since as φ is the identity in the variables of δ, $\varphi(\delta) = \delta$ and so $\varphi(\delta) \vdash \delta$. ∎

In the following necessary condition we assume that $\Sigma = \{\theta \to \delta\}$ is a singleton and $\mathbb{V}$ is the category of algebras for the language. We also assume that $\theta \to \delta$ is not a valid formula for otherwise $\mathbb{Q} = \mathbb{V}$.

3.4. Proposition. *Let $p : E \to B$ be a surjective homomorphism in $\mathbb{Q}$. Then 1. implies 2.:*

1. *$p : E \to B$ is an effective descent morphism in $\mathbb{Q}$.*
2. *For each assignment s such that $B \models \theta[s]$ there exist an assignment s_1 into E and a substitution $\varphi : V \to T(V)$ such that $p \circ s_1 = s \circ \varphi$,*

$$E \models \theta[s_1], \quad (\bar{\varphi}(t_1^i), \bar{\varphi}(t_2^i)) \in \mathbf{\Theta} \quad \textit{and} \quad (\bar{\varphi}(u_1), \bar{\varphi}(u_2)) \notin \mathbf{\Theta}$$

or equivalently, $E \models \theta[s_1]$, $\quad \theta \vdash \varphi(\theta) \quad$ and $\quad \theta, \neg\delta \nvdash \varphi(\delta)$.

Proof. Suppose that $B \models \theta[s]$. Since $\theta \to \delta$ is not valid, $T(V)/\mathbf{\Theta} \models \theta \wedge \neg\delta[\pi_{\mathbf{\Theta}}]$ and therefore $T(V)/\mathbf{\Theta}$ is not a model of Σ.

Moreover, $\bar{s}$ decomposes as $\beta \circ \pi_{\mathbf{\Theta}}$ for some homomorphism β. By Proposition 2.5 there exist s_1 and $\varphi : V \to T(V)$ such that

$$p \circ s_1 = \beta \circ \pi_{\mathbf{\Theta}} \circ \varphi = \bar{s} \circ \varphi, \quad E \models \theta[s_1] \quad \text{and} \quad T(V)/\mathbf{\Theta} \models \theta \wedge \neg\delta[\pi_{\mathbf{\Theta}} \circ \varphi],$$

that is, $(\bar{\varphi}(t_1^i), \bar{\varphi}(t_2^i)) \in \mathbf{\Theta}$ and $(\bar{\varphi}(u_1), \bar{\varphi}(u_2)) \notin \mathbf{\Theta}$ and therefore $\theta \vdash \varphi(\delta)$ and $\theta, \neg\delta \nvdash \varphi(\delta)$ as required. ∎

The gap between the above sufficient and necessary conditions seems to be the gap between non deducibility and deducibility of the negation (of $\varphi(\delta)$).

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