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DOUBLES FOR MONOIDAL CATEGORIES

Dedicated to Walter Tholen on his 60th birthday

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ABSTRACT. In a recent paper, Daisuke Tambara defined two-sided actions on an endomodule (= endodistributor) of a monoidal $\mathscr{V}$-category $\mathscr{A}$. When $\mathscr{A}$ is autonomous (= rigid = compact), he showed that the $\mathscr{V}$-category (that we call $\mathrm{Tamb}(\mathscr{A})$) of so-equipped endomodules (that we call Tambara modules) is equivalent to the monoidal centre $\mathcal{Z}[\mathscr{A},\mathscr{V}]$ of the convolution monoidal $\mathscr{V}$-category $[\mathscr{A},\mathscr{V}]$. Our paper extends these ideas somewhat. For general $\mathscr{A}$, we construct a promonoidal $\mathscr{V}$-category $\mathscr{D}\mathscr{A}$ (which we suggest should be called the double of $\mathscr{A}$) with an equivalence $[\mathscr{D}\mathscr{A},\mathscr{V}] \simeq \mathrm{Tamb}(\mathscr{A})$. When $\mathscr{A}$ is closed, we define strong (respectively, left strong) Tambara modules and show that these constitute a $\mathscr{V}$-category $\mathrm{Tamb}_s(\mathscr{A})$ (respectively, $\mathrm{Tamb}_{ls}(\mathscr{A})$) which is equivalent to the centre (respectively, lax centre) of $[\mathscr{A},\mathscr{V}]$. We construct localizations $\mathscr{D}_s\mathscr{A}$ and $\mathscr{D}_{ls}\mathscr{A}$ of $\mathscr{D}\mathscr{A}$ such that there are equivalences $\mathrm{Tamb}_s(\mathscr{A}) \simeq [\mathscr{D}_s\mathscr{A},\mathscr{V}]$ and $\mathrm{Tamb}_{ls}(\mathscr{A}) \simeq [\mathscr{D}_{ls}\mathscr{A},\mathscr{V}]$. When $\mathscr{A}$ is autonomous, every Tambara module is strong; this implies an equivalence $\mathcal{Z}[\mathscr{A},\mathscr{V}] \simeq [\mathscr{D}\mathscr{A},\mathscr{V}]$.

1. Introduction

For $\mathscr{V}$-categories $\mathscr{A}$ and $\mathscr{B}$, a *module* $T : \mathscr{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathscr{B}$ (also called "bimodule", "profunctor", and "distributor") is a $\mathscr{V}$-functor $T : \mathscr{B}^{\mathrm{op}} \otimes \mathscr{A} \longrightarrow \mathscr{V}$. For a monoidal $\mathscr{V}$-category $\mathscr{A}$, Tambara [Tam06] defined two-sided actions α of $\mathscr{A}$ on an endomodule $T : \mathscr{A} \longrightarrow\!\!\!\!\!\!+\;\, \mathscr{A}$. When $\mathscr{A}$ is autonomous (also called "rigid" or "compact") he showed that the $\mathscr{V}$-category $\mathrm{Tamb}(\mathscr{A})$ of Tambara modules (T, α) is equivalent to the monoidal centre $\mathcal{Z}[\mathscr{A},\mathscr{V}]$ of the convolution monoidal $\mathscr{V}$-category $[\mathscr{A},\mathscr{V}]$.

Our paper extends these ideas in four ways:

1. our base monoidal category $\mathscr{V}$ is quite general (as in [Kel82]) not just vector spaces;
2. our results are mainly for a closed monoidal $\mathscr{V}$-category $\mathscr{A}$, generalizing the autonomous case;
3. we show the connection with the lax centre as well as the centre; and,

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4. we introduce the double $\mathscr{D}\mathscr{A}$ of a monoidal $\mathscr{V}$-category $\mathscr{A}$ and some localizations of it, and relate these to Tambara modules.

Our principal goal is to give conditions under which the centre and lax centre of a $\mathscr{V}$-valued $\mathscr{V}$-functor monoidal $\mathscr{V}$-category is again such. Some results in this direction can be found in [DS07].

For general monoidal $\mathscr{A}$, we construct a promonoidal $\mathscr{V}$-category $\mathscr{D}\mathscr{A}$ with an equivalence $[\mathscr{D}\mathscr{A}, \mathscr{V}] \simeq \mathrm{Tamb}(\mathscr{A})$. When $\mathscr{A}$ is closed, we define when a Tambara module is (left) strong and show that these constitute a $\mathscr{V}$-category ($\mathrm{Tamb}_{ls}(\mathscr{A})$) $\mathrm{Tamb}_s(\mathscr{A})$ which is equivalent to the (lax) centre of $[\mathscr{A}, \mathscr{V}]$. We construct localizations $\mathscr{D}_s\mathscr{A}$ and $\mathscr{D}_{ls}\mathscr{A}$ of $\mathscr{D}\mathscr{A}$ such that there are equivalences $\mathrm{Tamb}_s(\mathscr{A}) \simeq [\mathscr{D}_s\mathscr{A}, \mathscr{V}]$ and $\mathrm{Tamb}_{ls}(\mathscr{A}) \simeq [\mathscr{D}_{ls}\mathscr{A}, \mathscr{V}]$. When $\mathscr{A}$ is autonomous, every Tambara module is strong, which implies an equivalence $\mathcal{Z}[\mathscr{A}, \mathscr{V}] \simeq [\mathscr{D}\mathscr{A}, \mathscr{V}]$. These results should be compared with those of [DS07] where the lax centre of $[\mathscr{A}, \mathscr{V}]$ is shown generally to be a full sub-$\mathscr{V}$-category of a functor $\mathscr{V}$-category $[\mathscr{A}_M, \mathscr{V}]$ which also becomes an equivalence $\mathcal{Z}[\mathscr{A}, \mathscr{V}] \simeq [\mathscr{A}_M, \mathscr{V}]$ when $\mathscr{A}$ is autonomous.

As we were completing this paper, Ignacio Lopez Franco sent us his preprint [LF07] which has some results in common with ours. As an example for $\mathscr{V}$-modules of his general constructions on pseudomonoids, he is also led to what we call the double monad.

2. Centres and convolution

We work with categories enriched in a base monoidal category $\mathscr{V}$ as used by Kelly [Kel82]. It is symmetric, closed, complete and cocomplete.

Let $\mathscr{A}$ denote a closed monoidal $\mathscr{V}$-category. We denote the tensor product by $A \otimes B$ and the unit by I in the hope that this will cause no confusion with the same symbols used for the base $\mathscr{V}$ itself. We have $\mathscr{V}$-natural isomorphisms

$$\mathscr{A}(A, {}^{B}C) \cong \mathscr{A}(A \otimes B, C) \cong \mathscr{A}(B, C^{A})$$

defined by evaluation and coevaluation morphisms

$$e_l : {}^{B}C \otimes B \longrightarrow C, \quad d_l : A \longrightarrow {}^{B}(A \otimes B), \quad e_r : A \otimes C^{A} \longrightarrow C \quad \text{and} \quad d_r : B \longrightarrow (A \otimes B)^{A}.$$

Consequently, there are canonical isomorphisms

$${}^{A\otimes B}C \cong {}^{A}({}^{B}C), \quad C^{A\otimes B} \cong (C^{A})^{B}, \quad ({}^{B}C)^{A} \cong {}^{B}(C^{A}) \quad \text{and} \quad {}^{I}C \cong C \cong C^{I}$$

which we write as if they were identifications just as we do with the associativity and unit isomorphisms. We also write ${}^{B}C^{A}$ for ${}^{B}(C^{A})$.

The Day convolution monoidal structure [Day70] on the $\mathscr{V}$-category $[\mathscr{A}, \mathscr{V}]$ of $\mathscr{V}$-

functors from $\mathscr{A}$ to $\mathscr{V}$ consists of the tensor product $F * G$ and unit J defined by

$$\begin{aligned}(F * G)A &= \int^{U,V} \mathscr{A}(U \otimes V, A) \otimes FU \otimes GV \\ &\cong \int^{V} F({}^{V}A) \otimes GV \\ &\cong \int^{U} FU \otimes G(A^{U})\end{aligned}$$

and

$$JA = \mathscr{A}(I, A).$$

In particular,

$$(F * \mathscr{A}(A, -))B \cong F({}^{A}B) \qquad \text{and} \qquad (\mathscr{A}(A, -) * G)B \cong G(B^{A}).$$

The centre of a monoidal category was defined in [JS91] and the lax centre was defined, for example, in [DPS07]. Since the representables are dense in $[\mathscr{A}, \mathscr{V}]$, an object of the *lax centre* $\mathscr{Z}_l[\mathscr{A}, \mathscr{V}]$ of $[\mathscr{A}, \mathscr{V}]$ is a pair (F, θ) consisting of $F \in [\mathscr{A}, \mathscr{V}]$ and a $\mathscr{V}$-natural family θ of morphisms

$$\theta_{A,B} : F({}^{A}B) \longrightarrow F(B^{A})$$

such that the diagrams

$$\begin{tikzcd}
F({}^{A\otimes B}C) \arrow[rr, "\theta_{A\otimes B,C}"] \arrow[d, "="'] & & F(C^{A\otimes B}) \\
F({}^{A}({}^{B}C)) \arrow[dr, "\theta_{A,{}^{B}C}"'] & & F((C^{A})^{B}) \arrow[u, "="'] \\
& F({}^{B}C^{A}) \arrow[ur, "\theta_{B,C^{A}}"'] &
\end{tikzcd}
\qquad \text{and} \qquad
\begin{tikzcd}
F({}^{I}A) \arrow[rr, "\theta_{I,A}"] \arrow[dr, "="'] & & F(A^{I}) \\
& FA \arrow[ur, "="'] &
\end{tikzcd}$$

commute. The hom object $\mathscr{Z}_l[\mathscr{A}, \mathscr{V}]((F, \theta), (G, \phi))$ is defined to be the equalizer of two obvious morphisms out of $[\mathscr{A}, \mathscr{V}](F, G)$. The *centre* $\mathscr{Z}[\mathscr{A}, \mathscr{V}]$ of $[\mathscr{A}, \mathscr{V}]$ is the full sub-$\mathscr{V}$-category of $\mathscr{Z}_l[\mathscr{A}, \mathscr{V}]$ consisting of those objects (F, θ) with θ invertible.

3. Tambara modules

Let $\mathscr{A}$ denote a monoidal $\mathscr{V}$-category. We do not need $\mathscr{A}$ to be closed for the definition of Tambara module although we will require this restriction again later.

A *left Tambara module* on $\mathscr{A}$ is a $\mathscr{V}$-functor $T : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ together with a family of morphisms

$$\alpha_l(A, X, Y) : T(X, Y) \longrightarrow T(A \otimes X, A \otimes Y)$$

which are $\mathscr{V}$-natural in each of the objects A, X and Y, satisfying the two equations $\alpha_l(I,X,Y) = 1_{T(X,Y)}$ and

$$\begin{tikzcd} T(X,Y) \arrow[rr, "\alpha_l(A',X,Y)"] \arrow[dr, "\alpha_l(A\otimes A',X,Y)"'] & & T(A'\otimes X, A'\otimes Y) \arrow[dl, "\alpha_l(A,A'\otimes X,A'\otimes Y)"] \\ & T(A\otimes A'\otimes X, A\otimes A'\otimes Y). & \end{tikzcd}$$

Similarly, a *right Tambara module* on $\mathscr{A}$ is a $\mathscr{V}$-functor $T : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ together with a family of morphisms

$$\alpha_r(B,X,Y) : T(X,Y) \longrightarrow T(X\otimes B, Y\otimes B)$$

which are $\mathscr{V}$-natural in each of the objects B, X and Y, satisfying the two equations $\alpha_r(I,X,Y) = 1_{T(X,Y)}$ and

$$\begin{tikzcd} T(X,Y) \arrow[rr, "\alpha_r(B,X,Y)"] \arrow[dr, "\alpha_r(B\otimes B',X,Y)"'] & & T(X\otimes B, Y\otimes B) \arrow[dl, "\alpha_r(B',B\otimes X,B\otimes Y)"] \\ & T(X\otimes B\otimes B', Y\otimes B\otimes B'). & \end{tikzcd}$$

A *Tambara module*(T,α) on $\mathscr{A}$ is a $\mathscr{V}$-functor $T : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ together with both left and right Tambara module structures satisfying the "bimodule" compatibility condition

$$\begin{tikzcd} T(X,Y) \arrow[r, "\alpha_l(A,X,Y)"] \arrow[d, "\alpha_r(B,X,Y)"'] & T(A\otimes X, A\otimes Y) \arrow[d, "\alpha_r(B,A\otimes X,A\otimes Y)"] \\ T(X\otimes B, Y\otimes B) \arrow[r, "\alpha_l(A,X\otimes B,Y\otimes B)"'] & T(A\otimes X\otimes B, A\otimes Y\otimes B). \end{tikzcd}$$

The morphism defined to be the diagonal of the last square is denoted by

$$\alpha(A,B,X,Y) : T(X,Y) \longrightarrow T(A\otimes X\otimes B, A\otimes Y\otimes B)$$

and we can express a Tambara module structure purely in terms of this, however, we need to refer to the left and right structures below.

3.1. PROPOSITION. *Suppose $\mathscr{A}$ is a monoidal $\mathscr{V}$-category and $T : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ is a $\mathscr{V}$-functor.*

(a) *If $\mathscr{A}$ is right closed, there is a bijection between $\mathscr{V}$-natural families of morphisms*

$$\alpha_l(A,X,Y) : T(X,Y) \longrightarrow T(A\otimes X, A\otimes Y)$$

and $\mathscr{V}$-natural families of morphisms

$$\beta_l(A,X,Y) : T(X,Y^A) \longrightarrow T(A\otimes X, Y).$$

(b) *Under the bijection of (a), the family α_l is a left Tambara structure if and only if the family β_l satisfies the two equations $\beta_l(I,X,Y) = 1_{T(X,Y)}$ and*

$$\begin{tikzcd} T(X, Y^{A\otimes A'}) \arrow[r, "\beta_l(A\otimes A', X, Y)"] \arrow[d, "="'] & T(A\otimes A'\otimes X, Y) \\ T(X, (Y^A)^{A'}) \arrow[r, "\beta_l(A', X, Y^A)"'] & T(A'\otimes X, Y^A). \arrow[u, "\beta_l(A, A'\otimes X, Y)"'] \end{tikzcd}$$

(c) *If $\mathscr{A}$ is left closed, there is a bijection between $\mathscr{V}$-natural families of morphisms*

$$\alpha_r(B,X,Y) : T(X,Y) \longrightarrow T(X\otimes B, Y\otimes B)$$

and $\mathscr{V}$-natural families of morphisms

$$\beta_r(B,X,Y) : T(X, {}^BY) \longrightarrow T(X\otimes B, Y).$$

(d) *Under the bijection of (c), the family α_r is a right Tambara structure if and only if the family β_r satisfies the two equations $\beta_r(I,X,Y) = 1_T(X,Y)$ and*

$$\begin{tikzcd} T(X, {}^{B\otimes B'}Y) \arrow[r, "\beta_r(B\otimes B', X, Y)"] \arrow[d, "="'] & T(X\otimes B\otimes B', Y) \\ T(X, {}^B({}^{B'}Y)) \arrow[r, "\beta_r(B, X, {}^{B'}Y)"'] & T(X\otimes B, {}^{B'}Y). \arrow[u, "\beta_r(B', X\otimes B, Y)"'] \end{tikzcd}$$

(e) *If $\mathscr{A}$ is closed, the families α_l and α_r form a Tambara module structure if and only if the families β_l and β_r, corresponding under (a) and (c), satisfy the condition*

$$\begin{tikzcd} T(X, {}^BY^A) \arrow[r, "\beta_l(A, X, {}^BY)"] \arrow[d, "\beta_r(B, X, Y^A)"'] & T(A\otimes X, {}^BY) \arrow[d, "\beta_r(B, A\otimes X, Y)"] \\ T(X\otimes B, Y^A) \arrow[r, "\beta_l(A, X\otimes B, Y)"'] & T(A\otimes X\otimes B, Y). \end{tikzcd}$$

Proof. The bijection of (a) is defined by the formulas

$$\beta_l(A,X,Y) = \Big(T(X,Y^A) \xrightarrow{\alpha_l(A,X,Y^A)} T(A\otimes X, A\otimes Y^A) \xrightarrow{T(A\otimes X, e_r)} T(A\otimes X, Y) \Big)$$

and

$$\alpha_l(A,X,Y) = \Big(T(X,Y) \xrightarrow{T(X,d_r)} T(X, (A\otimes Y)^A) \xrightarrow{\beta_l(A,X,A\otimes Y)} T(A\otimes X, A\otimes Y) \Big).$$

That the processes are mutually inverse uses the adjunction identities on the morphisms e and d. The bijection of (c) is obtained dually by reversing the tensor product. Translation of the conditions from the α to the β as required for (b), (d) and (e) is straightforward. ∎

A left (respectively, right) Tambara module T on $\mathscr{A}$ will be called *strong* when the morphisms

$$\beta_l(A, X, Y) : T(X, Y^A) \longrightarrow T(A \otimes X, Y)$$
$$\text{(respectively, } \beta_r(B, X, Y) : T(X, {}^BY) \longrightarrow T(X \otimes B, Y))$$

corresponding via Proposition 3.1 to the left (respectively, right) Tambara structure, are invertible. A Tambara module is called *left* (respectively, *right*) *strong* when it is strong as a left (respectively, right) module and *strong* when it is both left and right strong. In particular, notice that the hom $\mathscr{V}$-functor (= identity module) of $\mathscr{A}$ is a strong Tambara module.

3.2. Proposition. *Suppose $\mathscr{A}$ is a monoidal $\mathscr{V}$-category and $T : \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A} \longrightarrow \mathscr{V}$ is a $\mathscr{V}$-functor. If $\mathscr{A}$ is right (left) autonomous then every left (right) Tambara module is strong.*

Proof. If A^* denotes a right dual for A with unit $\eta : I \longrightarrow A^* \otimes A$ then an inverse for β_l is defined by the composite

$$T(A \otimes X, Y) \xrightarrow{\alpha_l(A^*, A\otimes X, Y)} T(A^* \otimes A \otimes X, A^* \otimes Y) \xrightarrow{T(\eta,1)} T(X, A^* \otimes Y)\,.$$

■

Write $\mathrm{LTamb}(\mathscr{A})$ for the $\mathscr{V}$-category whose objects are left Tambara modules $T = (T, \alpha_l)$ and whose hom $\mathrm{LTamb}(\mathscr{A})(T, T')$ in $\mathscr{V}$ is defined to be the intersection over all A, X and Y of the equalizers of the pairs of morphisms:

$$[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}](T, T') \underset{\mathscr{V}(1,\alpha_l)\circ \mathrm{pr}_{X,Y}}{\overset{\mathscr{V}(\alpha_l,1)\circ \mathrm{pr}_{A\otimes X, A\otimes Y}}{\rightrightarrows}} \mathscr{V}(T(X, Y), T'(A \otimes X, A \otimes Y))\,.$$

Equivalently, we can define the hom as an intersection of equalizers of pairs of morphisms:

$$[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}](T, T') \underset{\mathscr{V}(1,\beta_l)\circ \mathrm{pr}_{X,Y^A}}{\overset{\mathscr{V}(\beta_l,1)\circ \mathrm{pr}_{A\otimes X, Y}}{\rightrightarrows}} \mathscr{V}(T(X, Y^A), T'(A \otimes X, Y))\,.$$

Composition is defined so that we have a $\mathscr{V}$-functor $\iota : \mathrm{LTamb}(\mathscr{A}) \longrightarrow [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$ which forgets the left module structure on T. In fact, $\mathrm{LTamb}(\mathscr{A})$ becomes a monoidal $\mathscr{V}$-category in such a way that the forgetful $\mathscr{V}$-functor ι becomes strong monoidal. For this, the monoidal structure on $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$ is the usual tensor product (= composition) of endomodules:

$$(T \otimes_{\mathscr{A}} T')(X, Y) = \int^Z T(X, Z) \otimes T'(Z, Y).$$

When T and T' are left Tambara modules, the left Tambara structure

$$(T \otimes_{\mathscr{A}} T')(X, Y) \longrightarrow (T \otimes_{\mathscr{A}} T')(A \otimes X, A \otimes Y)$$

on $T \otimes_{\mathscr{A}} T'$ is defined by taking its composite with the coprojection copr_Z into the above coend to be the composite

$$T(X,Z) \otimes T'(Z,Y) \xrightarrow{\alpha_l \otimes \alpha_l} T(A \otimes X, A \otimes Z) \otimes T'(A \otimes Z, A \otimes Y) \xrightarrow{\mathrm{copr}_{A \otimes Z}} (T \otimes_{\mathscr{A}} T')(A \otimes X, A \otimes Y)\,.$$

Similarly we obtain monoidal $\mathscr{V}$-categories $\mathrm{RTamb}(\mathscr{A})$ and $\mathrm{Tamb}(\mathscr{A})$ of right Tambara and all Tambara modules on $\mathscr{A}$.

We write $\mathrm{LTamb}_s(\mathscr{A})$ for the full sub-$\mathscr{V}$-category of $\mathrm{LTamb}(\mathscr{A})$ consisting of the strong left Tambara modules. We write $\mathrm{Tamb}_{ls}(\mathscr{A})$, $\mathrm{Tamb}_{rs}(\mathscr{A})$ and $\mathrm{Tamb}_s(\mathscr{A})$ for the full sub-$\mathscr{V}$-categories of $\mathrm{Tamb}(\mathscr{A})$ consisting of the left strong, right strong and strong Tambara modules respectively.

If $\mathscr{A}$ is autonomous then $\mathrm{Tamb}(\mathscr{A}) = \mathrm{Tamb}_{ls}(\mathscr{A}) = \mathrm{Tamb}_{rs}(\mathscr{A}) = \mathrm{Tamb}_s(\mathscr{A})$ by Proposition 3.2.

4. The Cayley functor

Consider a right closed monoidal $\mathscr{V}$-category $\mathscr{A}$. There is a *Cayley $\mathscr{V}$-functor*

$$\Upsilon : [\mathscr{A}, \mathscr{V}] \longrightarrow [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$$

defined as follows. To each object $F \in [\mathscr{A}, \mathscr{V}]$, define $\Upsilon(F) = T_F$ by

$$T_F(X,Y) = F(Y^X).$$

The effect $\Upsilon_{F,G} : [\mathscr{A}, \mathscr{V}](F,G) \longrightarrow [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}](T_F, T_G)$ of Υ on homs is defined by taking its composite with the projection

$$\mathrm{pr}_{X,Y} : [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}](T_F, T_G) \longrightarrow \mathscr{V}(F(Y^X), G(Y^X))$$

to be the projection

$$\mathrm{pr}_{Y^X} : [\mathscr{A}, \mathscr{V}](F,G) \longrightarrow \mathscr{V}(F(Y^X), G(Y^X)).$$

4.1. Proposition. *The Cayley $\mathscr{V}$-functor Υ is strong monoidal; it takes Day convolution to composition of endomodules.*

Proof. We have the calculation:

$$
\begin{aligned}
(\Upsilon(F)\otimes_{\mathscr{A}}\Upsilon(G))(X,Y) &= \int^{Z}\Upsilon(F)(X,Z)\otimes\Upsilon(G)(Z,Y)\\
&= \int^{Z} F(Z^X)\otimes G(Y^Z)\\
&\cong \int^{Z,U,V}\mathscr{A}(U,Z^X)\otimes FU\otimes\mathscr{A}(V,Y^Z)\otimes GV\\
&\cong \int^{Z,U,V}\mathscr{A}(X\otimes U,Z)\otimes FU\otimes\mathscr{A}(Z\otimes V,Y)\otimes GV\\
&\cong \int^{U,V}\mathscr{A}(X\otimes U\otimes V,Y)\otimes FU\otimes GV\\
&\cong \int^{U,V}\mathscr{A}(U\otimes V,Y^X)\otimes FU\otimes GV\\
&\cong \Upsilon(F*G)(X,Y),
\end{aligned}
$$

and of course $\Upsilon(\mathscr{A}(I,-))(X,Y)=\mathscr{A}(I,Y^X)\cong\mathscr{A}(X,Y)$. ∎

In fact, Υ lands in the left Tambara modules by defining, for each $F\in[\mathscr{A},\mathscr{V}]$, the structure

$$\alpha_l(A,X,Y)=\Big(\, F(Y^X)\xrightarrow{F((d_r)^X)}F((A\otimes Y)^{A\otimes X})\,\Big)$$

on T_F. It is helpful to observe that the β_l corresponding to this α_l (via Proposition 3.1) is given by the identity

$$\beta_l(A,X,Y)=\Big(\, F(Y^{A\otimes X})\xrightarrow{1}F(Y^{A\otimes X})\,\Big),$$

showing that T_F becomes a strong left module. To see that there is a $\mathscr{V}$-functor $\hat{\Upsilon}:[\mathscr{A},\mathscr{V}]\longrightarrow\mathrm{LTamb}_s(\mathscr{A})$ satisfying $\iota\circ\hat{\Upsilon}=\Upsilon$, we merely observe that

$$\mathrm{pr}_{A\otimes X,Y}\circ\Upsilon_{F,G}=\mathrm{pr}_{Y^{A\otimes X}}=\mathrm{pr}_{(Y^A)^X}=\mathrm{pr}_{X,Y^A}\circ\Upsilon_{F,G}.$$

4.2. Proposition. *If $\mathscr{A}$ is a right closed monoidal $\mathscr{V}$-category then the $\mathscr{V}$-functor $\hat{\Upsilon}:[\mathscr{A},\mathscr{V}]\longrightarrow\mathrm{LTamb}_s(\mathscr{A})$ is an equivalence.*

Proof. Define $\zeta:\mathrm{LTamb}(\mathscr{A})(T_F,T_G)\longrightarrow[\mathscr{A},\mathscr{V}](F,G)$ by $\mathrm{pr}_Y\circ\zeta=\mathrm{pr}_{I,Y}\circ\iota_{T_F,T_G}$. Then

$$\mathrm{pr}_Y\circ\zeta\circ\hat{\Upsilon}_{F,G}=\mathrm{pr}_{I,Y}\circ\iota_{T_F,T_G}\circ\hat{\Upsilon}_{F,G}=\mathrm{pr}_{I,Y}\circ\Upsilon_{F,G}=\mathrm{pr}_Y$$

and

$$
\begin{aligned}
\mathrm{pr}_{X,Y}\circ\iota_{T_F,T_G}\circ\hat{\Upsilon}_{F,G}\circ\zeta &= \mathrm{pr}_{X,Y}\circ\Upsilon_{F,G}\circ\zeta\\
&= \mathrm{pr}_{Y^X}\circ\zeta\\
&= \mathrm{pr}_{I,Y^X}\circ\iota_{T_F,T_G}\\
&= \mathrm{pr}_{X,Y}\circ\iota_{T_F,T_G}.
\end{aligned}
$$

It follows that ζ is the inverse of $\hat{\Upsilon}_{F,G}$, so that $\hat{\Upsilon}$ is fully faithful. To see that $\hat{\Upsilon}$ is essentially surjective on objects, take a strong left module T. Put $FY = T(I,Y)$ as a $\mathscr{V}$-functor in Y. Then the isomorphism $\beta_l(X,I,Y)$ yields

$$T_F(X,Y) = F(Y^X) = T(I,Y^X) \cong T(X,Y);$$

so $\hat{\Upsilon}(F) \cong T$. ∎

Now suppose we have an object (F,θ) of the lax centre $\mathcal{Z}_l[\mathscr{A},\mathscr{V}]$ of $[\mathscr{A},\mathscr{V}]$. Then T_F becomes a right Tambara module by defining

$$\alpha_r(B,X,Y) = \Big(F(Y^X) \xrightarrow{F((d_l)^X)} F({}^B(Y\otimes B)^X) \xrightarrow{\theta_{B,(Y\otimes B)^X}} F(Y\otimes B)^{X\otimes B} \Big).$$

If $\mathscr{A}$ is left closed, the β_r corresponding to this α_r (via Proposition 3.1) is defined by

$$\beta_r(B,X,Y) = \Big(F({}^BY^X) \xrightarrow{\theta_{B,Y^X}} F(Y^{X\otimes B}) \Big).$$

It is easy to see that, in this way, $T_F = \hat{\Upsilon}(F)$ actually becomes a (two-sided) Tambara module which we write as $\hat{\Upsilon}(F,\theta)$, and we have a $\mathscr{V}$-functor

$$\hat{\Upsilon} : \mathcal{Z}_l[\mathscr{A},\mathscr{V}] \longrightarrow \mathrm{Tamb}_{ls}(\mathscr{A}).$$

4.3. Proposition. *If $\mathscr{A}$ is a closed monoidal $\mathscr{V}$-category then the $\mathscr{V}$-functor*

$$\hat{\Upsilon} : \mathcal{Z}_l[\mathscr{A},\mathscr{V}] \longrightarrow \mathrm{Tamb}_{ls}(\mathscr{A})$$

is an equivalence which restricts to an equivalence

$$\hat{\Upsilon} : \mathcal{Z}[\mathscr{A},\mathscr{V}] \longrightarrow \mathrm{Tamb}_{s}(\mathscr{A}).$$

Proof. The proof of full faithfulness proceeds along the lines of the beginning of the proof of Proposition 4.2. For essential surjectivity on objects, take a left strong Tambara module (T,α). Then $\beta_l(A,X,Y) : T(X,Y^A) \longrightarrow T(A\otimes X,Y)$ is invertible. Define the $\mathscr{V}$-functor $F : \mathscr{A} \longrightarrow \mathscr{V}$ by $FX = T(I,X)$ as in the proof of Proposition 4.2, and define $\theta_{A,Y} : F({}^AY) \longrightarrow F(Y^A)$ to be the composite

$$T(I,{}^AY) \xrightarrow{\beta_r(A,I,Y)} T(A,Y) \xrightarrow{\beta_l(A,I,Y)^{-1}} T(I,Y^A)\ .$$

This is easily seen to yield an object (F,θ) of the lax centre $\mathcal{Z}_l[\mathscr{A},\mathscr{V}]$ with $\hat{\Upsilon}(F,\theta) \cong T_F$. Thus we have the first equivalence. Clearly θ is invertible if and only if β_r is; the second equivalence follows. ∎

5. The double monad

Tambara modules are actually Eilenberg-Moore coalgebras for a fairly obvious comonad on $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$. We begin by looking at the case of left modules.

Let $\Theta_l : [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}] \longrightarrow [\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$ be the $\mathscr{V}$-functor defined by the end

$$\Theta_l(T)(X, Y) = \int_A T(A \otimes X, A \otimes Y).$$

There is a $\mathscr{V}$-natural family $\epsilon_T : \Theta_l(T) \longrightarrow T$ defined by the projections

$$\mathrm{pr}_I : \int_A T(A \otimes X, A \otimes Y) \longrightarrow T(X, Y).$$

There is a $\mathscr{V}$-natural family $\delta_T : \Theta_l(T) \longrightarrow \Theta_l(\Theta_l(T))$ defined by taking its composite with the projection

$$\mathrm{pr}_{B,C} : \int_{B,C} T(B \otimes C \otimes X, B \otimes C \otimes Y) \longrightarrow T(B \otimes C \otimes X, B \otimes C \otimes Y)$$

to be the projection

$$\mathrm{pr}_{B \otimes C} : \int_A T(A \otimes X, A \otimes Y) \longrightarrow T(B \otimes C \otimes X, B \otimes C \otimes Y).$$

It is now easily checked that $\Theta_l = (\Theta_l, \delta, \epsilon)$ is a comonad on $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$.

There is also a comonad Θ_r on $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$, a distributive law $\Theta_r\Theta_l \cong \Theta_l\Theta_r$, and a comonad $\Theta = \Theta_r\Theta_l$:

$$\Theta_r(T)(X, Y) = \int_B T(X \otimes B, Y \otimes B)$$

and

$$\Theta(T)(X, Y) = \int_{A,B} T(A \otimes X \otimes B, A \otimes Y \otimes B).$$

We can easily identify the $\mathscr{V}$-categories of Eilenberg-Moore coalgebras for these three comonads.

5.1. Proposition. *There are isomorphisms of $\mathscr{V}$-categories*

- $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]^{\Theta_l} \cong \mathrm{LTamb}(\mathscr{A})$,
- $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]^{\Theta_r} \cong \mathrm{RTamb}(\mathscr{A})$, *and*
- $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]^{\Theta} \cong \mathrm{Tamb}(\mathscr{A})$.

In fact, Θ_l, Θ_r and Θ are all monoidal comonads on $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]$. For example, the structure on Θ_l is provided by the $\mathscr{V}$-natural transformations $\Theta_l(T) \otimes_{\mathscr{A}} \Theta_l(T') \longrightarrow \Theta_l(T \otimes_{\mathscr{A}} T')$ and $\mathscr{A}(-,-) \longrightarrow \Theta_l(\mathscr{A}(-,-))$ with components

$$\int^Z \int_A T(A\otimes X, A\otimes Z)\otimes \int_B T'(B\otimes X, B\otimes Z) \longrightarrow \int_C \int^U T(C\otimes X, U)\otimes T'(U, C\otimes Y) \quad (1)$$

and

$$\mathscr{A}(X,Y) \longrightarrow \int_A \mathscr{A}(A\otimes X, A\otimes Y) \quad (2)$$

defined as follows. The morphism (1) is determined by its precomposite with the coprojection copr_Z and postcomposite with the projection pr_C; the result is defined to be the composite

$$\begin{aligned} &\int_A T(A\otimes X, A\otimes Z)\otimes \int_B T'(B\otimes X, B\otimes Z) \\ &\qquad \xrightarrow{\mathrm{pr}_C \otimes \mathrm{pr}_C} T(C\otimes X, C\otimes Z)\otimes T'(C\otimes Z, C\otimes Y) \\ &\qquad \xrightarrow{\mathrm{copr}_{C\otimes Z}} \int^U T(C\otimes X, U)\otimes T'(U, C\otimes Y)\ . \end{aligned}$$

The morphism (2) is simply the coprojection copr_I. It follows that $[\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}, \mathscr{V}]^{\Theta_l}$ becomes monoidal with the underlying functor becoming strong monoidal; see [Moe02] and [McC02]. Clearly we have:

5.2. Proposition. *The isomorphisms of Proposition 5.1 are monoidal.*

The next thing to observe is that Θ_l, Θ_r and Θ all have left adjoints Φ_l, Φ_r and Φ which therefore become opmonoidal monads whose $\mathscr{V}$-categories of Eilenberg-Moore algebras are monoidally isomorphic to $\mathrm{LTamb}(\mathscr{A})$, $\mathrm{RTamb}(\mathscr{A})$ and $\mathrm{Tamb}(\mathscr{A})$, respectively. Straightforward applications of the Yoneda Lemma, show that the formulas for these adjoints are

$$\begin{aligned} \Phi_l(S)(U,V) &= \int^{A,X,Y} \mathscr{A}(U, A\otimes X)\otimes \mathscr{A}(A\otimes Y, V)\otimes S(X,Y), \\ \Phi_r(S)(U,V) &= \int^{B,X,Y} \mathscr{A}(U, X\otimes B)\otimes \mathscr{A}(Y\otimes B, V)\otimes S(X,Y), \quad \text{and} \\ \Phi(S)(U,V) &= \int^{A,B,X,Y} \mathscr{A}(U, A\otimes X\otimes B)\otimes \mathscr{A}(A\otimes Y\otimes B, V)\otimes S(X,Y). \end{aligned}$$

Recall that left adjoint $\mathscr{V}$-functors $\Psi : [\mathscr{X}^{\mathrm{op}}, \mathscr{V}] \longrightarrow [\mathscr{Y}^{\mathrm{op}}, \mathscr{V}]$ are equivalent to $\mathscr{V}$-functors $\check{\Psi} : \mathscr{Y}^{\mathrm{op}} \otimes \mathscr{X} \longrightarrow \mathscr{V}$, which are also called modules $\check{\Psi} : \mathscr{X} \mathrel{\mkern3mu\vcenter{\hbox{$\scriptscriptstyle+$}}\mkern-12mu{\to}} \mathscr{Y}$ from $\mathscr{X}$ to $\mathscr{Y}$. The equivalence is defined by:

$$\check{\Psi}(Y,X) = \Psi(\mathscr{X}(-,X))(Y)$$

and

$$\Psi(M)(Y) = \int^X \check{\Psi}(Y, X) \otimes M(X).$$

It follows that Φ_l, Φ_r and Φ determine monads $\check{\Phi}_l$, $\check{\Phi}_r$ and $\check{\Phi}$ on $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ in the bicategory $\mathscr{V}$-**Mod**. The formulas are:

$$\check{\Phi}_l(X, Y, U, V) = \int^A \mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A \otimes Y, V),$$

$$\check{\Phi}_r(X, Y, U, V) = \int^B \mathscr{A}(U, X \otimes B) \otimes \mathscr{A}(Y \otimes B, V), \quad \text{and}$$

$$\check{\Phi}(X, Y, U, V) = \int^{A,B} \mathscr{A}(U, A \otimes X \otimes B) \otimes \mathscr{A}(A \otimes Y \otimes B, V).$$

6. Doubles

The bicategory $\mathscr{V}$-**Mod** admits the Kleisli construction for monads. Write $\mathscr{D}_l\mathscr{A}$, $\mathscr{D}_r\mathscr{A}$ and $\mathscr{D}\mathscr{A}$ for the Kleisli $\mathscr{V}$-categories for the monads $\check{\Phi}_l$, $\check{\Phi}_r$ and $\check{\Phi}$ on $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$ in the bicategory $\mathscr{V}$-**Mod**. We call them the *left double*, *right double* and *double* of the monoidal $\mathscr{V}$-category $\mathscr{A}$. They all have the same objects as $\mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$. The homs are defined by

$$\mathscr{D}_l\mathscr{A}((X, Y), (U, V)) = \int^A \mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A \otimes Y, V),$$

$$\mathscr{D}_r\mathscr{A}((X, Y), (U, V)) = \int^B \mathscr{A}(U, X \otimes B) \otimes \mathscr{A}(Y \otimes B, V), \quad \text{and}$$

$$\mathscr{D}\mathscr{A}((X, Y), (U, V)) = \int^{A,B} \mathscr{A}(U, A \otimes X \otimes B) \otimes \mathscr{A}(A \otimes Y \otimes B, V).$$

6.1. Proposition. *There are canonical equivalences of $\mathscr{V}$-categories:*

- $\Xi_l : \mathrm{LTamb}(\mathscr{A}) \simeq [\mathscr{D}_l\mathscr{A}, \mathscr{V}]$,
- $\Xi_r : \mathrm{RTamb}(\mathscr{A}) \simeq [\mathscr{D}_r\mathscr{A}, \mathscr{V}]$, *and*
- $\Xi : \mathrm{Tamb}(\mathscr{A}) \simeq [\mathscr{D}\mathscr{A}, \mathscr{V}]$.

It follows from the main result of Day [Day70] that these doubles $\mathscr{D}_l\mathscr{A}$, $\mathscr{D}_r\mathscr{A}$ and $\mathscr{D}\mathscr{A}$ all admit promonoidal structures (P_l, J_l), (P_r, J_r) and (P, J) for which the equivalences in Proposition 6.1 become monoidal when the right-hand sides are given the corresponding convolution structures. For example, we calculate that P_l and J_l are as follows:

$$\begin{aligned}
P_l((X, Y), (U, V); (H, K)) &= (\mathscr{D}_l\mathscr{A}((X, Y), -) \otimes_{\mathscr{A}} \mathscr{D}_l\mathscr{A}((U, V), -))(H, K) \\
&= \int^{Z,A,B} \mathscr{A}(H, A \otimes X) \otimes \mathscr{A}(A \otimes Y, Z) \otimes \mathscr{A}(Z, B \otimes U) \otimes \mathscr{A}(B \otimes V, K) \\
&= \int^{A,B} \mathscr{A}(H, A \otimes X) \otimes \mathscr{A}(A \otimes Y, B \otimes U) \otimes \mathscr{A}(B \otimes V, K)
\end{aligned}$$

and

$$J_l(H,K) = \mathscr{A}(H,K).$$

Furthermore, there are some special morphisms that exist in these doubles $\mathscr{D}_l\mathscr{A}$, $\mathscr{D}_r\mathscr{A}$ and $\mathscr{D}\mathscr{A}$. Let $\tilde{\alpha}_l : (X,Y) \longrightarrow (A\otimes X, A\otimes Y)$ denote the morphism in $\mathscr{D}_l\mathscr{A}$ defined by the composite

$$I \xrightarrow{j_{A\otimes X}\otimes j_{A\otimes Y}} \mathscr{A}(A\otimes X, A\otimes X)\otimes \mathscr{A}(A\otimes Y, A\otimes Y) \xrightarrow{\mathrm{copr}_A} \mathscr{D}_l\mathscr{A}((X,Y),(A\otimes X, A\otimes Y))\,.$$

The $\mathscr{V}$-functor Ξ_l has the property that $\Xi_l(T,\alpha_l)(X,Y) = T(X,Y)$ and $\Xi_l(T,\alpha_l)(\tilde{\alpha}_l) = \alpha_l$. When $\mathscr{A}$ is right closed, we let $\tilde{\beta}_l : (X, Y^A) \longrightarrow (A\otimes X, Y)$ denote the morphism in $\mathscr{D}_l\mathscr{A}$ defined by the composite

$$I \xrightarrow{j_{A\otimes X}\otimes e_r} \mathscr{A}(A\otimes X, A\otimes X)\otimes \mathscr{A}(A\otimes Y^A, Y) \xrightarrow{\mathrm{copr}_A} \mathscr{D}_l\mathscr{A}((X,Y^A),(A\otimes X, Y))\,.$$

Then $\Xi_l(T,\alpha_l)(\tilde{\beta}_l) = \beta_l$.

Similarly, we have the morphism $\tilde{\alpha}_r : (X,Y) \longrightarrow (X\otimes B, Y\otimes B)$ in $\mathscr{D}_r\mathscr{A}$, and also, when $\mathscr{A}$ is left closed, the morphism $\tilde{\beta}_r : (X, {}^BY) \longrightarrow (X\otimes B, Y)$.

There are $\mathscr{V}$-functors $\mathscr{D}_l\mathscr{A} \longrightarrow \mathscr{D}\mathscr{A} \longleftarrow \mathscr{D}_r\mathscr{A}$ which are the identity functions on objects and are defined on homs using projections with $B = I$ for the left leg and the projections $A = I$ for the second leg. In this way, the morphisms $\tilde{\alpha}_l$ and $\tilde{\alpha}_r$ can be regarded also as morphisms of $\mathscr{D}\mathscr{A}$. Under closedness assumptions, the morphisms $\tilde{\beta}_l$ and $\tilde{\beta}_r$ can also be regarded as morphisms of $\mathscr{D}\mathscr{A}$.

Let Σ_l denote the set of morphisms $\tilde{\beta}_l : (X, Y^A) \longrightarrow (A\otimes X, Y)$, let Σ_r denote the set of morphisms $\tilde{\beta}_r : (X, {}^BY) \longrightarrow (X\otimes B, Y)$, and let Σ denote the set of morphisms $\Sigma = \Sigma_l \cup \Sigma_r$. Under appropriate closedness assumptions on $\mathscr{A}$, we can form various $\mathscr{V}$-categories of fractions such as:

- $\mathrm{L}\mathscr{D}\mathscr{A} = \mathscr{D}_l\mathscr{A}[\Sigma_l^{-1}]$ and $\mathrm{R}\mathscr{D}\mathscr{A} = \mathscr{D}_r\mathscr{A}[\Sigma_r^{-1}]$,
- $\mathscr{D}_{ls}\mathscr{A} = \mathscr{D}\mathscr{A}[\Sigma_l^{-1}]$ and $\mathscr{D}_{rs}\mathscr{A} = \mathscr{D}\mathscr{A}[\Sigma_r^{-1}]$, and
- $\mathscr{D}_s\mathscr{A} = \mathscr{D}\mathscr{A}[\Sigma^{-1}]$.

The following result is now automatic.

6.2. Theorem. *For a closed monoidal $\mathscr{V}$-category $\mathscr{A}$, there are equivalences of $\mathscr{V}$-categories:*

- $[\mathrm{L}\mathscr{D}\mathscr{A}, \mathscr{V}] \simeq \mathrm{LTamb}_s(\mathscr{A}) \simeq [\mathscr{A}, \mathscr{V}]$,
- $[\mathscr{D}_{ls}\mathscr{A}, \mathscr{V}] \simeq \mathrm{Tamb}_{ls}(\mathscr{A}) \simeq \mathscr{Z}_l[\mathscr{A}, \mathscr{V}]$, *and*
- $[\mathscr{D}_s\mathscr{A}, \mathscr{V}] \simeq \mathrm{Tamb}_s(\mathscr{A}) \simeq \mathscr{Z}[\mathscr{A}, \mathscr{V}]$.

The first equivalence of Theorem 6.2 implies that $\mathrm{L}\mathscr{D}\mathscr{A}$ and $\mathscr{A}$ are Morita equivalent. This begs the question of whether there is a $\mathscr{V}$-functor relating them more directly. Indeed there is. We have a $\mathscr{V}$-functor

$$\Pi : \mathscr{D}_l\mathscr{A} \longrightarrow \mathscr{A}$$

defined on objects by $\Pi(X, Y) = Y^X$ and by defining the effect

$$\Pi : \mathscr{D}_l\mathscr{A}((X, Y), (U, V)) \longrightarrow \mathscr{A}(Y^X, V^U)$$

on hom objects to have its composite with the A-coprojection equal to the composite

$$\begin{aligned}
&\mathscr{A}(U, A \otimes X) \otimes \mathscr{A}(A \otimes Y, V) \\
&\qquad \xrightarrow{V^{(-)}\otimes(-)^{A\otimes X}} \mathscr{A}(V^{A\otimes X}, V^U) \otimes \mathscr{A}((A \otimes Y)^{A\otimes X}, V^{A\otimes X}) \\
&\qquad \xrightarrow{\text{composition}} \mathscr{A}((A \otimes Y)^{A\otimes X}, V^U) \\
&\qquad \xrightarrow{\mathscr{A}((d_r)^X, V^U)} \mathscr{A}(Y^X, V^U)\ .
\end{aligned}$$

It is easy to see that Π takes the morphisms $\tilde{\beta}_l : (X, Y^A) \longrightarrow (A \otimes X, Y)$ to isomorphisms. So Π induces a $\mathscr{V}$-functor

$$\hat{\Pi} : \mathrm{L}\mathscr{D}_l\mathscr{A} \longrightarrow \mathscr{A};$$

this induces the first equivalence of Theorem 6.2.

For closed monoidal $\mathscr{A}$, the second and third equivalences of Theorem 6.2 show that both the lax centre and the centre of the convolution monoidal $\mathscr{V}$-category $[\mathscr{A}, \mathscr{V}]$ are again functor $\mathscr{V}$-categories $[\mathscr{D}_{ls}\mathscr{A}, \mathscr{V}]$ and $[\mathscr{D}_s\mathscr{A}, \mathscr{V}]$. Since $\mathcal{Z}_l[\mathscr{A}, \mathscr{V}]$ and $\mathcal{Z}[\mathscr{A}, \mathscr{V}]$ are monoidal with the tensor products colimit preserving in each variable, using the correspondence in [Day70], there are lax braided and braided promonoidal structures on $\mathscr{D}_{ls}\mathscr{A}$ and $\mathscr{D}_s\mathscr{A}$ which are such that $[\mathscr{D}_{ls}\mathscr{A}, \mathscr{V}]$ and $[\mathscr{D}_s\mathscr{A}, \mathscr{V}]$ become closed monoidal under convolution, and the equivalences of Theorem 6.2 become lax braided and braided monoidal equivalences.

6.3. Remark.

- We are grateful to Brian Day for pointing out that the $\mathscr{V}$-category $\mathscr{A}_M$ appearing in [DS07] is equivalent to the full sub-$\mathscr{V}$-category of $\mathscr{D}\mathscr{A}$ consisting of the objects of the form (I, Y).
- He also pointed out that a consequence of Theorem 6.2 is that the centre of $\mathscr{V}$ as a $\mathscr{V}$-category is equivalent to $\mathscr{V}$ itself. This also can be seen directly by using the $\mathscr{V}$-naturality in X of the centre structure $u_X : A \otimes X \longrightarrow X \otimes A$ on an object A of $\mathscr{V}$, and the fact that $u_I = 1$, to deduce that $u_X = c_{A,X}$ (the symmetry of $\mathscr{V}$). Generally, the centre of $\mathscr{V}$ as a monoidal **Set**-category is not equivalent to $\mathscr{V}$.

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