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LIMIT PRESERVING FULL EMBEDDINGS

Dedicated to Professor Walter Tholen on his 60th birthday

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Abstract. We prove that every small strongly connected category k has a full embedding preserving all limits existing in k into a category of unary universal algebras. The number of unary operations can be restricted to $|\operatorname{mor} k|$ in case when k has a terminal object and only preservation of limits over finitely many objects is desired. And all limits existing in such a category k are preserved by a full embedding of k into the category of all algebraic systems with $|\operatorname{mor} k|$ unary operation and one unary relation.

1. Introduction

Let Set be the category of all sets and mappings and let k be a small category. The well-known *Eilenberg-Mac Lane representation* $M : k \to \mathrm{Set}$ of [3], and the *Yoneda embedding* $Y : k \to \mathrm{Set}^{k^{\mathrm{op}}}$ of k into the category $\mathrm{Set}^{k^{\mathrm{op}}}$ of all contravariant functors from k to Set and all their natural transformations are faithful functors with somewhat opposite properties. For any $a \in \operatorname{obj} k$, the representing object Ma is a model of simplicity – merely a set of moderate size, while the object Ya is a many-sorted algebra in the sense of [1] with $|\operatorname{obj} k|$ many sorts and $|\operatorname{mor} k|$ many (heterogeneous) operations. On the other hand, the functor Y is nicer because it is full and preserves all limits existing in k, while M has neither of these two properties.

This paper investigates the existence of full and faithful functors $\Phi : k \to \mathcal{K}$ for categories $\mathcal{K}$ whose objects have considerably simpler structure than the objects of $\mathrm{Set}^{k^{\mathrm{op}}}$, namely categories of (mono-sorted) universal algebras or algebraic systems (cf. [2, 6, 9]) that preserve at least some limits existing in k. Such functors thus occupy an 'intermediate ground' between M and Y.

Since for any category $\mathcal{K}$ of algebraic systems the forgetful functor $V : \mathcal{K} \to \mathrm{Set}$ preserves limits, the existence of a limit preserving faithful representation $\Phi : k \to \mathcal{K}$ implies that the faithful composite functor $V \circ \Phi : k \to \mathrm{Set}$ must preserve all limits existing

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in k. But we show this to be impossible for the small category k given in Example 2.1 in Section 2.

A category k is *connected* if any pair of k-objects belongs to the transitive closure of the binary relation R on $\operatorname{obj} k$ given by aRb iff $k(a,b) \cup k(b,a) \neq \emptyset$. And k is *strongly connected* if every hom-set $k(a,b)$ with $a,b \in \operatorname{obj} k$ is non-void.

For any cardinal λ, let $\mathrm{Alg}(1 \times \lambda)$ denote the category of all universal algebras with λ unary operations.

Theorem 2.2 of Section 2 states that every strongly connected small category k has a full embedding into some $\mathrm{Alg}(1 \times \lambda)$ that preserves all limits existing in k. The number λ of unary operations is quite large, but in Section 3 it is reduced in case when the strongly connected category k has a terminal object and only preservation of limits of diagrams over finitely many objects is required. Preservation of all limits is restored and the reduced number of unary operations retained when representing unary algebras are enriched by one unary relation, and this is done in Section 4. The final Section 5 applies these results to countable categories and concludes with some open problems.

2. A general representation result

2.1. Example. There exists a connected finite category k for which no faithful functor $F : k \to \mathrm{Set}$ can preserve all (finite) products existing in k.

Proof. The category k has the object set

$$\operatorname{obj} k = \{q, a, p, a_1, a_2, a_3\},$$

and all the non-void hom-sets of k are as follows:

(1) $k(p, a_i) = \{\pi_i\}$ for $i = 1, 2, 3$,

(2) $k(q, r) = \{\mu_r\}$ for all $r \in \operatorname{obj} k \setminus \{q\}$,

(3) $k(q, q) = \{1_q, \epsilon\}$ with $\epsilon^2 = 1_q$ and $k(r, r) = \{1_r\}$ for all $r \in \operatorname{obj} k \setminus \{q\}$,

(4) $k(a, a_1) = \{\phi, \psi\}$,

With the obvious composition, it is clear that k is a category.

Since there are no other k-morphisms than those listed above, from (1) and (2) it follows that $(p, \{\pi_1, \pi_2, \pi_3\}) = \prod\{a_1, a_2, a_3\}$ and $(p, \{\pi_2, \pi_3\}) = \prod\{a_2, a_3\}$.

Let $F : k \to \mathrm{Set}$ be a faithful functor. Then $|Fq| \geq 2$ because of (3) and hence $|Fr| \geq 1$ for all $r \in \operatorname{obj} k \setminus \{q\}$ because of (2), so that $Fs \neq \emptyset$ for every $s \in \operatorname{obj} k$. If F preserves finite products, then the Cartesian product $Fa_1 \times Fa_2 \times Fa_3$ of these sets must be isomorphic to $Fa_2 \times Fa_3$. Since the sets Fa_i are non-void, this is possible only when Fa_1 is a singleton. And since F is faithful, this contradicts (4). ∎

DEFINITION For any strongly connected small category k, let a functor $G : k \to \mathrm{Set}$ be defined by

$$Ga = \prod\{k(c,a) \mid c \in \operatorname{obj} k\} \text{ for every } a \in \operatorname{obj} k,$$

and for any k-morphism $p : a \to b$ by

$$Gp(x) = \{p \circ x_c \mid c \in \operatorname{obj} k\} \text{ for every } x = \{x_c \in k(c,a) \mid c \in \operatorname{obj} k\} \in Ga.$$

For any $x = \{x_c \mid c \in \operatorname{obj} k\} \in Ga$ we write $p \circ x = \{p \circ x_c \mid c \in \operatorname{obj} k\}$, so that $(p \circ x)_c = p \circ x_c$. We call $x_c \in k(c,a)$ the *c-component* of $x = \{x_c \mid c \in \operatorname{obj} k\} \in Ga$.

The functor G is obviously well-defined and faithful and, as a product of hom-functors, it preserves all limits existing in k.

2.2. THEOREM. *For any strongly connected small category k, there is a cardinal λ and a full embedding $\Psi : k \to \mathrm{Alg}(1 \times \lambda)$ that preserves all limits existing in k.*

PROOF. The proposed full embedding

$$\Psi : k \to \mathrm{Alg}(1 \times \lambda)$$

is carried by the product $G = \prod\{k(c, -) \mid c \in \operatorname{obj} k\}$ of hom-functors just defined. The actual algebra Ψa has the form

$$\Psi a = (Ga\,; \{\omega_z^a \mid z \in Gc \text{ and } c \in \operatorname{obj} k\}),$$

where each operation $\omega_z^a : Ga \to Ga$ is defined for $x = \{x_c \mid c \in \operatorname{obj} k\} \in Ga$ by

$$(\omega_z^a(x))_d = x_c \circ z_d \text{ for every } z \in Gc \text{ and } d \in \operatorname{obj} k.$$

This makes sense because components z_d of $z \in Gc$ belong to $k(d,c)$ and $x_c \in k(c,a)$, so that $\omega_z^a(x) \in Ga$ again.

It is clear that Ψ is a well-defined one-to-one functor that preserves all limits. We outline only a proof that Ψ is full. So let

$$g : (Ga\,; \{\omega_z^a \mid z \in Gc \text{ and } c \in \operatorname{obj} k\}) \to (Gb\,; \{\omega_z^b \mid z \in Gc \text{ and } c \in \operatorname{obj} k\})$$

be a homomorphism of these algebras. Choose an $x \in Ga$ with $x_a = 1_a$ and denote $p = (g(x))_a$, so that $p \in k(a,b)$. We aim to show that $g = Gp\,[= \Psi p]$. We have $(\omega_x^a(x))_d = x_a \circ x_d = 1_a \circ x_d = x_d$ for every $d \in \operatorname{obj} k$. Thus $\omega_x^a(x) = x$, that is, $x \in Ga$ is a fixpoint of the operation ω_x^a. Since g is a homomorphism, the element $g(x) \in Gb$ is a fixpoint of the operation ω_x^b, that is, $(\omega_x^b(g(x)))_d = (g(x))_d$ for every $d \in \operatorname{obj} k$. But $(\omega_x^b(g(x)))_d = (g(x))_a \circ x_d = p \circ x_d$, and hence $(g(x))_d = p \circ x_d$ for every $d \in \operatorname{obj} k$. Now let $y \in Ga$ be arbitrary. We have $(\omega_y^a(x))_d = x_a \circ y_d = y_d$ for every $d \in \operatorname{obj} k$, that is, $\omega_y^a(x) = y$. But then $g(y) = g(\omega_y^a(x)) = \omega_y^b(g(x))$. Hence $(g(y))_d = (\omega_y^b(g(x)))_d = (g(x))_a \circ y_d = p \circ y_d$ for every $d \in \operatorname{obj} k$, that is, $g(y) = (Gp)(y)$. Therefore $g = Gp$ as claimed. ∎

Although the number of needed operations is finite for any finite k, to represent a given infinite strongly connected category k with $|\operatorname{mor} k| = \kappa$ the proof of Theorem 2.2 may need $\lambda = \kappa^\kappa$ unary operations. As will be seen below, this number can be reduced back to κ when some additional assumptions are made.

3. Limits of diagrams over finitely many objects

In this section we prove the following result.

3.1. THEOREM. *For every strongly connected small category k with a terminal object and infinite $|\operatorname{mor} k| = \kappa$, there is a full embedding $\Phi_0 : k \to \mathrm{Alg}(1 \times \kappa)$ preserving all existing limits of diagrams over finitely many objects. For every $a \in \operatorname{obj} k$, the algebra $\Phi_0 a \in \mathrm{Alg}(1 \times \kappa)$ is idempotent.*

Theorem 3.1 appears weaker than Theorem 2.2 because it requires that k have a terminal object and does not assert the preservation of all limits. The advantage of Theorem 3.1, however, is that it represents any small category k with infinite $\kappa = |\operatorname{mor} k|$ by algebras with only κ unary operations while Theorem 2.2 requires

$$\lambda = \sum \Big\{ \Big\{ \prod |k(d,c)| \mid d \in \operatorname{obj} k \Big\} \mid c \in \operatorname{obj} k \Big\}$$

operations; this number λ can be uncountable even when $|\operatorname{mor} k|$ is countable. Theorem 3.1, on the other hand, enables us to similarly represent any strongly connected countable category k with a terminal object in the category $\mathrm{Alg}(1 \times 2)$. The latter result is the best possible because the number of operations cannot be further reduced. We prove it in Section 5.

The functor Φ_0 will be carried by a functor $F : k \to \mathrm{Set}$ we now describe.

DEFINITION Let t denote a terminal object of k and $k(c,t) = \{\nabla_c\}$ for every $c \in \operatorname{obj} k$. The functor $F : k \to \mathrm{Set}$ is a subfunctor of the functor G from Section 2. Specifically, for any $a \in \operatorname{obj} k$ we let

$$Fa \subseteq \prod \{k(c,a) \mid c \in \operatorname{obj} k\} = Ga$$

consist of all sequences $x \in \prod\{k(c,a) \mid c \in \operatorname{obj} k\}$ for which there is a $\gamma \in k(t,a)$ such that $x_c = \gamma \circ \nabla_c$ for all but finitely many $c \in \operatorname{obj} k$. For any $p \in k(a,b)$, the map $Fp : Fa \to Fb$ is given by

$$(Fp)(x) = p \circ x \text{ for every } x \in Fa.$$

For any $x \in Fa$, the element $p \circ x \in Gb$ belongs to Fb because $(p \circ x)_c = p \circ x_c = p \circ \gamma \circ \nabla_c$ for all but finitely many $c \in \operatorname{obj} k$ and $p \circ \gamma \in k(t,b)$. Therefore F is a well-defined functor. Choosing any $x \in Fa$ with $x_a = 1_a$, for any two distinct $p, q \in k(a,b)$ we get $(Fp)(x)_a = p \neq q = (Fq)(x)_a$ and hence $Fp \neq Fq$. The functor F is therefore faithful.

Next we show that F has the required preservation property.

3.2. Proposition. *The functor $F : k \to \mathrm{Set}$ preserves existing limits of diagrams over finitely many objects.*

Proof. Let $D : S \to k$ be a diagram with a finite $\mathrm{obj}\, S$ for which $\lim D$ exists in k. Denote $\lim D = (l, \{\lambda_s \mid s \in \mathrm{obj}\, S\})$, with the morphisms $\lambda_s \in k(l, Ds)$ forming a limit cone.

Recall that F is a subfunctor of G such that every Fa with $a \in \mathrm{obj}\, k$ consists of all those $x \in Ga = \prod\{k(c, a) \mid c \in \mathrm{obj}\, k\}$ for which there is a $\gamma \in k(t, a)$ and a finite $A \subseteq \mathrm{obj}\, k$ such that $x_c = \gamma \circ \nabla_c$ for all $c \in \mathrm{obj}\, k \setminus A$, and that $Gp(x) = p \circ x$ for any $p \in k(a, b)$ and every $x \in Ga$.

The concrete form of limits in Set implies that we need to show that for every system $\{x^{(s)} \in FDs \mid s \in \mathrm{obj}\, S\}$ satisfying $FD\sigma(x^{(s)}) = x^{(s')}$ for every S-morphism $\sigma : s \to s'$ there is a unique $x \in Fl$ such that $\lambda_s \circ x = F\lambda_s(x) = x^{(s)}$.

So let $\{x^{(s)} \in FDs \mid s \in \mathrm{obj}\, S\}$ be such a system. For every $c \in \mathrm{obj}\, k$ and every S-morphism $\sigma : s \to s'$ we then have $D\sigma \circ x_c^{(s)} = (D\sigma \circ x^{(s)})_c = FD\sigma(x^{(s)})_c = x_c^{(s')}$, so that $\{x_c^{(s)} \in k(c, Ds) \mid s \in \mathrm{obj}\, S\}$ is a cone over D in k. Since $\lim D = (l, \{\lambda_s \mid s \in \mathrm{obj}\, S\})$ in k, for each $c \in \mathrm{obj}\, k$ there is a unique $x_c \in k(c, l)$ such that $x_c^{(s)} = \lambda_s \circ x_c$. Thus $x = \{x_c \mid c \in \mathrm{obj}\, k\} \in Gl$ and $G\lambda_s(x) = \lambda_s \circ x = x^{(s)}$ for every $s \in \mathrm{obj}\, S$. Since $\{G\lambda_s \mid s \in \mathrm{obj}\, S\}$ is a limit cone, the element $x \in Gl$ is the only one for which $\lambda_s \circ x = x^{(s)}$ for all $s \in \mathrm{obj}\, S$. It thus remains to show that $x \in Fl$.

Since $x^{(s)} \in FDs$ for $s \in \mathrm{obj}\, S$, we have a finite set $A_s \subseteq \mathrm{obj}\, k$ and $\gamma_s \in k(t, Ds)$ such that $x_c^{(s)} = \gamma_s \circ \nabla_c$ for all $c \in \mathrm{obj}\, k \setminus A_s$. Then the set $A = \bigcup\{A_s \mid s \in \mathrm{obj}\, S\}$ is finite because $\mathrm{obj}\, S$ is finite, and $x_c^{(s)} = \gamma_s \circ \nabla_c$ for every $s \in \mathrm{obj}\, S$ and all $c \in \mathrm{obj}\, k \setminus A$.

Also, if $c \in \mathrm{obj}\, k \setminus A$ then from $D\sigma \circ \lambda_s = \lambda_{s'}$ for every S-morphism $\sigma : s \to s'$ it follows that $\gamma_{s'} \circ \nabla_c = \lambda_{s'} \circ x_c = D\sigma \circ \lambda_s \circ x_c = D\sigma \circ \gamma_s \circ \nabla_c$. Since k is strongly connected, each terminal morphism ∇_c is an epi, and we obtain $\gamma_{s'} = D\sigma \circ \gamma_s$ for every $\sigma : s \to s'$ in S. This means that $(t, \{\gamma_s \mid s \in \mathrm{obj}\, S\})$ is a cone over D in k, and hence there is a unique $\gamma \in k(t, l)$ such that $\lambda_s \circ \gamma = \gamma_s$ for every $s \in \mathrm{obj}\, S$.

Altogether, for every $c \in \mathrm{obj}\, k \setminus A$ and every $s \in \mathrm{obj}\, S$ we have $\lambda_s \circ x_c = x_c^{(s)} = \gamma_s \circ \nabla_c = \lambda_s \circ \gamma \circ \nabla_c$. Since $\{\lambda_s \mid s \in \mathrm{obj}\, S\}$ is a limit cone, it follows that $x_c = \gamma \circ \nabla_c$ for every $c \in \mathrm{obj}\, k \setminus A$. Since A is finite and $\gamma \in k(t, l)$, this means that $x \in Fl$. ∎

3.3. Remark. Adding a terminal object to the finite category k of Example 2.1 clearly produces a finite category l that is not strongly connected and has a terminal object, for which no faithful functor $H : l \to \mathrm{Set}$ preserving (finite) products exists. This naturally leads to a question of characterization of those small categories k with a terminal object for which there is a faithful functor $H : k \to \mathrm{Set}$ that preserves all limits existing in k.

The following claim proves the essential part of Theorem 3.1.

3.4. Lemma. *Any small strongly connected category k with a terminal object t has a full embedding preserving all its existing limits over finitely many objects into the category*

$\mathrm{Alg}(1 \times \kappa)$ *of multiunary algebras, where*

$$\kappa = |\operatorname{mor} k| + \left|\bigcup\{k(t,c) \mid c \in \operatorname{obj} k\}\right|.$$

It is clear that $\kappa = |\operatorname{mor} k|$ in case when $|\operatorname{mor} k|$ is infinite.

We prove Lemma 3.4 in 3.5–3.11 below.

3.5. Since a full embedding $\Phi_0 : k \to \mathrm{Alg}(1 \times \kappa)$ will be carried by the functor $F : k \to \mathrm{Set}$, it is clear that Φ_0 is going to be faithful.

In 3.6 and 3.7, on each set Fa with $a \in \operatorname{obj} k$ we define a multiunary algebra $\Phi_0 a$.

3.6. First, for any $\nu \in k(c, c')$, we define a unary operation $o^a_\nu : Fa \to Fa$ so that for any $x = \{x_d \in k(d,a) \mid d \in \operatorname{obj} k\} \in Fa$, the element $o^a_\nu(x) = x' = \{x'_d \in k(d,a) \mid d \in \operatorname{obj} k\}$ has the components

$$x'_d = \begin{cases} x_{c'} \circ \nu & \text{for } d = c, \\ x_d & \text{for } d \in (\operatorname{obj} k) \setminus \{c\}. \end{cases}$$

3.7. Second, for any $c \in \operatorname{obj} k$ and any $\gamma \in k(t,c)$, we define a unary operation $\omega^a_{c,\gamma}$ on each Fa. For any $x = \{x_d \in k(d,a) \mid d \in \operatorname{obj} k\} \in Fa$, $\omega^a_{c,\gamma}(x) = x' = \{x'_d \in k(d,a) \mid d \in \operatorname{obj} k\}$ will have the components

$$x'_d = \begin{cases} x_c & \text{for } d = c, \\ x_c \circ \gamma \circ \nabla_d & \text{for } d \in (\operatorname{obj} k) \setminus \{c\}. \end{cases}$$

The operation $\omega^a_{c,\gamma}$ thus leaves the c-component x_c of x unchanged while acting as the 'right translation' of x_c by the appropriate morphism $\gamma \circ \nabla_d$ for every other component of x.

To see that Fp with $p \in k(a,b)$ preserves any such operation, note that

$$((Fp)(\omega^a_{c,\gamma}(x)))_d = p \circ (\omega^a_{c,\gamma}(x))_d = \begin{cases} p \circ x_c & \text{if } d = c, \\ p \circ (x_c \circ \gamma \circ \nabla_d) & \text{if } d \neq c \end{cases}$$

and

$$(\omega^b_{c,\gamma}(Fp)(x))_d = \begin{cases} ((Fp)(x))_c & \text{if } d = c, \\ ((Fp)(x))_c \circ \gamma \circ \nabla_d & \text{if } d \neq c. \end{cases}$$

From $((Fp)(x))_c = p \circ x_c$ it then follows that $Fp \circ \omega^a_{c,\gamma} = \omega^b_{c,\gamma} \circ Fp$. Therefore all defined operations are preserved. Consequently, setting

$$\begin{aligned} \Phi_0 a &= (Fa; \{o^a_\nu \mid \nu \in \operatorname{mor} k\} \cup \{\omega^a_{c,\gamma} \mid c \in \operatorname{obj} k \text{ and } \gamma \in k(t,c)\}) \text{ for every } a \in \operatorname{obj} k \text{ and} \\ \Phi_0 p &= Fp \text{ for every } p \in k(a,b) \end{aligned}$$

produces a well-defined functor $\Phi_0 : k \to \mathrm{Alg}(1 \times \kappa)$.

3.8. The functor $\Phi_0 : k \to \mathrm{Alg}(1\times\kappa)$ preserves all limits of diagrams with finitely many objects existing in k because $F : k \to \mathrm{Set}$ preserves them, see Proposition 3.2.

3.9. To prove the fullness of Φ_0, let $g : \Phi_0 a \to \Phi_0 b$ be a homomorphism of these algebras. We select and fix some $\alpha \in k(t,a)$, and use it to define a particular (a,α)-*discrete* element $z \in Fa$ whose components are given as

$$z_c = \begin{cases} 1_a & \text{if } c = a, \\ \alpha \circ \nabla_c & \text{if } c \in (\mathrm{obj}\, k) \setminus \{a\}. \end{cases}$$

We denote

$$p = g(z)_a. \tag{1}$$

Then $p \in k(a,b)$, and we aim to show that $g = Fp$. In order to do this, we first describe the specific element $g(z) \in Fb$. Noting that $\omega^a_{a,\alpha}(z)_a = z_a = 1_a$ and $\omega^a_{a,\alpha}(z)_c = z_a \circ \alpha \circ \nabla_c = \alpha \circ \nabla_c$ for any $c \neq a$, we conclude that $\omega^a_{a,\alpha}(z) = z$ and, since g preserves the operation $\omega_{a,\alpha}$ we also get $\omega^b_{a,\alpha}(g(z)) = g(z)$. Thus, first of all, $\omega^b_{a,\alpha}(g(z))_a = g(z)_a = p$ and, for any k-object $c \neq a$ we have $\omega^b_{a,\alpha}(g(z))_c = g(z)_c = g(z)_a \circ \alpha \circ \nabla_c = p \circ \alpha \circ \nabla_c$ with $p \circ \alpha \in k(t,b)$. Altogether, this means that $g(z) = (Fp)(z)$ for the (a,α)-discrete element z defined in (z) above with the use of the one specified morphism $\alpha \in k(t,a)$.

3.10. Next we show that any other $\alpha' \in k(t,a)$ determines the same element $p \in k(a,b)$ as that given in (1), so that for any (a,α')-discrete element $z' \in Fa$ given by its c-components

$$z'_c = \begin{cases} 1_a & \text{if } c = a, \\ \alpha' \circ \nabla_c & \text{if } c \in (\mathrm{obj}\, k) \setminus \{a\}, \end{cases}$$

we also have $g(z') = (Fp)(z')$. We simply observe that $\omega^a_{a,\alpha'}(z) = z'$, from which it follows that $\omega^b_{a,\alpha'}(g(z)) = g(\omega^a_{a,\alpha'}(z)) = g(z')$, and hence $p = g(z)_a = (\omega^b_{a,\alpha'}(g(z)))_a = g(z')_a$. And for $c \neq a$, we have $g(z')_c = \omega^b_{a,\alpha'}(g(z))_c = g(z)_a \circ \alpha' \circ \nabla_c = p \circ \alpha' \circ \nabla_c$. Altogether, there is a unique $p \in k(a,b)$ such that $g(z) = (Fp)(z)$ for *every* (a,α)-discrete element $z \in Fa$ with $\alpha \in k(t,a)$.

3.11. However, we need to show that $g(y) = (Fp)(y)$ for every $y \in Fa$. Recall that for any $y \in Fa$, there exists some $\gamma \in k(t,a)$ such that all but finitely many components of y have the form $y_d = \gamma \circ \nabla_d$ with $d \in \mathrm{obj}\, k$. An inductive argument using the operations o_μ with $\mu \in \mathrm{mor}\, k$ will complete the proof of this fact. To this end, let $S \subseteq \mathrm{obj}\, k$ be a finite set such that $y_d = \gamma \circ \nabla_d$ for every $d \in (\mathrm{obj}\, k) \setminus S$. Explicitly then,

$$y_d = \begin{cases} \text{some } \mu_d \in k(d,a) & \text{if } d \in S, \\ \gamma \circ \nabla_d & \text{if } d \in (\mathrm{obj}\, k) \setminus S. \end{cases}$$

If $a \notin S$, then also $y_d = \gamma \circ \nabla_d$ for every $d \in (\mathrm{obj}\, k) \setminus S'$ for the larger set $S' = S \cup \{a\}$, so that we may just as well assume that $a \in S$.

So let $a \in S$. Order the finite set S linearly in such a way that a is its last element. We proceed inductively along the order $c_1 < c_2 \ldots < c_m < a$ of $S = \{c_1, \ldots, c_m, a\}$ as follows.

Let $c_1 \in S \setminus \{a\}$, and denote $\mu_1 = y_{c_1} \in k(c_1, a)$. Let $z \in Fa$ be the (a,γ)-discrete element, and denote $z_1 = o^a_{\mu_1}(z)$. Thus

$$(z_1)_d = \begin{cases} 1_a & \text{if } d = a, \\ \mu_1 & \text{if } d = c_1, \\ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{a, c_1\}. \end{cases} \tag{2}$$

Since $g(z_1) = g(o^a_{\mu_1}(z)) = o^b_{\mu_1}(g(z))$ and because, for every $w = \{w_d \mid d \in \operatorname{obj} k\}$, the operation $o^b_{\mu_1}$ with $\mu_1 \in k(c_1, a)$ is given by

$$o^b_{\mu_1}(w)_d = \begin{cases} w_a \circ \mu_1 & \text{if } d = c_1, \\ w_d & \text{if } d \in (\operatorname{obj} k) \setminus \{c_1\}, \end{cases}$$

for $w = g(z_1)$ we obtain

$$g(z_1)_d = o^b_{\mu_1}(g(z))_d = \begin{cases} p & \text{if } d = a \\ p \circ \mu_1 & \text{if } d = c_1, \\ p \circ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{a, c_1\}. \end{cases}$$

Therefore $(z_1)_d = y_d$ and $g(z_1)_d = (Fp)(z_1)_d = p \circ y_d$ for every $d \in \{c_1\} \cup ((\operatorname{obj} k) \setminus S)$. This constitutes the initial induction step. Next suppose that $c_2 \in S \setminus \{a, c_1\}$ exists. Define $z_2 = o^a_{\mu_2}(z_1)$ with $\mu_2 = y_{c_2} \in k(c_2, a)$; thus

$$(z_2)_d = \begin{cases} 1_a & \text{if } d = a, \\ \mu_i & \text{if } d = c_i \text{ for some } i \in \{1, 2\}, \\ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{a, c_1, c_2\}. \end{cases} \tag{3}$$

Using the operation o_{μ_2}, we now similarly obtain

$$g(z_2)_d = \begin{cases} p & \text{if } d = a \\ p \circ \mu_i & \text{if } d = c_i \text{ for some } i \in \{1, 2\}, \\ p \circ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{a, c_1, c_2\}. \end{cases}$$

Thus $(z_2)_d = y_d$ and $p \circ y_d = (Fp)(z_2)_d = g(z_2)_d$ for every $d \in \{c_1, c_2\} \cup ((\operatorname{obj} k) \setminus S)$. Continuing inductively along the order of $S \setminus \{a\} = \{c_1, \dots, c_m\}$, we conclude that for the element $z_m = (o^a_{\mu_m} \circ \dots \circ o^a_{\mu_1})(z)$ we have $(z_m)_d = y_d$ and $g(z_m)_d = (Fp)(y)_d$ for every $d \in \{c_1, c_2, \dots c_m\} \cup ((\operatorname{obj} k) \setminus S) = (\operatorname{obj} k) \setminus \{a\}$.

In the final step, we apply the operation o_μ with $\mu = y_a \in k(a, a)$ to the element $z_m = (o^a_{\mu_m} \circ \dots \circ o^a_{\mu_1})(z)$ (or to the element $z_0 = z$ in the case of $S = \{a\}$) in the same argument, to conclude that $g(y) = (Fp)(y)$. Since this holds true for any given $y \in Fa$, we have $g = \Phi_0 p$ for the k-morphism $p : a \to b$ defined by (1) – as was to be shown.

This completes the proof of Lemma 3.4.

3.12. Observation. All operations of any algebra $\Phi_0 a$ with $a \in \operatorname{obj} k$ are idempotent.

PROOF. We consider the operations $o_\nu^a : Fa \to Fa$ first. Recall that for a given $\nu \in k(c, c')$ and for any $x = \{x_d \in k(d, a) \mid d \in \operatorname{obj} k\}$, the element $x' = o_\nu^a(x)$ has the components

$$x'_d = o_\nu^a(x)_d = \begin{cases} x_{c'} \circ \nu & \text{for } d = c, \\ x_d & \text{for } d \neq c. \end{cases}$$

Denote $x'' = o_\nu^a(x')$. We then have $x''_c = o_\nu^a(x')_c = x'_{c'} \circ \nu$. If $c' \neq c$ then $x'_{c'} = x_{c'}$ and hence $x''_c = x_{c'} \circ \nu = x'_c$ while for $c = c'$ we have $x''_c = x'_{c'} \circ \nu = x'_c$ again. For $d \neq c$, we have $x''_d = x'_d = x_d$. Altogether $x'' = x'$ and hence o_ν^a is idempotent.

Recall that the remaining operations $\omega_{c,\gamma}^a : Fa \to Fa$ of Φa with $c \in \operatorname{obj} k$ and $\gamma \in k(t, c)$ are given by

$$x'_d = \omega_{c,\gamma}^a(x)_d = \begin{cases} x_c & \text{if } d = c, \\ x_c \circ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{c\}. \end{cases}$$

Denote $x'' = \omega_{c,\gamma}^a(x')$. Obviously $x''_c = x'_c = x_c$ and for $d \neq c$ we have $x''_d = x'_c \circ \gamma \circ \nabla_d$ and $x'_d = x_c \circ \gamma \circ \nabla_d$ and hence $x''_d = x'_d$ follows from the equality $x'_c = x_c$ shown just above. Therefore $x'' = x'$, and hence $\omega_{c,\gamma}^a$ is also idempotent. ■

The proof of Theorem 3.1 is now complete.

A weaker form of Theorem 3.1 dealing only with finite products was proved and used already in [13].

4. Representations by algebraic systems

In this section we construct a limit preserving representation of any infinite strongly connected small category k with a terminal object into the category $\mathcal{S}(1 \times \kappa\,; 1)$ of all algebraic systems with $\kappa = |\operatorname{mor} k|$ unary operations and one unary relation. Adding the unary relation thus improves both Theorem 2.2 and Theorem 3.1 for infinite strongly connected small categories with a terminal object.

4.1. LEMMA. *Let k be a small strongly connected category with a terminal object. Then k can be fully embedded with the preservation of all its existing limits into the category $\mathcal{S}$ of algebraic systems having as many unary operations o_ν as k has morphisms, one unary relation U, and one α-ary relation R for which the cardinal α is one more than the cardinality of the set of objects of k.*

Lemma 4.1 will be proved in 4.2–4.11 below.

4.2. To define a limit preserving functor $\Phi : k \to \mathcal{S}$, we use the product $G : k \to \mathrm{Set}$ of hom-functors from Section 2.

4.3. On each $Ga = \prod\{k(c, a) \mid c \in \operatorname{obj} k\}$, we define the structure of an $\mathcal{S}$-object

$$\Phi a = (Ga; U_a, R_a, \{o_\nu^a \mid \nu \in \operatorname{mor} k\})$$

having a unary relation $U_a \subseteq Ga$, α-ary relation $R_a \subseteq (Ga)^\alpha$ and unary operations $o_\nu^a : Ga \to Ga$ for $\nu \in \operatorname{mor} k$ defined in 4.4–4.6 below.

4.4. First we define the unary relation U_a on Ga. To do this, for any $c \in \operatorname{obj} k$ we let $\nabla_c : c \to t$ denote the unique k-morphism to the terminal object t of k and then set

$$U_a = \{\{\gamma \circ \nabla_c \in k(c, a) \mid c \in \operatorname{obj} k\} \mid \gamma \in k(t, a)\}.$$

We show that $(Gp)(U_a) \subseteq U_b$ for any $p \in k(a, b)$. Indeed, for each element $e = \{\gamma \circ \nabla_c \mid c \in \operatorname{obj} k\}$, the element $e' = (Gp)(e)$ has every its c-component of the form $e'_c = p \circ \gamma \circ \nabla_c$ in which $p \circ \gamma \in k(t, b)$, so that $e' \in U_b \subseteq Gb$ as claimed.

4.5. Secondly, for any $\nu \in k(c, c')$, we define a unary operation $o^a_\nu : Ga \to Ga$ by setting, for any $x = \{x_d \in k(d, a) \mid d \in \operatorname{obj} k\} \in Ga$,

$$o^a_\nu(x)_d = \begin{cases} x_d & \text{for } d \neq c, \\ x_{c'} \circ \nu & \text{for } d = c. \end{cases}$$

It is clear that any $Gp : Ga \to Gb$ with $p \in k(a, b)$ preserves these operations.

4.6. Finally we define an α-ary relation R_a on Ga. This α-ary relation R_a is indexed by the disjoint union $\mathcal{G} = \{0\} \cup \operatorname{obj} k$. We set

$$R_a = \{R_{x,\gamma} \mid x \in Ga \text{ and } \gamma \in k(t, a)\},$$

where the entries of each element $R_{x,\gamma} \in R_a \subseteq (Ga)^{\mathcal{G}}$ are $R_{x,\gamma}(0) = x \in Ga$ and, for any $c \in \operatorname{obj} k$ the element $R_{x,\gamma}(c) \in Ga$ is given by

$$R_{x,\gamma}(c)_d = \begin{cases} x_d & \text{if } d = c, \\ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{c\}. \end{cases} \tag{4}$$

To verify that $(Gp)^{\mathcal{G}}(R_a) \subseteq R_b$ for every $p \in k(a, b)$, choose any $R_{x,\gamma} \in R_a$. Then $(Gp)^{\mathcal{G}}(R_{x,\gamma}) \in (Gb)^{\mathcal{G}}$ has the following entries. First, the entry at $0 \in \mathcal{G}$ is $(Gp) \circ (R_{x,\gamma}(0)) = G(p)(x) = \{p \circ x_d \mid d \in \operatorname{obj} k\}$. For any $c \in \operatorname{obj} k$, by (4) we have

$$[(Gp)(R_{x,\gamma}(c))]_d = \begin{cases} p \circ x_c & \text{if } d = c, \\ (p \circ \gamma) \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{c\}. \end{cases}$$

Hence $(Gp)^{\mathcal{G}}(R_{x,\gamma}) \in R_b$, and $(Gp)^{\mathcal{G}}(R_a) \subseteq R_b$ follows.

4.7. From 4.2–4.6 it follows that setting

$$\Phi a = (Ga; U_a, R_a, \{o^a_\nu \mid \nu \in \operatorname{mor} k\}) \text{ for } a \in \operatorname{obj} k \quad \text{and} \quad \Phi p = Gp \text{ for } p \in k(a, b)$$

produces a well-defined one-to-one functor from k to the category $\mathcal{S}$ of all algebraic systems with the unary operations o_ν and the relations U and R.

4.8. To prove that Φ is full, let $g : Ga \to Gb$ be any mapping that preserves all operations and both relations. Select and fix some arbitrary $\gamma \in k(t,a)$. Using this γ, let $z \in Ga$ be the element defined by

$$z_d = \begin{cases} 1_a & \text{for } d = a, \\ \gamma \circ \nabla_d & \text{for } d \in (\operatorname{obj} k) \setminus \{a\}. \end{cases}$$

Then $g(z) \in Gb$ has the component $g(z)_a \in k(a,b)$, and we denote

$$p = g(z)_a.$$

For $\gamma \circ \nabla_a \in k(a,a)$, the mapping g preserves its corresponding operation $o_{\gamma\circ\nabla_a}$, implying that $g(o^a_{\gamma\circ\nabla_a}(z)) = o^b_{\gamma\circ\nabla_a}(g(z))$. From the definition of $o^a_{\gamma\circ\nabla_a}$ and of the element $z \in Ga$ it follows that for every $c \in \operatorname{obj} k$, the element $o^a_{\gamma\circ\nabla_a}(z)$ has the c-component $o^a_{\gamma\circ\nabla_a}(z)_c = \gamma \circ \nabla_c$. Therefore $o^a_{\gamma\circ\nabla_a}(z) \in U_a$ and hence $o^b_{\gamma\circ\nabla_a}(g(z)) = g(o^a_{\gamma\circ\nabla_a}(z)) \in U_b$. Thus there is a $\delta \in k(t,b)$ such that $o^b_{\gamma\circ\nabla_a}(g(z))_c = \delta \circ \nabla_c$ for all $c \in \operatorname{obj} k$. The operation $o^b_{\gamma\circ\nabla_a}$ does not change the d-component of its argument for $d \neq a$, so that $g(z)_d = \delta \circ \nabla_d$ for all $d \neq a$, with the same δ. For the object $a \in \operatorname{obj} k$, the a-component $o^b_{\gamma\circ\nabla_a}(g(z))_a$ of $o^b_{\gamma\circ\nabla_a}(g(z))$ is $g(z)_a \circ \gamma \circ \nabla_a = p \circ \gamma \circ \nabla_a$ and from $o^b_{\gamma\circ\nabla_a}(g(z)) \in U_b$ it follows that $\delta \circ \nabla_c = p \circ \gamma \circ \nabla_c$ for every $c \in \operatorname{obj} k$. Therefore the element $g(z) \in Gb$ has the components

$$g(z)_d = \begin{cases} p \in k(a,b) & \text{if } d = a, \\ p \circ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{a\}. \end{cases} \tag{5}$$

To proceed further, we need the following

4.9. Claim. *Let $c \in \operatorname{obj} k$ and $\mu \in k(c,a)$ be given. Let the element $y = \{y_d \mid d \in \operatorname{obj} k\} \in Ga$ be defined by*

$$y_d = \begin{cases} \mu & \text{if } d = c, \\ \gamma \circ \nabla_d & \text{if } d \in (\operatorname{obj} k) \setminus \{c\}. \end{cases}$$

Then $g(y) = \{p \circ y_d \mid d \in \operatorname{obj} k\}$, that is, $g(y) = (Gp)(y)$ for any such element $y \in Ga$.

Proof. First, let us suppose that $a \neq c$, and let $z \in Ga$ be the element defined in 4.8. Thus the components of $g(z)$ are as in (5) above. Write $z' = o^a_{\gamma\circ\nabla_a}(o^a_\mu(z))$. The definition of operations o_μ and $o_{\gamma\circ\nabla_a}$ implies that $z' = y$. Hence $g(y) = o^b_{\gamma\circ\nabla_a}(o^b_\mu(g(z)))$. Since o^b_μ only replaces the c-component of $g(z)$ by $(g(z))_a \circ \mu = p \circ \mu$ and $o^b_{\gamma\circ\nabla_a}$ only replaces the a-component of $g(z)$ by $(g(z))_a \circ \gamma \circ \nabla_a = p \circ \gamma \circ \nabla_a$, from (5) it follows that $g(y) = \{p \circ y_d \mid d \in \operatorname{obj} k\}$.

Second, if $a = c$, then $y = o^a_\mu(z)$ and the calculation is similar but easier. This concludes the proof of Claim 4.9. ∎

4.10. To complete the proof of fullness of Φ, let $x = \{x_d \in k(d,a) \mid d \in \operatorname{obj} k\} \in Ga$ be arbitrary. Let $R_{x,\gamma} \in R_a \subseteq (Ga)^{\mathcal{G}}$ be the element of R_a with the previously specified $\gamma \in k(t,a)$. Recall that the $\mathcal{G}$-tuple $R_{x,\gamma}$ from R_a has the entry $R_{x,\gamma}(0) = x$ and, for every $c \in \operatorname{obj} k$, the entry $R_{x,\gamma}(c) \in Ga$ is given by its components

$$R_{x,\gamma}(c)_d = \begin{cases} x_d & \text{for } d = c, \\ \gamma \circ \nabla_d & \text{for } d \in (\operatorname{obj} k) \setminus \{c\}, \end{cases}$$

so that each $R_{x,\gamma}(c) \in Ga$ has the form to which Claim 4.9 above applies. Therefore, by Claim 4.9, $g(R_{x,\gamma}(c)) = (Gp)(R_{x,\gamma}(c))$, meaning that $g(R_{x,\gamma}(c))_d = p \circ (R_{x,\gamma}(c))_d$ for every $d \in \operatorname{obj} k$. But $(Gg)^{\mathcal{G}}$ sends R_a into R_b, and this implies that $g(R_{x,\gamma}) = R_{v,\delta} \in R_b$ for some $v \in Gb$ and some $\delta \in k(t,b)$. For any $c \in \operatorname{obj} k$, the element $R_{v,\delta}(c) \in Gb$ has the components

$$R_{v,\delta}(c)_d = \begin{cases} v_c & \text{for } d = c, \\ \delta \circ \nabla_d & \text{for } d \in (\operatorname{obj} k) \setminus \{c\}, \end{cases}$$

and $R_{v,\delta}(0) = v \in Gb$. But $R_{v,\delta}(c)_d = p \circ (R_{x,\gamma}(c))_d$ for all $d \in \operatorname{obj} k$, so that $v_d = p \circ x_d$ for $d = c$, and $\delta \circ \nabla_d = p \circ \gamma \circ \nabla_d$ for all $d \in (\operatorname{obj} k) \setminus \{c\}$. What is then the element $v \in Gb$? The element $v = R_{v,\delta}(0)$ has the d-component $v_d = R_{v,\delta}(d)_d = p \circ x_d \in k(d,b)$. This determines $v \in Gb$ uniquely as the element $v = \{p \circ x_d \in k(d,b) \mid d \in \operatorname{obj} k\}$, and hence implies that $v = (Gp)(x)$. Since $x \in Ga$ was arbitrary, we conclude that $g = Gp$. Altogether, this shows that the functor Φ is full.

Remark. Having concluded the proof of fullness of Φ, we comment that the final part of the argument removes the proof's dependence on the initially selected morphism $\gamma \in k(t,a)$; any other $\gamma' \in k(t,a)$ would serve the purpose equally well. Indeed, if we choose another $\gamma' \in k(t,a)$, if $z' \in Ga$ is the element with $z'_a = 1_a$ and $z'_d = \gamma' \circ \nabla_d$ for $d \neq a$, and if $p' = (g(z'))_a$, then the proof's procedure shows that $g(x) = (Gp')(x)$ for every $x \in Ga$. Choosing an $x \in Ga$ with $x_a = 1_a$ then gives $p' = ((Gp')(x))_a = (g(x))_a = ((Gp)(x))_a = p$.

4.11. To show that Φ preserves all limits existing in k, let $D : S \to k$ be a diagram with a limit $\lim D = (l, \{\lambda_s \in k(l, Ds) \mid s \in \operatorname{obj} S\})$ in k. We claim that $\lim(\Phi \circ D) = (\Phi l, \{\Phi\lambda_s \in \mathcal{S}(\Phi l, \Phi Ds) \mid s \in \operatorname{obj} S\})$ in $\mathcal{S}$. The category $\mathcal{S}$ is complete, so that $\lim(\Phi \circ D) = (L, \{\Lambda_s \in \mathcal{S}(L, \Phi Ds) \mid s \in \operatorname{obj} S\})$ exists in $\mathcal{S}$. Let $h : \Phi l \to L$ denote the unique $\mathcal{S}$-morphism satisfying $\Lambda_s \circ h = \Phi\lambda_s$ for every $s \in \operatorname{obj} S$.

We know that $\lim(G \circ D) = (Gl, \{G\lambda_s : Gl \to GDs \mid s \in \operatorname{obj} S\})$ in Set because G preserves all limits. Let $V : \mathcal{S} \to \mathrm{Set}$ denote the standard forgetful functor. There is a unique mapping $f : VL \to Gl$ such that $G\lambda_s \circ f = V\Lambda_s$ for every $s \in \operatorname{obj} S$. But then $G\lambda_s \circ f \circ Vh = V\Lambda_s \circ Vh = G\lambda_s$ for every $s \in \operatorname{obj} S$, and hence $f \circ Vh = 1_{Gl}$ because the maps $G\lambda_s$ form a limit cone.

The functor $V : \mathcal{S} \to \mathrm{Set}$ preserves all limits, so that $(VL, \{V\Lambda_s : VL \to GDs \mid s \in \operatorname{obj} S\})$ is a limit in Set. But then from $V\Lambda_s \circ Vh \circ f = G\lambda_s \circ f = V\Lambda_s$ for every $s \in \operatorname{obj} S$ it follows that $Vh \circ f = 1_{VL}$. Therefore the mapping f is a bijection, and we only need to show that f carries an $\mathcal{S}$-morphism $L \to l$.

To see that f preserves the unary relation, it suffices to note that $U_l \neq \emptyset$. A similar observation shows that f also preserves the relation R. The unary operations are preserved because any bijection f between two algebras whose inverse h is a homomorphism must be an isomorphism. This completes the proof of Lemma 4.1.

The main result of this section will now easily follow.

4.12. THEOREM. *If k is an infinite strongly connected small category with a terminal object and $|\operatorname{mor} k| = \alpha$, then there is a full embedding $\Phi_1 : k \to \mathcal{S}(1 \times \alpha\,; 1)$ preserving all limits existing in k.*

PROOF. From Lemma 4.1 it follows that there is a full embedding $\Phi : k \to \mathcal{S}$ preserving all limits such that $\Phi a = (Ga; U_a, R_a, \{o_\nu \mid \nu \in \alpha\}) \in \mathcal{S}(1 \times \alpha\,; 1, \alpha)$ is an algebraic system in which U_a is a unary relation, R_a is an α-ary relation and the α operations o_ν on Ga are unary. Indeed, this is because k is infinite, so that $|\operatorname{obj} k| + 1 \leq |\operatorname{mor} k| = \alpha$ and hence we can uniformly enlarge the arity $|\operatorname{obj} k| + 1$ of all R_a with $a \in \operatorname{obj} k$ to α when necessary. Henceforth we assume that $R_a \subseteq (Ga)^\alpha$ for every $a \in \operatorname{obj} k$.

Noting that the relations U_a and R_a are always non-void, in the first step we replace them by the non-void relation $V_a = U_a \times R_a$ which is α-ary, thereby defining an algebraic system $\Theta a = (Ga; V_a, \{o_\nu \mid \nu \in \alpha\}) \in \mathcal{S}(1 \times \alpha; \alpha)$ with one α-ary relation V_a and α unary operations $o_\nu : Ga \to Ga$.

In the second step, we define the algebraic system $\Phi_1 a$ as the extension of the system $(\Theta a)^\alpha$ by α operations that are the composites $\Delta \circ p_i$ of the projections $p_i : (\Theta a)^\alpha \to \Theta a$ with $i \in \alpha$ and the diagonal map $\Delta : Ga \to (Ga)^\alpha$. Since α is infinite, the algebraic system thus obtained, that is, the system

$$\Phi_1 a = ((Ga)^\alpha; V_a^\alpha, \{o_\nu^\alpha \mid \nu \in \alpha\} \cup \{\Delta \circ p_i \mid i \in \alpha\})$$

belongs to $\mathcal{S}(1 \times \alpha\,; 1)$. Setting $\Phi_1 p = (Gp)^\alpha$ for every $p \in k(a, b)$ then gives rise to a limit preserving one-to-one functor $\Phi_1 : k \to \mathcal{S}(1 \times \alpha\,; 1)$. A standard argument using the 'projection' unary operations $\Delta \circ p_i$ with $i \in \alpha$ then ensures that Φ_1 is full, cf. [10]. ∎

5. Representations of countable categories

A category $\mathcal{U}$ is called *alg-universal* if for every category $\mathrm{Alg}(\Sigma)$ of algebras of mono-sorted similarity type Σ there is a full and faithful functor $\mathrm{Alg}(\Sigma) \to \mathcal{U}$. There are numerous categories $\mathcal{U}$ of algebras or algebraic systems that are alg-universal, see [10]. One consequence of alg-universality of $\mathcal{U}$ is that for every small category k there is a full embedding $\Psi_k : k \to \mathcal{U}$, see [10]. If $\mathcal{U}$ is an alg-universal category of algebras or algebraic systems, then these full embeddings are *small* in the sense that for an infinite k the size of representing objects $\Psi_k a \in \operatorname{obj} \mathcal{U}$ satisfies $|\Psi_k a| \leq |\operatorname{mor} k|$ for every $a \in \operatorname{obj} k$ and some are such that every $\Psi_k a$ is finite whenever $|\operatorname{mor} k|$ is finite (we then say that Ψ_k *preserves finiteness*). Small full embeddings do exist, for instance, into the alg-universal category of all commutative rings with unit [4] and into numerous varieties of non-commutative semigroups [8], but no such full embedding can preserve finiteness. Small full embeddings $\Psi_k : k \to \mathcal{U}$ that also preserve finiteness exist into the category of directed or undirected graphs, into $\mathrm{Alg}(1 \times 2)$, see [10], and even into certain finitely generated varieties of $(0,1)$-lattices [5] and of distributive double p-algebras [7].

But if $\mathcal{U}$ is any alg-universal category of algebras or algebraic systems whatsoever, then none of the full embeddings $k \to \mathcal{U}$ can preserve finite products for the finite category k of Example 2.1. As we shall see below, any countable strongly connected category with a terminal object can be fully embedded into $\mathrm{Alg}(1 \times 2)$ by a functor preserving limits over finitely many objects, but full embeddings into any alg-universal variety of $(0,1)$-lattices or distributive double p-algebras preserving finite products do not exist because of congruence distributivity, see [12]. This is why the present paper is restricted to limit preserving full embeddings into alg-universal categories of unary algebras or algebraic systems.

To reduce the similarity type of algebras (or algebraic systems) representing countable strongly connected categories, we need the following result of [11].

5.1. Proposition. [11] *There are limit preserving full embeddings*

(1) $\Psi_\omega : \mathrm{Alg}(1 \times \omega) \to \mathrm{Alg}(1 \times 2)$,

(2) *for any* $n \in \omega$, *a finiteness preserving* $\Psi_n : \mathrm{Alg}(1 \times n) \to \mathrm{Alg}(1 \times 2)$,

such that for each $\lambda \in \omega \cup \{\omega\}$, *the functor* Ψ_λ *is carried by the hom-functor* $\mathrm{Set}(\gamma_\lambda, -)$ *with some suitable cardinal* γ_λ *which is finite whenever* λ *is.*

We now apply the above results to countable categories.

5.2. Corollary. *Let* k *be a strongly connected category with a terminal object and countable* $|\operatorname{mor} k|$. *Then there is a full embedding* $\Phi'_0 : k \to \mathrm{Alg}(1 \times 2)$ *preserving all limits of diagrams with finitely many objects existing in* k.

Proof. We set $\Phi'_0 = \Psi_\omega \circ \Phi_0$, where $\Phi_0 : k \to \mathrm{Alg}(1 \times \omega)$ is the functor from Theorem 3.12 and $\Psi_\omega : \mathrm{Alg}(1 \times \omega) \to \mathrm{Alg}(1 \times 2)$ is the functor from Proposition 5.1(1). ■

5.3. Corollary. *Let* k *be a strongly connected category with a terminal object and countable* $|\operatorname{mor} k|$. *Then there is a full embedding* $\Phi'_1 : k \to \mathcal{S}(1 \times 2\,; 1)$ *preserving all limits existing in* k.

Proof. Let Ψ_ω be the functor from Proposition 5.1(1). The functor $\Phi_1 : k \to \mathcal{S}(1 \times \omega\,; 1)$ from Theorem 4.12 has the unary relation $W = V^\omega$ on the objects in its range. Define $\Phi'_1 : k \to \mathcal{S}(1 \times 2\,; 1)$ on k-objects by $\Phi'_1 a = (\Psi_\omega a\,; Z)$, with $Z = W^\omega$, and by $\Phi'_1 p = \Phi_1 p$ on k-morphisms. ■

5.4. Corollary. *For any finite strongly connected category* k, *there is a limit preserving full embedding* $\Lambda : k \to \mathrm{Alg}(1 \times 2)$ *such that* Λa *is a finite algebra for every* $a \in \operatorname{obj} k$.

Proof. If k is finite, then the limit preserving full embedding $\Psi : k \to \mathrm{Alg}(1 \times \lambda)$ of Theorem 2.2 has λ finite and the objects Ψa are finite for all $a \in \operatorname{obj} k$. Set $\Lambda = \Psi_\lambda \circ \Psi$, where $\Psi_\lambda ; \mathrm{Alg}(1 \times \lambda) \to \mathrm{Alg}(1 \times 2)$ is as in Proposition 5.1(2).n ■

In conclusion, we ask some naturally arising questions.

Problem 1. Is there a full embedding of $\mathrm{Alg}(1 \times 2)$ into some $\mathrm{Alg}(n \times 1)$ that preserves limits over finitely many algebras (or at least products of finitely many algebras)?

Problem 2. For which cardinals $\kappa > \omega$, if any, is there a full embedding of $\mathrm{Alg}(1 \times \kappa)$ into $\mathrm{Alg}(1 \times 2)$ that preserves limits over finitely many algebras (or products of finitely many algebras)?

Problem 3. Fully characterize the small categories k for which there is a faithful functor $H : k \to \mathrm{Set}$ preserving all limits (or products) existing in k.

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