Theory and Applications of Categories, Vol. 21, No. 1, 2008, pp. 7–20.

A CONVENIENT CATEGORY FOR DIRECTED HOMOTOPY

Dedicated to Walter Tholen at the occasion of his sixtieth birthday

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Abstract. We propose a convenient category for directed homotopy consisting of "directed" topological spaces generated by "directed" cubes. Its main advantage is that, like the category of topological spaces generated by simplices suggested by J. H. Smith, it is locally presentable.

1. Introduction

We propose a convenient category for doing directed homotopy whose main advantage is being locally presentable. Our proposal is based on the suggestion of J. H. Smith to use Δ-generated topological spaces as a convenient category for usual homotopy. His suggestion was written down by D. Dugger [7] but it has turned out that it is not clear how to prove that the resulting category is locally presentable. We will present the missing proof and, in fact, we prove a more general result saying that for each fibre-small topological category $\mathcal{K}$ and each small full subcategory $\mathcal{I}$, the category $\mathcal{K}_{\mathcal{I}}$ of $\mathcal{I}$-generated objects in $\mathcal{K}$ is locally presentable. In the case of Δ-generated topological spaces, we take as $\mathcal{K}$ the category **Top** of topological spaces and continuous maps and as $\mathcal{I}$ the full subcategory consisting of simplices Δ_n, $n = 0, 1, \ldots$. We recall that a category $\mathcal{K}$ is topological if it is equipped with a faithful functor $U : \mathcal{K} \to \mathbf{Set}$ to the category of sets such that one can mimic the formation of "initially generated topological spaces" (see [2]). The category **d-Space** of d-spaces (in the sense of [11]) is topological and its full subcategory generated by suitably ordered cubes is our proposed convenient category for directed homotopy.

The idea of suitably generated topological spaces is quite old and goes back to [19] and [18] where the aim was to get a cartesian closed replacement of **Top**. The classical choice of $\mathcal{I}$ is the category of compact Hausdorff spaces. The insight of J. H. Smith is that the smallness of $\mathcal{I}$ makes $\mathbf{Top}_{\mathcal{I}}$ locally presentable. By [19] 3.3, $\mathbf{Top}_{\Delta}$ is even cartesian closed.

The authors would like to acknowledge helpful comments of P. Gaucher improving our presentation.

* Supported by the Ministry of Education of the Czech Republic under the projects MSM 0021622409 and 1M0545. The hospitality of the Aalborg University is gratefully acknowledged.

Received by the editors 2007-09-27 and, in revised form, 2008-03-26.

Published on 2008-05-07 in the Tholen Festschrift.

2000 Mathematics Subject Classification: Primary 18C35; Secondary 55P99, 68Q85.

Key words and phrases: locally presentable category, simplex-generated spaces, directed homotopy, dicovering.

2. Locally presentable categories

A category $\mathcal{K}$ is *locally λ-presentable* (where λ is a regular cardinal) if it is cocomplete and has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$. A category which is locally λ-presentable for some regular cardinal λ is called *locally presentable.* Recall that an object K is λ-presentable if its hom-functor $\hom(K,-) : \mathcal{K} \to \mathbf{Set}$ preserves λ-filtered colimits. We will say that K is *presentable* if it is λ-presentable for some regular cardinal λ. A useful characterization is that a category $\mathcal{K}$ is locally presentable if and only if it is cocomplete and has a small dense full subcategory consisting of presentable objects (see [3], 1.20).

A distinguished advantage of locally presentable categories is given by the following two results. Recall that, given morphisms $f : A \to B$ and $g : C \to D$ in a category $\mathcal{K}$, we write

$$f\Box g \qquad (f \perp g)$$

if, in each commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

there is a (unique) diagonal $d : B \to C$ with $df = u$ and $gd = v$.

For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$$\mathcal{H}^{\Box} = \{g|f\Box g \text{ for each } f \in \mathcal{H}\},$$
$$^{\Box}\mathcal{H} = \{f|f\Box g \text{ for each } g \in \mathcal{H}\},$$
$$\mathcal{H}^{\perp} = \{g|f \perp g \text{ for each } f \in \mathcal{H}\},$$
$$^{\perp}\mathcal{H} = \{f|f \perp g \text{ for each } g \in \mathcal{H}\}.$$

The smallest class of morphisms of $\mathcal{K}$ containing isomorphisms and being closed under transfinite compositions, pushouts of morphisms from $\mathcal{H}$ and retracts (in the category $\mathcal{K}^{\to}$ of morphisms of $\mathcal{K}$) is denoted as $\mathrm{cof}(\mathcal{H})$ while the smallest class of morphisms of $\mathcal{K}$ closed under all colimits (in $\mathcal{K}^{\to}$) and containing $\mathcal{H}$ is denoted as $\mathrm{colim}(\mathcal{H})$.

Given two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$, the pair $(\mathcal{L}, \mathcal{R})$ is called a *weak factorization system* if

1. $\mathcal{R} = \mathcal{L}^{\Box}$, $\mathcal{L} = {}^{\Box}\mathcal{R}$

and

(2) any morphism h of $\mathcal{K}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

The pair $(\mathcal{L}, \mathcal{R})$ is called a *factorization system* if condition (1) is replaced by

(1') $\mathcal{R} = \mathcal{L}^{\perp}$, $\mathcal{L} = {}^{\perp}\mathcal{R}$.

While the first result below can be found in [4] (or [1]), we are not aware of any published proof of the second one.

2.1. Theorem. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $(\mathrm{cof}(\mathcal{C}), \mathcal{C}^{\Box})$ is a weak factorization system in $\mathcal{K}$.*

2.2. Theorem. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system in $\mathcal{K}$.*

Proof. It is easy to see (and well known) that

$$\mathrm{colim}(\mathcal{C}) \subseteq {}^{\perp}(\mathcal{C}^{\perp}).$$

It is also easy to see that $g : C \to D$ belongs to $\mathcal{C}^{\perp}$ if and only if it is orthogonal in $\mathcal{K} \downarrow D$ to each morphism $f : (A, vf) \to (B, v)$ with $f \in \mathcal{C}$. By [3], 4.4, it is equivalent to g being injective to a larger set of morphisms of $\mathcal{K} \downarrow D$. Since this larger set is constructed using pushouts and pushouts in $\mathcal{K} \downarrow D$ are given by pushouts in $\mathcal{K}$, $g : C \to D$ belongs to $\mathcal{C}^{\perp}$ if and only if it is injective in $\mathcal{K} \downarrow D$ to each morphism $f : (A, vf) \to (B, v)$ with $f \in \mathcal{C}$ and to each morphism $f^* : (A, vf^*) \to (B, v)$ with $f \in \mathcal{C}$. The morphism f^* is given as follows. We form the pushout of f and f and f^* is a unique morphism making the following diagram commutative

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[d, "p_2"] \arrow[ddr, "\mathrm{id}_B"] & \\
B \arrow[r, "p_1"'] \arrow[drr, "\mathrm{id}_B"'] & A^* \arrow[dr, "f^*"] & \\
& & B
\end{tikzcd}$$

Then f^* belongs to $\mathrm{colim}(\mathcal{C})$ because it is the pushout of $f : f \to \mathrm{id}_B$ and $f : f \to \mathrm{id}_B$ in $\mathcal{K}^{\to}$ and $f, \mathrm{id}_B \in \mathrm{colim}(\mathcal{C})$:

$$\begin{tikzcd}
B \arrow[rrr, "\mathrm{id}_B"] \arrow[ddd, "\mathrm{id}_B"'] & & & B \arrow[ddd, "\mathrm{id}_B"] \\
& A \arrow[ul, "f"'] \arrow[r, "f"] \arrow[d, "f"'] & B \arrow[ur, "\mathrm{id}_B"] \arrow[d, "p_2"] & \\
& B \arrow[r, "p_1"'] \arrow[dl, "\mathrm{id}_B"] & A^* \arrow[dr, "f^*"] & \\
B \arrow[rrr, "\mathrm{id}_B"'] & & & B
\end{tikzcd}$$

Since $\bar{\mathcal{C}} = \mathcal{C} \cup \{f^* \backslash f \in \mathcal{C}\}$ is a set, $(\mathrm{cof}(\bar{\mathcal{C}}), \bar{\mathcal{C}}^{\Box})$ is a weak factorization system (by 2.1). We have shown that

$$\bar{\mathcal{C}}^{\Box} = \mathcal{C}^{\perp}$$

and

$$\bar{\mathcal{C}} \subseteq \operatorname{colim}(\mathcal{C}).$$

The consequence is that

$$\operatorname{cof}(\bar{\mathcal{C}}) \subseteq \operatorname{colim} \mathcal{C}.$$

It follows from the fact that each pushout of a morphism f belongs to $\operatorname{colim}(\{f\})$ (see [13], (the dual of) M13) and a transfinite composition of morphisms belongs to their colimit closure. In fact, given a smooth chain of morphisms $(f_{ij} : K_i \to K_j)_{i<j<\lambda}$ (i.e., λ is a limit ordinal, $f_{jk} f_{ij} = f_{ik}$ for $i < j < k$ and $f_{ij} : K_i \to K_j$ is a colimit cocone for any limit ordinal $j < \lambda$), let $f_i : K_i \to K$ be a colimit cocone. Then f_0, which is the transfinite composition of f_{ij} is a colimit in $\mathcal{K}^{\to}$ of the chain

$$\begin{tikzcd}
K_0 \arrow[r, "\mathrm{id}_{K_0}"] \arrow[d, "f_{00}"'] & K_0 \arrow[r] \arrow[d, "f_{01}"] & \cdots & K_0 \arrow[d, "f_0"] \\
K_0 \arrow[r, "f_{01}"'] & K_1 \arrow[r] & \cdots & K
\end{tikzcd}$$

Thus we have

$$\operatorname{cof}(\bar{\mathcal{C}}) \subseteq {}^{\perp}(\mathcal{C}^{\perp}).$$

Conversely

$${}^{\perp}(\mathcal{C}^{\perp}) \subseteq {}^{\square}(\mathcal{C}^{\perp}) = {}^{\square}(\bar{\mathcal{C}}^{\square}) = \operatorname{cof}(\bar{\mathcal{C}}).$$

We have proved that $(\operatorname{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ is a factorization system. ■

3. Generated spaces

A functor $U : \mathcal{K} \to \mathbf{Set}$ is called *topological* if each cone

$$(f_i : X \to UA_i)_{i \in I}$$

in **Set**, where I is a class, has a unique U-initial lift $(\bar{f}_i : A \to A_i)_{i \in I}$ (see [2]). It means that

1. $UA = X$ and $U\bar{f}_i = f_i$ for each $i \in I$ and
2. given $h : UB \to X$ with $f_i h = U\bar{h}_i$, $\bar{h}_i : B \to A_i$ for each $i \in I$ then $h = U\bar{h}$ for a unique $\bar{h} : B \to A$.

Each topological functor is faithful and thus the pair $(\mathcal{K}, U)$ is a concrete category. Such concrete categories are called topological. The motivating example of a topological category is **Top**.

Topological functors can be characterized as functors U such that each cocone $(f_i : UA_i \to X)_{i \in I}$ has a unique U-final lift $(\bar{f}_i : A_i \to A)_{i \in I}$ (see [2], 21.9). It means that

(1') $UA = X$ and $U\bar{f}_i = f_i$ for each $i \in I$ and

(2') given $h : X \to UB$ with $hf_i = U\bar{h}_i$, $\bar{h}_i : A_i \to B$ for each $i \in I$ then $h = U\bar{h}$ for a unique $\bar{h} : A \to B$.

3.1. Example. (1) A preordered set $(A, \leq)$ is a set A equipped with a reflexive and transitive relation $\leq$. It means that it satisfies the formulas

$$(\forall x)(x \leq x)$$

and

$$(\forall x, y, z)(x \leq y \wedge y \leq z \to x \leq z).$$

Morphisms of preordered sets are isotone maps, i.e., maps preserving the relation $\leq$. The category of preordered sets is topological. The U-initial lift of a cone $(f_i : X \to UA_i)_{i \in I}$ is given by putting $a \leq b$ on X if and only if $f_i(a) \leq f_i(b)$ for each $i \in I$.

(2) An ordered set is a preordered set $(A, \leq)$ where $\leq$ is also antisymmetric, i.e., if it satisfies

$$(\forall x, y)(x \leq y \wedge y \leq x \to x = y).$$

The category of ordered sets is not topological because the underlying functor to sets does not preserve colimits.

All three formulas from the example are strict universal Horn formulas and the difference between the first two and the third one is that antisymmetry uses the equality. It was shown in [16] that this situation is typical. But one has to use the logic $L_{\infty,\infty}$ (see [6]). It means that one has a class Σ of relation symbols whose arities are arbitrary cardinal numbers and one uses conjunctions of an arbitrary set of formulas and quantifications over an arbitrary set of variables. A *relational universal strict Horn theory T without equality* then consists of formulas

$$(\forall x)(\varphi(x) \to \psi(x))$$

where x is a set of variables and φ, ψ are conjunctions of atomic formulas without equality. The category of models of a theory T is denoted by $\mathbf{Mod}(T)$. Σ is called the *type* of T.

3.2. Theorem. *Each fibre-small topological category $\mathcal{K}$ is isomorphic (as a concrete category) to a category of models of a relational universal strict Horn theory T without equality.*

This result was proved in [16], 5.3. When the type Σ of T is a set then T is a set and $\mathbf{Mod}(T)$ is locally presentable (see [3], 5.30). The theory for **Top**, as presented by Manes [15], is based on ultrafilter convergence (see [16], 5.4) and does not consist of a set of formulas. The category **Top** is far from being locally presentable because it does not have a small dense full subcategory (see [3], 1.24(7)) and no non-discrete space is presentable ([3], 1.14(6)).

A cone $(\bar{f}_i : A \to A_i)_{i \in I}$ in **Top** is *U-initial* if it satisfies condition (2) above. A cocone $(\bar{f}_i : A_i \to A)_{i \in I}$ in **Top** is called *U-final* if it satisfies the condition (2').

3.3. Definition. Let $(\mathcal{K}, U)$ be a topological category and $\mathcal{I}$ a full subcategory of $\mathcal{K}$. An object K of $\mathcal{K}$ is called *$\mathcal{I}$-generated* if the cocone $(C \to K)_{C \in \mathcal{I}}$ consisting of all morphisms from objects of $\mathcal{I}$ to K is U-final.

Let $\mathcal{K}_{\mathcal{I}}$ denote the full subcategory of $\mathcal{K}$ consisting of $\mathcal{I}$-generated objects. Using the terminology of [2], $\mathcal{K}_{\mathcal{I}}$ is the *final closure* of $\mathcal{I}$ in $\mathcal{K}$ and $\mathcal{I}$ is *finally dense* in $\mathcal{K}_{\mathcal{I}}$.

3.4. Remark. Let $\mathcal{I}$ be a full subcategory of **Top**. A topological space X is *$\mathcal{I}$-generated* if it has the property that a subset $S \subseteq X$ is open if and only if $f^{-1}(S)$ is open for every continuous map $f : Z \to X$ with $Z \in \mathcal{I}$. Thus we get $\mathcal{I}$-generated spaces of [7] in this case.

We follow the terminology of [7] although it is somewhat misleading because, in the classical case of $\mathcal{I}$ consisting of compact Hausdorff spaces, the resulting $\mathcal{I}$-generated spaces are called k-spaces. A compactly generated space should also be weakly Hausdorff (see, e.g., [12]).

3.5. Proposition. *Let $(\mathcal{K}, U)$ be a topological category and $\mathcal{I}$ a full subcategory. Then $\mathcal{K}_{\mathcal{I}}$ is coreflective in $\mathcal{K}$ and contains $\mathcal{I}$ as a dense subcategory.*

Proof. By [2], 21.31, $\mathcal{K}_{\mathcal{I}}$ is coreflective in $\mathcal{K}$. Since $\mathcal{I}$ is finally dense in $\mathcal{K}_{\mathcal{I}}$, it is dense. ∎

The coreflector $R : \mathcal{K} \to \mathcal{K}_{\mathcal{I}}$ assigns to K the smallest $\mathcal{I}$-generated object RK on UK in the following sense: $URK = UK$, id_{UK} carries the morphism $c_K : RK \to K$ and for each $A \in \mathcal{K}_{\mathcal{I}}$ and each $f : A \to K$ in $\mathcal{K}$ there is a unique $g : A \to RK$ in $\mathcal{K}_{\mathcal{I}}$ such that $c_K g = f$.

A concrete category $(\mathcal{K}, U)$ is called *fibre-small* provided that, for each set X, there is only a set of objects K in $\mathcal{K}$ with $UK = X$.

3.6. Theorem. *Let $(\mathcal{K}, U)$ be a fibre-small topological category and let $\mathcal{I}$ be a full small subcategory of $\mathcal{K}$. Then the category $\mathcal{K}_{\mathcal{I}}$ is locally presentable.*

Proof. By 3.2, $\mathcal{K}$ is concretely isomorphic to $\mathbf{Mod}(T)$ where T is a relational universal strict Horn theory without equality of type Σ. We can assume that T contains all its consequences. It means that a universal strict Horn sentence without equality belongs to T provided that it holds for all models of T. We can express Σ as a union of an increasing chain

$$\Sigma_0 \subseteq \Sigma_1 \subseteq \dots \Sigma_i \subseteq \dots$$

of subsets Σ_i indexed by all ordinals. Let T_i be the subset of T consisting of all sentences of type Σ_i. This yields T as the union of the increasing chain

$$T_0 \subseteq T_1 \subseteq \dots T_i \subseteq \dots$$

of subsets T_i indexed by all ordinals. The inclusions $T_i \subseteq T_j$, $i \leq j$ induce functors $H_{ij} : \mathbf{Mod}(T_j) \to \mathbf{Mod}(T_i)$ given by reducts. It means that we forget all relation symbols from Σ_j not belonging to Σ_i. Analogously, we get functors $H_i : \mathbf{Mod}(T) \to \mathbf{Mod}(T_i)$

for each i. All these functors are concrete (i.e., preserve underlying sets) and have left adjoints

$$F_{ij} : \mathbf{Mod}(T_i) \to \mathbf{Mod}(T_j)$$

and

$$F_i : \mathbf{Mod}(T_i) \to \mathbf{Mod}(T).$$

These left adjoints are also concrete and $F_{ij}(A)$ is given by the U-initial lift of the cone

$$f : U(A) \to U(B)$$

consisting of all maps f such that $f : A \to H_{ij}(B)$ is a morphism in $\mathbf{Mod}(T_i)$. It means that new relations from Σ_j are precisely consequences of the theory T_i. The functors F_i are given in the same way. Since these left adjoints are concrete, they are faithfull. Since T contains all its consequences, they are also full. Thus we have expressed $\mathbf{Mod}(T)$ as a union of an increasing chain of full coreflective subcategories

$$\mathbf{Mod}(T_0) \subseteq \mathbf{Mod}(T_1) \subseteq \dots \mathbf{Mod}(T_i) \subseteq \dots$$

indexed by all ordinals. Moreover, all these coreflective subcategories are locally presentable.

Let $\mathcal{I}$ be a full small subcategory of $\mathcal{K}$. Then there is an ordinal i such that $\mathcal{I} \subseteq \mathbf{Mod}(T_i)$. Consequently, $\mathcal{K}_\mathcal{I} \subseteq \mathbf{Mod}(T_i)$ and thus $\mathcal{K}_\mathcal{I}$ is a full coreflective subcategory of a locally presentable $\mathbf{Mod}(T_i)$ having a small dense full subcategory $\mathcal{I}$. Since $\mathcal{I}$ is a set, there is a regular cardinal λ such that all objects from $\mathcal{I}$ are λ-presentable in $\mathbf{Mod}(T_i)$ (see [3], 1.16). Since $\mathcal{K}_\mathcal{I}$ is closed under colimits in $\mathbf{Mod}(T_i)$, each object from $\mathcal{I}$ is λ-presentable in $\mathcal{K}_\mathcal{I}$. Hence $\mathcal{K}_\mathcal{I}$ is locally λ-presentable. ■

3.7. Corollary. *Let $\mathcal{I}$ be a small full subcategory of* **Top**. *Then the category* $\mathbf{Top}_\mathcal{I}$ *is locally presentable.*

3.8. Remark. Let $\mathcal{K}$ be a category such that the coreflective closure $\mathcal{K}_\mathcal{I}$ of each small full subcategory $\mathcal{I}$ of $\mathcal{K}$ is locally presentable. Then $\mathcal{K}$ is a union of a chain

$$\mathcal{K}_0 \subseteq \mathcal{K}_1 \subseteq \dots \mathcal{K}_i \subseteq$$

of full coreflective subcategories which are locally presentable. It suffices to express $\mathcal{K}$ as a union of a chain

$$\mathcal{I}_0 \subseteq \mathcal{I}_1 \subseteq \dots \mathcal{I}_i \subseteq$$

of small full subcategories and pass to

$$\mathcal{K}_{\mathcal{I}_0} \subseteq \mathcal{K}_{\mathcal{I}_1} \subseteq \dots \mathcal{K}_{\mathcal{I}_i} \subseteq$$

3.9. Theorem. *Let $\mathcal{I}$ be a full subcategory of* **Top** *containing discs D_n and spheres S_n, $n = 0, 1, \dots$. Then the category* $\mathbf{Top}_\mathcal{I}$ *admits a cofibrantly generated model structure, where cofibrations and weak equivalences are the same as in* **Top**.

Proof. Analogous to [12], 2.4.23. ■

4. Generated directed spaces

In order to get our convenient category for directed homotopy, we have to replace **Top** by a suitable category of "directed" topological spaces. A natural candidate is the category **PTop** of preordered topological spaces. Its objects are topological spaces whose underlying sets are preordered. Morphisms are isotone continuous maps. **PTop** is a topological category but, since we require transitivity, the counterclockwise ordering of the circle S^1 induces that $x \leq y$ for all points x, y on S^1. This makes **PTop** less convenient. We thus prefer the category **d-Space** of d-spaces in the sense of [11]. In what follows, we will denote the unit interval with the discrete order by I and the unit interval with the standard order by $\vec{I}$.

4.1. Definition. A *d-space* is a pair $(X, \vec{P}(X))$ consisting of a topological space X and and a subset $\vec{P}(X)$ of the set X^I of all continuous maps $I \to X$ such

1. all constant paths are in $\vec{P}(X)$ and
2. $\vec{P}(X)$ is closed under concatenation and increasing reparametrization.

The second condition means that, for $\gamma, \mu \in \vec{P}(X)$ and $f : \vec{I} \to \vec{I}$ isotone and continuous, $\gamma * \mu \in \vec{P}(X)$ and $\gamma f \in \vec{P}(X)$.

$\vec{P}(X)$ is called the set of dipaths or directed paths. A *d-map* $f : (X, \vec{P}(X)) \to (Y, \vec{P}(Y))$ is a continuous map $f : X \to Y$ such that $\gamma \in \vec{P}(X)$ implies $f\gamma \in \vec{P}(Y)$.

In **d-Space**, the directions are represented by the allowed paths and not as a relation on the space itself. Thus directed circles behave well here.

On a d-space $(X, \vec{P}(X))$, we define the preorder relation by means of $x \leq y$ if and only if there is $\gamma \in \vec{P}(X)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. This gives a functor from **d-Space** to **PTop**. On the other hand, the isotone continuous maps from $\vec{I}$ to a space in **PTop** yield a set of dipaths, hence a functor from **PTop** to **d-Space**.

In what follows, we will denote a d-space $(X, \vec{P}(X))$ just by X.

4.2. Theorem. **d-Space** *is a topological category.*

Proof. Let T be a relational universal strict Horn theory without equality giving **Top** and using relation symbols R_j, $j \in J$. We add a new continuum-ary relation symbol R whose interpretation is the set of directed paths. We add to T the following axioms:

(1) $(\forall x)R(x)$ where x is the constant,

(2) $(\forall x)(R((x_{\frac{i}{2}})_i) \wedge R((x_{\frac{i+1}{2}})_i) \to R(x))$,

(3) $(\forall x)(R(x) \to R(xt)$ where t is an increasing reparametrization,

(4) $(\forall x)(R(x) \to R_j(xa))$ where $j \in J$ and I satisfies R_j for a.

The resulting relational universal strict Horn theory axiomatizes d-spaces. In fact, (1) makes each constant path directed, (2) says that directed paths are closed under concatenation, (3) says that they are closed under increasing reparametrization and (4) says that they are continuous. ■

4.3. Remark. (i) A d-space is called *saturated* if it satisfies the converse implication to (3) for surjective increasing reparametrizations:

(5) $(\forall x)(R(xt) \to R(x)$ where t is a surjective increasing reparametrization.

This means that a path is directed whenever some of its increasing surjective reparametrizations is directed. Thus saturated d-spaces also form a topological category.

(ii) There is, of course, a direct proof of 4.2. Let U : **d-Space** $\to$ **Set** be the forgetful functor. By [2], 21.9, it suffices to see that for any cocone $(f_i : UA_i \to X)_{i\in I}$ there is a unique U*-final lift* $(\bar{f}_i : A_i \to A)_{i\in I}$, i.e., there is a unique **d-Space** structure on X such that $h : X \to UB$ is a d-morphism whenever hf_i is a d-morphism for all $i \in I$. The topology is defined by declaring a set V to be open if and only if $f_i^{-1}(V)$ open for all $i \in I$. Let $\vec{P}(A)$ be the closure under concatenation and increasing reparametrization of the set of all constant paths and all $f_i\gamma$ where $\gamma \in \vec{P}(A_i)$ and $i \in I$. It is not difficult to see that this provides a U-final lift.

4.4. Corollary. *Let $\mathcal{I}$ be a small full subcategory of* **d-Space**. *Then the category* **d-Space**$_{\mathcal{I}}$ *is locally presentable.*

4.5. Definition. Let $\mathcal{B}$ be the full subcategory of **d-Space** with objects all cubes $I_1 \times I_2 \times \ldots \times I_n$ where I_k is either the unit interval with the discrete order (i.e., equality) or the unit interval with the standard order. The order on $I_1 \times I_2 \times \ldots \times I_n$ is the product relation. The dipaths are the increasing paths with respect to this relation.

4.6. Corollary. *The category* **d-Space**$_{\mathcal{B}}$ *is locally presentable.*

We consider the category **d-Space**$_{\mathcal{B}}$ as a suitable framework for studying the directed topology problems arising in concurrency. One reason for this is that the geometric realization of a cubical complex belongs to **d-Space**$_{\mathcal{B}}$. These are geometric models of Higher Dimensional Automata (see [9]). In [9], the directions on the spaces are given via a *local partial order* and not as d-spaces but the increasing paths with respect to the local partial order are precisely the dipaths in the d-space structure.

4.7. Definition. Given two d-maps $f, g : X \to Y$, a *d-homotopy* (see [11]) is a d-map $H : X \times \vec{I} \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$. The d-homotopy relation is reflexive and transitive but it is not symmetric. Its symmetric and transitive hull is the *d-homotopy equivalence relation.* A d-homotopy of dipaths γ, μ with common initial and final points is a d-map $H : \vec{I} \times \vec{I} \to Y$ such that $H(t, 0) = \gamma(t)$, $H(t, 1) = \mu(t)$ and $H(0, s) = \gamma(0) = \mu(0)$ and $H(1, s) = \gamma(1) = \mu(1)$.

A *dihomotopy* (see [9]) is not assumed to respect the order in the homotopy coordinate, which means that it is a d-map $H : X \times I \to Y$. The resulting dihomotopy relation is an equivalence relation. Dihomotopy of dipaths is defined as above, i.e., we require endpoints to be fixed.

Since $\mathcal{B}$ contains both the interval with the discrete order and the interval with the natural order, the category $\mathbf{d\text{-}Space}_{\mathcal{B}}$ is convenient for the both kinds of directed homotopy.

Globes have been considered as models for higher dimensional automata in [10]. A globe on a non-empty (d-)space X is the unreduced suspension $X \times \vec{I}/(x,1) \sim *_1, (x,0) \sim *_0$. If X is in $\mathbf{d\text{-}Space}_{\mathcal{B}}$ then the same holds for the globe of X because it is given as a coequalizer. The globe of the empty set is the d-space of two disjoint points which also belongs to $\mathbf{d\text{-}Space}_{\mathcal{B}}$. An elementary globe is the globe of an unordered ball. Since an unordered ball is homeomorphic to an unordered cube, the globes of these are d-homeomorphic. And globes of unordered cubes are in our category.

5. Dicoverings

In [8], dicoverings, i.e., coverings of directed topological spaces are introduced as the analogy of coverings in the undirected case. Directed topological spaces are understood there as locally partially ordered topological ones. It turns out that it is not obvious which category one should choose to get universal dicoverings. In our framework, we have a setting which is on one hand much more general than the almost combinatorial one of cubical sets and, on the other hand, it is not as general as locally partially ordered topological spaces where dicoverings are certainly not well behaved. The first part of the following definition is analogous to that used in [8].

5.1. Definition. Let $g : Y \to X$ be a morphism in $\mathbf{d\text{-}Space}$ and $x \in X$. Then g is a *dicovering with respect to* x if, for all $y \in g^{-1}(x)$ and all $\gamma \in \vec{P}(X)$ with $\gamma(0) = x$, there is a unique lift $\hat{\gamma}$ with $\hat{\gamma}(0) = y$. This means the commutative square below has a unique filler $\hat{\gamma}$:

$$\begin{tikzcd} \{0\} \arrow[r] \arrow[d, hook] & Y \arrow[d, "g"] \\ \vec{I} \arrow[r, "\gamma"] \arrow[ur, dashed, "\hat{\gamma}"] & X \end{tikzcd}$$

Moreover, for all $H : I \times \vec{I} \to X$ with $H(s,0) = x$ there is a unique lift $\hat{H}$ with $\hat{H}(s,0) = y$. This means that the commutative square below has a unique filler $\hat{H}$:

$$\begin{tikzcd} I \times \{0\} \arrow[r] \arrow[d, hook] & Y \arrow[d, "g"] \\ I \times \vec{I} \arrow[r, "H"] \arrow[ur, dashed, "\hat{H}"] & X \end{tikzcd}$$

We say that a morphism $g : Y \to X$ in $\mathbf{d\text{-}Space}$ is a *dicovering* if it is a dicovering with respect to each $x \in X$.

5.2. Remark. A dicovering does not need to be surjective. In fact, the unique morphism $\emptyset \to X$ is always a dicovering.

We will show that dicoverings form (the right part) of a factorization system. In what follows, we will use the following notation.

$$f_0 : \{0\} \to \vec{I}$$

will denote the inclusion. Let J be the coequalizer

$$I \overset{u}{\underset{v}{\rightrightarrows}} I \times \vec{I} \longrightarrow J$$

where $u(s) = (0, 0)$ and $v(s) = (s, 0)$. Then

$$f_1 : \{(0, 0)\} \to J$$

will be the inclusion and let

$$\mathcal{C} = \{f_0, f_1\}.$$

5.3. Proposition. *A morphism $g : Y \to X$ in* **d-Space** *is a dicovering if and only if belongs to $\mathcal{C}^\perp$.*

Proof. It follows from the fact that morphisms $H : I \times \vec{I} \to X$ with $H(-, 0)$ constant uniquely correspond to morphisms $J \to X$. ∎

5.4. Definition. A *pointed d-space* is a pair (X, x) consisting of a d-space X and a point $x \in X$. A morphism of pointed d-spaces $(X, x) \to (Y, y)$ is a d-map $f : X \to Y$ such that $f(x) = y$.

A *pointed dicovering* is a pointed d-map which is a dicovering.

Pointed d-spaces will be called *pd-spaces* and their morphisms *pd-maps*. **pd-Space** will denote the category of pd-spaces and $\mathbf{pd\text{-}Space}_\mathcal{B}$ the category of pd-spaces (X, x) such that $X \in \mathbf{d\text{-}Space}_\mathcal{B}$. Pointed dicoverings will be called *p-dicoverings*.

5.5. Definition. A *universal p-dicovering* of $X \in \mathbf{pd\text{-}Space}_\mathcal{B}$ is a p-dicovering $v : \tilde{X} \to X$ such that for any p-dicovering $g : Y \to X$ in $\mathbf{pd\text{-}Space}_\mathcal{B}$ there is a unique pd-map $t : \tilde{X} \to Y$ such that $v = gt$.

5.6. Corollary. *A universal p-dicovering exists for every pd-space in* $\mathbf{pd\text{-}Space}_\mathcal{B}$.

Proof. Consider a pd-space (X, x) in $\mathbf{pd\text{-}Space}_\mathcal{B}$ and denote by $*$ an initial pd-space $(\{0\}, 0)$. Consider the d-map $f : * \to X$ such that $f(0) = x$. Following 2.2 and 4.6, there is a $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^\perp)$ factorization

$$* \xrightarrow{w} \tilde{X} \xrightarrow{v} X$$

of f. Let $\tilde{x} = w(0)$. Then

$$v : (\tilde{X}, \tilde{x}) \to (X, x)$$

is a universal p-dicovering of (X, x). In fact, it is a p-dicovering and, given a p-dicovering

$$g : (Y, y) \to (X, x),$$

we apply the unique right lifting property to

$$\begin{tikzcd} * \arrow[r, "u"] \arrow[d] & Y \arrow[d, "g"] \\ \tilde{X} \arrow[r, "v"'] & X \end{tikzcd}$$

where $u(0) = y$. ∎

5.7. Remark. It immediately follows from the definition that the universal p-dicovering is unique up to isomorphism. There is a universal dicovering of a d-space X as well but it is not interesting because it is given as

$$0 \to X$$

where 0 is the empty d-space.

The pd-map $t : \tilde{X} \to Y$ from 5.5 is a dicovering too. This follows from $(\mathrm{colim}(\mathcal{C}), \mathcal{C}^{\perp})$ being a factorization system (see, e.g., [17]).

A universal p-dicovering of (X, x) does not need to be surjective in general. But it is surjective in the following important case.

5.8. Definition. A pointed d-space (X, x) will be called *well pointed* if each point $z \in X$ is *in the future of* x in the sense that there is a dipath $\gamma : \vec{I} \to X$ with $\gamma(0) = x$ and $\gamma(1) = z$.

Well pointed d-spaces will be called *wpd-spaces* and $\mathbf{wpd\text{-}Space}_{\mathcal{B}}$ will denote the full subcategory of $\mathbf{pd\text{-}Space}_{\mathcal{B}}$ consisting of wpd-spaces.

5.9. Lemma. *Let $g : (Y, y) \to (X, x)$ be a p-dicovering and (X, x) be a wpd-space. Then g is surjective and (Y, y) is a wpd-space.*

Proof. It immediately follows from the fact that p-dicoverings lift dipaths. ∎

Consequently, universal p-dicoverings exist and are surjective in the category of $\mathcal{B}$-generated wpd-spaces. Also, for wpd-spaces (Y, y) and (X, x), a p-dicovering $(Y, y) \to (X, x)$ with respect to x is a p-dicovering. Moreover, p-dicoverings of wpd-spaces correspond to simple dicoverings from [8]. This paper also constructs a "universal" dicovering $\pi : (\tilde{X}, \tilde{x}) \to (X, x)$ by endowing the set of dihomotopy classes of dipaths starting in x with a topology and a local partial order. The construction makes $(\tilde{X}, \tilde{x})$ a wpd-space but does not provide a universal object. The reason is that, given a dicovering $g : Y \to X$, the induced map $t : U\tilde{X} \to UY$ preserves dipaths but it does not need to be continuous. We expect that this is true provided that (X, x) is a wpd-space but we do not have a proof of this yet.

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