Theory and Applications of Categories, Vol. 18, No. 21, 2007, pp. 654–664.

QUOTIENTS OF TANNAKIAN CATEGORIES

J.S. MILNE

ABSTRACT. We classify the "quotients" of a tannakian category in which the objects of a tannakian subcategory become trivial, and we examine the properties of such quotient categories.

Introduction

Given a tannakian category T and a tannakian subcategory S, we ask whether there exists a quotient of T by S, by which we mean an exact tensor functor $q\colon \mathsf{T} \to \mathsf{Q}$ from T to a tannakian category Q such that

a. the objects of T that become trivial in Q (i.e., isomorphic to a direct sum of copies of $1\!\!1$ in Q) are precisely those in S, and

b. every object of Q is a subquotient of an object in the image of q.

When T is the category $\mathsf{Rep}(G)$ of finite-dimensional representations of an affine group scheme G the answer is obvious: there exists a unique normal subgroup H of G such that the objects of S are the representations on which H acts trivially, and there exists a canonical functor q satisfying (a) and (b), namely, the restriction functor $\mathsf{Rep}(G) \to \mathsf{Rep}(H)$ corresponding to the inclusion $H \hookrightarrow G$. By contrast, in the general case, there need not exist a quotient, and when there does there will usually not be a canonical one. In fact, we prove that there exists a q satisfying (a) and (b) if and only if S is neutral, in which case the q are classified by the k-valued fibre functors on S. Here $k \overset{\text{def}}{=} \mathrm{End}(1\!\!1)$ is assumed to be a field.

From a slightly different perspective, one can ask the following question: given a subgroup H of the fundamental group $\pi(\mathsf{T})$ of T, does there exist an exact tensor functor $q\colon \mathsf{T} \to \mathsf{Q}$ such that the resulting homomorphism $\pi(\mathsf{Q}) \to q(\pi(\mathsf{T}))$ maps $\pi(\mathsf{Q})$ isomorphically onto $q(H)$? Again, there exists such a q if and only if the subcategory T^H of T, whose objects are those on which H acts trivially, is neutral, in which case the functors q correspond to the k-valued fibre functors on T^H.

The two questions are related by the "tannakian correspondence" between tannakian subcategories of T and subgroups of $\pi(\mathsf{T})$ (see 1.7).

In addition to proving the above results, we determine the fibre functors, polarizations, and fundamental groups of the quotient categories Q.

Received by the editors 2007-07-07 and, in revised form, 2007-12-02.
Transmitted by Ieke Moerdijk. Published on 2007-12-14.
2000 Mathematics Subject Classification: 18D10.
Key words and phrases: gerbes, motives, tannakian categories.

The original motivation for these investigations came from the theory of motives (see Milne 2002, 2007).

Notation: The notation $X \approx Y$ means that X and Y are isomorphic, and $X \simeq Y$ means that X and Y are canonically isomorphic (or that there is a given or unique isomorphism).

1. Preliminaries

For tannakian categories, we use the terminology of Deligne and Milne 1982. In particular, we write $\mathbb{1}$ for any identity object of a tannakian category — recall that it is uniquely determined up to a unique isomorphism. We fix a field k and consider only tannakian categories with $k = \mathrm{End}(\mathbb{1})$ and only functors of tannakian categories that are k-linear.

GERBES

1.1. We refer to Giraud 1971, Chapitre IV, for the theory of gerbes. All gerbes will be for the flat (i.e., fpqc) topology on the category Aff_k of affine schemes over k. The band (= lien) of a gerbe $\mathcal{G}$ is denoted $\mathrm{Bd}(\mathcal{G})$. A commutative band can be identified with a sheaf of groups.

1.2. Let $\alpha\colon \mathcal{G}_1 \to \mathcal{G}_2$ be a morphism of gerbes over Aff_k, and let ω_0 be an object of $\mathcal{G}_{2,k}$. Define $(\omega_0 \downarrow \mathcal{G}_1)$ to be the fibred category over Aff_k whose fibre over $S \xrightarrow{s} \operatorname{Spec} k$ has as objects the pairs (ω, a) consisting of an object ω of $\mathrm{ob}(\mathcal{G}_{1,S})$ and an isomorphism $a\colon s^*\omega_0 \to \alpha(\omega)$ in $\mathcal{G}_{2,S}$; the morphisms $(\omega, a) \to (\nu, b)$ are the isomorphisms $\varphi\colon \omega \to \nu$ in $\mathcal{G}_{1,S}$ giving rise to a commutative triangle. Thus,

$$\begin{tikzcd}
\omega \arrow[dd, "\varphi"] & & & \alpha(\omega) \arrow[dd, "\alpha(\varphi)"] \\
& & s^*(\omega_0) \arrow[ur, "a"] \arrow[dr, "b"'] & \\
\nu & & & \alpha(\nu) \\
\mathcal{G}_{1,S} & & \mathcal{G}_{2,S} &
\end{tikzcd}$$

If the map of bands defined by α is an epimorphism, then $(\omega_0 \downarrow \mathcal{G}_1)$ is a gerbe, and the sequence of bands

$$1 \to \mathrm{Bd}(\omega_0 \downarrow \mathcal{G}_1) \to \mathrm{Bd}(\mathcal{G}_1) \to \mathrm{Bd}(\mathcal{G}_2) \to 1 \tag{1}$$

is exact (Giraud 1971, IV 2.5.5(i)).

1.3. Recall (Saavedra Rivano 1972, III 2.2.2) that a gerbe is said to be tannakian if its band is locally defined by an affine group scheme. It is clear from the exact sequence (1) that if $\mathcal{G}_1$ and $\mathcal{G}_2$ are tannakian, then so also is $(\omega_0 \downarrow \mathcal{G}_1)$.

1.4. The fibre functors on a tannakian category T form a gerbe $\text{FIB}(\mathsf{T})$ over Aff_k (Deligne 1990, 1.13). Each object X of T defines a representation $\omega \mapsto \omega(X)$ of $\text{FIB}(\mathsf{T})$, and in this way we get an equivalence $\mathsf{T} \to \mathsf{Rep}(\text{FIB}(\mathsf{T}))$ of tannakian categories (Deligne 1989, 5.11; Saavedra Rivano 1972, III 3.2.3, p200). Every gerbe whose band is tannakian arises in this way from a tannakian category (Saavedra Rivano 1972, III 2.2.3).

FUNDAMENTAL GROUPS

1.5. We refer to Deligne 1989, §§5,6, for the theory of algebraic geometry in a tannakian category T and, in particular, for the fundamental group $\pi(\mathsf{T})$ of T. It is the affine group scheme[1] in T such that $\omega(\pi(\mathsf{T})) \simeq \underline{\mathrm{Aut}}^{\otimes}(\omega)$ functorially in the fibre functor ω on T. The group $\pi(\mathsf{T})$ acts on each object X of T, and ω transforms this action into the natural action of $\underline{\mathrm{Aut}}^{\otimes}(\omega)$ on $\omega(X)$. The various realizations $\omega(\pi(\mathsf{T}))$ of $\pi(\mathsf{T})$ determine the band of T (i.e., the band of $\text{FIB}(\mathsf{T})$).

1.6. An exact tensor functor $F\colon \mathsf{T}_1 \to \mathsf{T}_2$ of tannakian categories defines a homomorphism $\pi(F)\colon \pi(\mathsf{T}_2) \to F(\pi(\mathsf{T}_1))$ (Deligne 1989, 6.4). Moreover:

a. F induces an equivalence of T_1 with a category whose objects are the objects of T_2 endowed with an action of $F(\pi(\mathsf{T}_1))$ compatible with that of $\pi(\mathsf{T}_2)$ (Deligne 1989, 6.5);

b. $\pi(F)$ is flat and surjective if and only if F is fully faithful and every subobject of $F(X)$, for X in T_1, is isomorphic to the image of a subobject of X (cf. Deligne and Milne 1982, 2.21);

c. $\pi(F)$ is a closed immersion if and only if every object of T_2 is a subquotient of an object in the image of q (ibid.).

1.7. For a subgroup[2] $H \subset \pi(\mathsf{T})$, we let T^H denote the full subcategory of T whose objects are those on which H acts trivially. It is a tannakian subcategory of T (i.e., it is a strictly full subcategory closed under the formation of subquotients, direct sums, tensor products, and duals) and it follows from Deligne 1990, 8.17, that every tannakian subcategory arises in this way from a unique subgroup of $\pi(\mathsf{T})$ (see Bertolin 2003, 1.6)[3]. The objects of $\mathsf{T}^{\pi(\mathsf{T})}$ are exactly the trivial objects of T, and there exists a unique (up to a unique isomorphism) fibre functor

$$\gamma^{\mathsf{T}}\colon \mathsf{T}^{\pi(\mathsf{T})} \to \mathsf{Vec}_k,$$

namely, $\gamma^{\mathsf{T}}(X) = \mathrm{Hom}(\mathbb{1}, X)$.

[1] "T-schéma en groupes affines" in Deligne's terminology.

[2] Note that every subgroup H of $\pi(\mathsf{T})$ is normal. For example, the fundamental group π of the category $\mathsf{Rep}(G)$ of representations of the affine group scheme $G = \mathrm{Spec}(A)$ is A regarded as an object of $\mathrm{Ind}(\mathsf{Rep}(G))$. The action of G on A is that defined by inner automorphisms. A subgroup of π is a quotient $A \to B$ of A (as a bi-algebra) such that the action of G on A defines an action of G on B. Such quotients correspond to normal subgroups of G.

[3] Théorème 8.17 of Deligne 1990 applies without restriction on the field k because we are assuming that T is tannakian (see ibid. 6.9, 8.1). Equally, the blanket hypothesis that k is of characteristic zero in Bertolin 2003 is not required for §1 of the paper.

1.8. For a subgroup H of $\pi(\mathsf{T})$ and an object X of T, we let X^H denote the largest subobject of X on which the action of H is trivial. Thus $X = X^H$ if and only if X is in T^H.

1.9. When H is contained in the centre of $\pi(\mathsf{T})$, then it is an affine group scheme in $\mathsf{T}^{\pi(\mathsf{T})}$, and so γ^{T} identifies it with an affine group scheme over k in the usual sense. For example, γ^{T} identifies the centre of $\pi(\mathsf{T})$ with $\underline{\mathrm{Aut}}^{\otimes}(\mathrm{id}_{\mathsf{T}})$ (cf. Saavedra Rivano 1972, II 3.3.3.2, p150).

MORPHISMS OF TANNAKIAN CATEGORIES

1.10. For a group G, a right G-object X, and a left G-object Y, $X \wedge^G Y$ denotes the contracted product of X and Y, i.e., the quotient of $X \times Y$ by the diagonal action of G, $(x, y)g = (xg, g^{-1}y)$. When $G \to H$ is a homomorphism of groups, $X \wedge^G H$ is the H-object obtained from X by extension of the structure group. In this last case, if X is a G-torsor, then $X \wedge^G H$ is also an H-torsor. See Giraud 1971, III 1.3, 1.4.

1.11. Let T be a tannakian category over k, and assume that the fundamental group π of T is commutative. A torsor P under π in T defines a tensor equivalence $\mathsf{T} \to \mathsf{T}$, $X \mapsto P \wedge^{\pi} X$, bound by the identity map on $\mathrm{Bd}(\mathsf{T})$, and every such equivalence arises in this way from a torsor under π (cf. Saavedra Rivano 1972, III 2.3). For any k-algebra R and R-valued fibre functor ω on T, $\omega(P)$ is an R-torsor under $\omega(\pi)$ and $\omega(P \wedge^{\pi} X) \simeq \omega(P) \wedge^{\omega(\pi)} \omega(X)$.

2. Quotients

For any exact tensor functor $q\colon \mathsf{T} \to \mathsf{T}'$, the full subcategory T^q of T whose objects become trivial in T' is a tannakian subcategory of T (obviously).

We say that an exact tensor functor $q\colon \mathsf{T} \to \mathsf{Q}$ of tannakian categories is a *quotient functor* if every object of Q is a subquotient of an object in the image of q; equivalently, if the homomorphism $\pi(q)\colon \pi(\mathsf{Q}) \to q(\pi\mathsf{T})$ is a closed immersion (see 1.6c). If, in addition, the homomorphism $\pi(q)$ is normal (i.e., its image is a normal subgroup of $q(\mathsf{T})$), then we say that q is *normal*.

2.1. EXAMPLE. Consider the exact tensor functor $\omega^f\colon \mathsf{Rep}(G) \to \mathsf{Rep}(H)$ defined by a homomorphism $f\colon H \to G$ of affine group schemes. The objects of $\mathsf{Rep}(G)^{\omega^f}$ are those on which H (equivalently, the intersection of the normal subgroups of G containing $f(H)$) acts trivially. The functor ω^f is a quotient functor if and only if f is a closed immersion, in which case it is normal if and only if $f(H)$ is normal in G.

2.2. PROPOSITION. *An exact tensor functor $q\colon \mathsf{T} \to \mathsf{Q}$ of tannakian categories is a normal quotient functor if and only if there exists a subgroup H of $\pi(\mathsf{T})$ such that $\pi(q)$ induces an isomorphism $\pi(\mathsf{Q}) \to q(H)$.*

PROOF. $\Longleftarrow$: Because q is exact, $q(H) \to q(\pi\mathsf{T})$ is a closed immersion. Therefore $\pi(q)$ is a closed immersion, and its image is the normal subgroup $q(H)$ of $q(\pi\mathsf{T})$.

$\Longrightarrow$: Because q is a quotient functor, $\pi(q)$ is a closed immersion. Let H be the kernel of the homomorphism $\pi(\mathsf{T}) \to \pi(\mathsf{T}^q)$ defined by the inclusion $\mathsf{T}^q \hookrightarrow \mathsf{T}$. The image of $\pi(q)$ is contained in $q(H)$, and equals it if and only if q is normal. To see this, let $G = q\pi(\mathsf{T})$, and identify T with the category of objects of Q with an action of G compatible with that of $\pi(\mathsf{Q}) \subset G$. Then q becomes the forgetful functor, and $\mathsf{T}^q = \mathsf{T}^{\pi(\mathsf{Q})}$. Thus, $q(H)$ is the subgroup of G acting trivially on those objects on which $\pi(\mathsf{Q})$ acts trivially. It follows that $\pi(\mathsf{Q}) \subset q(H)$, with equality if and only if $\pi(\mathsf{Q})$ is normal in G. ■

In the situation of the proposition, we sometimes call Q *a quotient of* T *by* H (cf. Milne 2002, 1.3).

Let $q\colon \mathsf{T} \to \mathsf{Q}$ be an exact tensor functor of tannakian categories. By definition, q maps T^q into $\mathsf{Q}^{\pi(\mathsf{Q})}$, and so we acquire a k-valued fibre functor $\omega^q \overset{\text{def}}{=} \gamma^{\mathsf{Q}} \circ (q|\mathsf{T}^q)$ on T^q:

$$\begin{tikzcd}
\mathsf{T}^q \arrow[r, "q|\mathsf{T}_q"'] \arrow[rr, bend left, "\omega^q"] \arrow[d, hook] & \mathsf{Q}^{\pi(\mathsf{Q})} \arrow[r, "\gamma^{\mathsf{Q}}"'] \arrow[d, hook] & \mathsf{Vec}_k \\
\mathsf{T} \arrow[r, "q"] & \mathsf{Q}.
\end{tikzcd} \qquad \omega^q(X) = \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, qX).$$

In particular, T^q is neutral. A fibre functor ω on Q, defines a fibre functor $\omega \circ q$ on T, and the (unique) isomorphism $\gamma^{\mathsf{Q}} \to \omega|\mathsf{Q}^{\pi(\mathsf{Q})}$ defines an isomorphism $a(\omega)\colon \omega^q \to (\omega \circ q)|\mathsf{T}^q$.

2.3. PROPOSITION. *Let* $q\colon \mathsf{T} \to \mathsf{Q}$ *be a normal quotient, and let* H *be the subgroup of* $\pi(\mathsf{T})$ *such that* $\pi(\mathsf{Q}) \simeq q(H)$.

a. For X, Y *in* T*, there is a canonical functorial isomorphism*

$$\mathrm{Hom}_{\mathsf{Q}}(qX, qY) \simeq \omega^q(\underline{\mathrm{Hom}}(X,Y)^H).$$

b. The map $\omega \mapsto (\omega \circ q, a(\omega))$ *defines an equivalence of gerbes*

$$r(q)\colon \mathrm{FIB}(\mathsf{Q}) \to (\omega^q{\downarrow}\mathrm{FIB}(\mathsf{T})).$$

PROOF. (a) From the various definitions and Deligne and Milne 1982,

$$\begin{aligned}
\mathrm{Hom}_{\mathsf{Q}}(qX, qY) &\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, \underline{\mathrm{Hom}}(qX, qY)^{\pi(\mathsf{Q})}) && \text{(ibid. 1.6.4)}\\
&\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, (q\underline{\mathrm{Hom}}(X,Y))^{q(H)}) && \text{(ibid. 1.9)}\\
&\simeq \mathrm{Hom}_{\mathsf{Q}}(\mathbb{1}, q(\underline{\mathrm{Hom}}(X,Y)^H))\\
&\simeq \omega^q(\underline{\mathrm{Hom}}(X,Y)^H) && \text{(definition of } \omega^q\text{)}.
\end{aligned}$$

(b) The functor $\mathrm{FIB}(\mathsf{T}) \to \mathrm{FIB}(\mathsf{T}^H)$ gives rise to an exact sequence

$$1 \to \mathrm{Bd}(\omega_Q{\downarrow}\mathrm{FIB}(\mathsf{T})) \to \mathrm{Bd}(\mathsf{T}) \to \mathrm{Bd}(\mathsf{T}^H) \to 0$$

(see 1.2). On the other hand, we saw in the proof of (2.2) that $H = \mathrm{Ker}(\pi(\mathsf{T}) \to \pi(\mathsf{T}^H))$. On comparing these statements, we seee that the morphism $r(q)$ of gerbes is bound by an isomorphism of bands, which implies that it is an equivalence of gerbs (Giraud 1971, IV 2.2.6). ∎

2.4. PROPOSITION. *Let* (Q, q) *be a normal quotient of* T. *An exact tensor functor* $q' \colon \mathsf{T} \to \mathsf{T}'$ *factors through* q *if and only* $\mathsf{T}^{q'} \supset \mathsf{T}^q$ *and* $\omega^q \approx \omega^{q'}|\mathsf{T}^q$.

PROOF. The conditions are obviously necessary. For the sufficiency, choose an isomorphism $b \colon \omega^q \to \omega^{q'}|\mathsf{T}^q$. A fibre functor ω on T' then defines a fibre functor $\omega \circ q'$ on T and an isomorphism $a(\omega)|\mathsf{T}^q \circ b \colon \omega^q \to (\omega \circ q')|\mathsf{T}^q$. In this way we get a homomorphism

$$\mathrm{FIB}(\mathsf{T}') \to (\omega^q \downarrow \mathrm{FIB}(\mathsf{T})) \simeq \mathrm{FIB}(\mathsf{Q})$$

and we can apply (1.4) to get a functor $\mathsf{Q} \to \mathsf{T}'$ with the correct properties. ∎

2.5. THEOREM. *Let* T *be a tannakian category over* k, *and let* ω_0 *be a* k*-valued fibre functor on* T^H *for some subgroup* $H \subset \pi(\mathsf{T})$. *There exists a quotient* (Q, q) *of* T *by* H *such that* $\omega^q \simeq \omega_0$.

PROOF. The gerbe $(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$ is tannakian (see 1.3). From the morphism of gerbes

$$(\omega, a) \mapsto \omega \colon (\omega_0 \downarrow \mathrm{FIB}(\mathsf{T})) \to \mathrm{FIB}(\mathsf{T}),$$

we obtain a morphism of tannakian categories

$$\mathsf{Rep}(\mathrm{FIB}(\mathsf{T})) \to \mathsf{Rep}(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$$

(see 1.4). We define Q to be $\mathsf{Rep}(\omega_0 \downarrow \mathrm{FIB}(\mathsf{T}))$ and we define q to be the composite of the above morphism with the equivalence (see 1.4)

$$\mathsf{T} \to \mathsf{Rep}(\mathrm{FIB}(\mathsf{T})).$$

Since a gerbe and its tannakian category of representations have the same band, an argument as in the proof of Proposition 2.3 shows that $\pi(q)$ maps $\pi(\mathsf{Q})$ isomorphically onto $q(H)$. A direct calculation shows that ω^q is canonically isomorphic to ω_0. ∎

We sometimes write T/ω for the quotient of T defined by a k-valued fibre functor ω on a subcategory of T.

2.6. EXAMPLE. Let $(\mathsf{T}, w, \mathbb{T})$ be a Tate triple, and let S be the full subcategory of T of objects isomorphic to a direct sum of integer tensor powers of the Tate object $\mathbb{T}$. Define ω_0 to be the fibre functor on S,

$$X \mapsto \varinjlim_n \mathrm{Hom}\Big(\bigoplus_{-n \le r \le n} \mathbb{1}(r), X\Big).$$

Then the quotient tannakian category T/ω_0 is that defined in Deligne and Milne 1982, 5.8.

2.7. REMARK. Let $q\colon \mathsf{T} \to \mathsf{Q}$ be a normal quotient functor. Then T can be recovered from Q, the homomorphism $\pi(\mathsf{Q}) \to q(\pi(\mathsf{T}))$, and the actions of $q(\pi(\mathsf{T}))$ on the objects of Q (apply 1.6a).

2.8. REMARK. A fixed k-valued fibre functor on a tannakian category T determines a Galois correspondence between the subsets of $\mathrm{ob}(\mathsf{T})$ and the equivalence classes of quotient functors $\mathsf{T} \to \mathsf{Q}$.

2.9. EXERCISE. Use (1.10, 1.11) to express the correspondence between fibre functors on tannakian subcategories of T and normal quotients of T in the language of 2-categories.

2.10. ASIDE. Let G be the fundamental group $\pi(\mathsf{T})$ of a tannakian category T, and let H be a subgroup of G. We use the same letter to denote an affine group scheme in T and the band it defines. Then, under certain hypotheses, for example, if all the groups are commutative, there will be an exact sequence

$$\cdots \to H^1(k,G) \to H^1(k,G/H) \to H^2(k,H) \to H^2(k,G) \to H^2(k,G/H).$$

The category T defines a class $c(\mathsf{T})$ in $H^2(k,G)$, namely, the G-equivalence class of the gerbe of fibre functors on T, and the image of $c(\mathsf{T})$ in $H^2(k,G/H)$ is the class of T^H. Any quotient of T by H defines a class in $H^2(k,H)$ mapping to $c(\mathsf{T})$ in $H^2(k,G)$. Thus, the exact sequence suggests that a quotient of T by H will exist if and only if the cohomology class of T^H is neutral, i.e., if and only if T^H is neutral as a tannakian category, in which case the quotients are classified by the elements of $H^1(k,G/H)$ (modulo $H^1(k,G)$). When T is neutral and we fix a k-valued fibre functor on it, then the elements of $H^1(k,G/H)$ classify the k-valued fibre functors on T^H. Thus, the cohomology theory suggests the above results, and in the next subsection we prove that a little more of this heuristic picture is correct.

THE COHOMOLOGY CLASS OF THE QUOTIENT

For an affine group scheme G over a field k, $H^r(k,G)$ denotes the cohomology group computed with respect to the flat topology. When G is not commutative, this is defined only for $r = 0,1,2$ (Giraud 1971).

2.11. PROPOSITION. *Let (Q,q) be a quotient of T by a subgroup H of the centre of $\pi(\mathsf{T})$. Suppose that T is neutral, with k-valued fibre functor ω. Let $G = \underline{\mathrm{Aut}}^{\otimes}(\omega)$, and let $\wp(\omega^q)$ be the $G/\omega(H)$-torsor $\underline{\mathrm{Hom}}(\omega|\mathsf{T}^H, \omega^q)$. Under the connecting homomorphism*

$$H^1(k,G/H) \to H^2(k,H)$$

the class of $\wp(\omega^q)$ in $H^1(k,G/H)$ maps to the class of Q in $H^2(k,H)$.

PROOF. Note that $H = \mathrm{Bd}(\mathsf{Q})$, and so the statement makes sense. According to Giraud 1971, IV 4.2.2, the connecting homomorphism sends the class of $\wp(\omega^q)$ to the class of the gerbe of liftings of $\wp(\omega^q)$, which can be identified with $(\omega^q{\downarrow}\mathrm{FIB}(\mathsf{T}))$. Now Proposition 2.3 shows that the H-equivalence class of $(\omega^q{\downarrow}\mathrm{FIB}(\mathsf{T}))$ equals that of $\mathrm{FIB}(\mathsf{Q})$ which (by definition) is the cohomology class of Q. ∎

SEMISIMPLE NORMAL QUOTIENTS

Everything can be made more explicit when the categories are semisimple. Throughout this subsection, k has characteristic zero.

2.12. PROPOSITION. *Every normal quotient of a semisimple tannakian category is semisimple.*

PROOF. A tannakian category is semisimple if and only if the identity component of its fundamental group is pro-reductive (cf. Deligne and Milne 1982, 2.28), and a normal subgroup of a reductive group is reductive (because its unipotent radical is a characteristic subgroup). ∎

Let T be a semisimple tannakian category over k, and let ω_0 be a k-valued fibre functor on a tannakian subcategory S of T. We can construct an explicit quotient T/ω_0 as follows. First, let $(\mathsf{T}/\omega_0)'$ be the category with one object $\bar{X}$ for each object X of T, and with

$$\mathrm{Hom}_{(\mathsf{T}/\omega_0)'}(\bar{X},\bar{Y}) = \omega_0(\underline{\mathrm{Hom}}(\bar{X},\bar{Y})^H)$$

where H is the subgroup of $\pi(\mathsf{T})$ defining S. There is a unique structure of a k-linear tensor category on $(\mathsf{T}/\omega_0)'$ for which $q\colon \mathsf{T} \to (\mathsf{T}/\omega_0)'$ is a tensor functor. With this structure, $(\mathsf{T}/\omega_0)'$ is rigid, and we define T/ω_0 to be its pseudo-abelian hull. Thus, T/ω_0 has

$$\begin{aligned} \text{objects: } &\text{pairs } (\bar{X}, e) \text{ with } X \in \mathrm{ob}(\mathsf{T}) \text{ and } e \text{ an idempotent in } \mathrm{End}(\bar{X}),\\ \text{morphisms: } &\mathrm{Hom}_{\mathsf{T}/\omega_0}((\bar{X},e),(\bar{Y},f)) = f \circ \mathrm{Hom}_{(\mathsf{T}/\omega_0)'}(\bar{X},\bar{Y}) \circ e. \end{aligned}$$

Then $(\mathsf{T}/\omega_0, q)$ is a quotient of T by H, and $\omega^q \simeq \omega_0$.

Let ω be a fibre functor on T, and let a be an isomorphism $\omega_0 \to \omega|\mathsf{T}^H$. The pair (ω, a) defines a fibre functor ω_a on T/ω_0 whose action on objects is determined by

$$\omega_a(\bar{X}) = \omega(X)$$

and whose action on morphisms is determined by

$$\begin{tikzcd}
\mathrm{Hom}(\bar{X},\bar{Y}) \arrow[rr, dashed, "\omega_a"] \arrow[d, equal, "\mathrm{def}"] & & \mathrm{Hom}(\omega_a(\bar{X}),\omega_a(\bar{Y})) \\
\omega_0(\underline{\mathrm{Hom}}(X,Y)^H) \arrow[r, "a"] & \omega(\underline{\mathrm{Hom}}(X,Y)^H) \arrow[r, "\simeq"] & \underline{\mathrm{Hom}}(\omega(X),\omega(Y))^{\omega(H)}. \arrow[u, hook]
\end{tikzcd}$$

The map $(\omega, a) \mapsto \omega_a$ defines an equivalence $(\omega_0 {\downarrow} \mathrm{F{\small IB}}(\mathsf{T})) \to \mathrm{F{\small IB}}(\mathsf{T}/\omega_0)$.

Let $H_1 \subset H_0 \subset \pi(\mathsf{T})$, and let ω_0 and ω_1 be k-valued fibre functors on T^{H_0} and T^{H_1} respectively. A morphism $\alpha\colon \omega_0 \to \omega_1|\mathsf{T}^{H_0}$ defines an exact tensor functor $\mathsf{T}/\omega_0 \to \mathsf{T}/\omega_1$ whose action on objects is determined by

$$\bar{X} \text{ (in } \mathsf{T}^{H_0}) \mapsto \bar{X} \text{ (in } \mathsf{T}^{H_1}),$$

and whose action on morphisms is determined by

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{T}/\omega_0}(\bar{X},\bar{Y}) \arrow[rr, dashed] \arrow[d, equal, "\mathrm{def}"] & & \mathrm{Hom}_{\mathsf{T}/\omega_1}(\bar{X},\bar{Y}) \arrow[d, equal, "\mathrm{def}"] \\
\omega_0(\underline{\mathrm{Hom}}_{\mathsf{T}}(X,Y)^{H_0}) \arrow[r, "\alpha"] & \omega_1(\underline{\mathrm{Hom}}_{\mathsf{T}}(X,Y)^{H_0}) \arrow[r, hook] & \omega_1(\underline{\mathrm{Hom}}_{\mathsf{T}}(X,Y)^{H_1})).
\end{tikzcd}$$

When $H_1 = H_0$, this is an isomorphism (!) of tensor categories $\mathsf{T}/\omega_0 \to \mathsf{T}/\omega_1$.

Let (Q_1, q_1) and (Q_2, q_2) be quotients of T by H. For simplicity, assume that $\pi \stackrel{\mathrm{def}}{=} \pi(\mathsf{T})$ is commutative. Then $\underline{\mathrm{Hom}}(\omega^{q_1}, \omega^{q_2})$ is π/H-torsor, and we assume that it lifts to a π-torsor P in T, so $P \wedge^{\pi} (\pi/H) = \underline{\mathrm{Hom}}(\omega^{q_1}, \omega^{q_2})$. Then

$$\mathsf{T} \xrightarrow{X \mapsto P \wedge^{\pi} X} \mathsf{T} \xrightarrow{q_2} \mathsf{Q}_2$$

realizes Q_2 as a quotient of T by H, and the corresponding fibre functor on T^H is $P \wedge^{\pi} \omega^{q_2} \simeq \omega^{q_1}$. Therefore, there exists a commutative diagram of exact tensor functors

$$\begin{tikzcd}
\mathsf{T} \arrow[r, "X \mapsto P \wedge^{\pi} X"] \arrow[d, "q_1"] & \mathsf{T} \arrow[d, "q_2"] \\
\mathsf{Q}_1 \arrow[r] & \mathsf{Q}_2,
\end{tikzcd}$$

which depends on the choice of P lifting $\underline{\mathrm{Hom}}(\omega^{q_1}, \omega^{q_2})$ in an obvious way.

3. Polarizations

We refer to Deligne and Milne 1982, 5.12, for the notion of a (graded) polarization on a Tate triple over $\mathbb{R}$. We write V for the category of $\mathbb{Z}$-graded complex vector spaces endowed with a semilinear automorphism a such that $a^2 v = (-1)^n v$ if $v \in V^n$. It has a natural structure of a Tate triple (ibid. 5.3). The canonical polarization on V is denoted Π^{V}.

A morphism $F\colon (\mathsf{T}_1, w_1, \mathbb{T}_1) \to (\mathsf{T}_2, w_2, \mathbb{T}_2)$ of Tate triples is an exact tensor functor $F\colon \mathsf{T}_1 \to \mathsf{T}_2$ preserving the gradations together with an isomorphism $F(\mathbb{T}_1) \simeq \mathbb{T}_2$. We say that such a morphism is *compatible* with graded polarizations Π_1 and Π_2 on T_1 and T_2 (denoted $F\colon \Pi_1 \mapsto \Pi_2$) if

$$\psi \in \Pi_1(X) \Rightarrow F\psi \in \Pi_2(FX),$$

in which case, for any X homogeneous of weight n, $\Pi_1(X)$ consists of the sesquilinear forms $\psi\colon X \otimes \bar{X} \to \mathbb{1}(-n)$ such that $F\psi \in \Pi_2(FX)$. In particular, given F and Π_2, there exists at most one graded polarization Π_1 on T_1 such that $F\colon \Pi_1 \mapsto \Pi_2$.

Let $\mathsf{T} = (\mathsf{T}, w, \mathbb{T})$ be an algebraic Tate triple over $\mathbb{R}$ such that $w(-1) \neq 1$. Given a graded polarization Π on T, there exists a morphism of Tate triples $\xi_{\Pi}\colon \mathsf{T} \to \mathsf{V}$ (well defined up to isomorphism) such that $\xi_{\Pi}\colon \Pi \mapsto \Pi^{\mathsf{V}}$ (Deligne and Milne 1982, 5.20). Let ω_{Π} be the composite

$$\mathsf{T}^{w(\mathbb{G}_m)} \xrightarrow{\xi_{\Pi}} \mathsf{V}^{w(\mathbb{G}_m)} \xrightarrow{\gamma^{\mathsf{V}}} \mathsf{Vec}_{\mathbb{R}};$$

it is a fibre functor on $\mathsf{T}^{w(\mathbb{G}_m)}$.

A CRITERION FOR THE EXISTENCE OF A POLARIZATION

3.1. PROPOSITION. *Let* $\mathsf{T} = (\mathsf{T}, w, \mathbb{T})$ *be an algebraic Tate triple over* $\mathbb{R}$ *such that* $w(-1) \neq 1$, *and let* $\xi\colon \mathsf{T} \to \mathsf{V}$ *be a morphism of Tate triples. There exists a graded polarization* Π *on* T *(necessarily unique) such that* $\xi\colon \Pi \mapsto \Pi^{\mathsf{V}}$ *if and only if the real algebraic group* $\underline{\mathrm{Aut}}^{\otimes}(\gamma^{\mathsf{V}} \circ \xi|\mathsf{T}^{w(\mathbb{G}_m)})$ *is anisotropic.*

PROOF. Let $G = \underline{\mathrm{Aut}}^{\otimes}(\gamma^{\mathsf{V}} \circ \xi|\mathsf{T}^{w(\mathbb{G}_m)})$. Assume Π exists. The restriction of Π to $\mathsf{T}^{w(\mathbb{G}_m)}$ is a symmetric polarization, which the fibre functor $\gamma^{\mathsf{V}} \circ \xi$ maps to the canonical polarization on $\mathsf{Vec}_{\mathbb{R}}$. This implies that G is anisotropic (Deligne 1972, 2.6).

For the converse, let X be an object of weight n in $\mathsf{T}_{(\mathbb{C})}$. A sesquilinear form $\psi\colon \xi(X) \otimes \overline{\xi(X)} \to \mathbb{1}(-n)$ arises from a sesquilinear form on X if and only if it is fixed by G. Because G is anisotropic, there exists a $\psi \in \Pi^{\mathsf{V}}(\xi(X))$ fixed by G (ibid., 2.6), and we define $\Pi(X)$ to consist of all sesquilinear forms ϕ on X such that $\xi(\phi) \in \Pi^{\mathsf{V}}(\xi(X))$. It is now straightforward to check that $X \mapsto \Pi(X)$ is a polarization on T. ∎

3.2. COROLLARY. *Let* $F\colon (\mathsf{T}_1, w_1, \mathbb{T}_1) \to (\mathsf{T}_2, w_2, \mathbb{T}_2)$ *be a morphism of Tate triples, and let* Π_2 *be a graded polarization on* T_2. *There exists a graded polarization* Π_1 *on* T_1 *such that* $F\colon \Pi_1 \mapsto \Pi_2$ *if and only if the real algebraic group* $\underline{\mathrm{Aut}}^{\otimes}(\gamma^{\mathsf{V}} \circ \xi_{\Pi_2} \circ F|\mathsf{T}_1^{w(\mathbb{G}_m)})$ *is anisotropic.*

POLARIZATIONS ON QUOTIENTS

The next proposition gives a criterion for a polarization on a Tate triple to pass to a quotient Tate triple.

3.3. PROPOSITION. *Let* $\mathsf{T} = (\mathsf{T}, w, \mathbb{T})$ *be an algebraic Tate triple over* $\mathbb{R}$ *such that* $w(-1) \neq 1$. *Let* (Q, q) *be a quotient of* T *by* $H \subset \pi(\mathsf{T})$, *and let* ω^q *be the corresponding fibre functor on* T^H. *Assume* $H \supset w(\mathbb{G}_m)$, *so that* Q *inherits a Tate triple structure from that on* T, *and that* Q *is semisimple. Given a graded polarization* Π *on* T, *there exists a graded polarization* Π' *on* Q *such that* $q\colon \Pi \mapsto \Pi'$ *if and only if* $\omega^q \approx \omega_\Pi|\mathsf{T}^H$.

PROOF. $\Rightarrow$: Let Π' be such a polarization on Q, and consider the functors

$$\mathsf{T} \xrightarrow{q} \mathsf{Q} \xrightarrow{\xi_{\Pi'}} \mathsf{V}, \quad \xi_{\Pi'}\colon \Pi' \mapsto \Pi^{\mathsf{V}}.$$

Both $\xi_{\Pi'} \circ q$ and ξ_Π are compatible with Π and Π^{V} and with the Tate triple structures on T and V, and so $\xi_{\Pi'} \circ q \approx \xi_\Pi$ (Deligne and Milne 1982, 5.20). On restricting everything to $\mathsf{T}^{w(\mathbb{G}_m)}$ and composing with γ^V, we get an isomorphism $\omega_{\Pi'} \circ (q|\mathsf{T}^{w(\mathbb{G}_m)}) \approx \omega_\Pi$. Now restrict this to T^H, and note that

$$\left(\omega_{\Pi'} \circ (q|\mathsf{T}^{w(\mathbb{G}_m)})\right)|\mathsf{T}^H = (\omega_{\Pi'}|\mathsf{Q}^{\pi(\mathsf{Q})}) \circ (q|\mathsf{T}^H) \simeq \omega^q$$

because $\omega_{\Pi'}|\mathsf{Q}^{\pi(\mathsf{Q})} \simeq \gamma^Q$.

$\Leftarrow$: The choice of an isomorphism $\omega^q \to \omega_\Pi|\mathsf{T}^H$ determines an exact tensor functor

$$\mathsf{T}/\omega^q \to \mathsf{T}/\omega_\Pi.$$

As the quotients T/ω^q and T/ω_Π are tensor equivalent respectively to Q and V, this shows that there is an exact tensor functor $\xi\colon \mathsf{Q} \to \mathsf{V}$ such that $\xi \circ q \approx \xi_\Pi$. Evidently $\underline{\mathrm{Aut}}^{\otimes}(\gamma^{\mathsf{V}} \circ \xi | \mathsf{Q}^{w(\mathbb{G}_m)})$ is isomorphic to a subgroup of $\underline{\mathrm{Aut}}^{\otimes}(\gamma^{\mathsf{V}} \circ \xi_\Pi | \mathsf{T}^{w(\mathbb{G}_m)})$. Since the latter is anisotropic, so also is the former (Deligne 1972, 2.5). Hence ξ defines a graded polarization Π' on Q (Proposition 3.1), and clearly $q\colon \Pi \mapsto \Pi'$. ∎

References

[Bertolin, (2003)] BERTOLIN, C. 2003. Motivic Galois theory for motives of niveau ≤ 1. Preprint arXiv:math.NT/0309379.

[Deligne, (1972)] DELIGNE, P. 1972. La conjecture de Weil pour les surfaces $K3$. *Invent. Math.* 15:206–226.

[Deligne, (1989)] DELIGNE, P. 1989. Le groupe fondamental de la droite projective moins trois points, pp. 79–297. *In* Galois groups over $\mathbf{Q}$ (Berkeley, CA, 1987), volume 16 of *Math. Sci. Res. Inst. Publ.* Springer, New York.

[Deligne, (1990)] DELIGNE, P. 1990. Catégories tannakiennes, pp. 111–195. *In* The Grothendieck Festschrift, Vol. II, Progr. Math. Birkhäuser Boston, Boston, MA.

[Deligne and Milne, (1982)] DELIGNE, P. AND MILNE, J. S. 1982. Tannakian categories, pp. 101–228. *In* Hodge cycles, motives, and Shimura varieties, Lecture Notes in Mathematics. Springer-Verlag, Berlin.

[Giraud, (1971)] GIRAUD, J. 1971. Cohomologie non abélienne. Springer-Verlag, Berlin.

[Milne, (2002)] MILNE, J. S. 2002. Polarizations and Grothendieck's standard conjectures. *Ann. of Math. (2)* 155:599–610.

[Milne, (2007)] MILNE, J. S. 2007. Rational Tate classes. Preprint arXiv:0707.3167

[Saavedra Rivano, (1972)] SAAVEDRA RIVANO, N. 1972. Catégories Tannakiennes. Springer-Verlag, Berlin.

Mathematics Dept., University of Michigan, Ann Arbor, MI, USA
Website: `www.jmilne.org/math/`
Email: `jmilne@umich.edu`

This article may be accessed via WWW at `http://www.tac.mta.ca/tac/` or by anonymous ftp at `ftp://ftp.tac.mta.ca/pub/tac/ html/volumes/18/21/18-21.{dvi,ps,pdf}`